Brief paper

On leaderless and leader-following consensus for interacting clusters of second-order multi-agent systems

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Abstract
This paper investigates the group consensus phenomenon for multiple interacting clusters of double-integrator agents in the presence of both cooperative and competitive inter-cluster couplings under two different frameworks, viz., the framework that all agents share the same position and velocity interaction topology and the framework that the position and velocity topologies are modeled by totally independent graphs. Both the case without leaders and the leader-following (meaning there is a single leader for each cluster of agents) case are systematically investigated. Different systems models are systematically analyzed accordingly using various different techniques. Theoretical analysis shows that for most cases, there holds a consistent structural result that group consensus can be achieved if the underlying topology for each cluster of agents satisfies certain connectivity assumptions and further, the intra-cluster couplings are sufficiently strong. Some lower bounds for such strengths are explicitly specified as well.

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1. Introduction

The last decade has witnessed the rapid progresses in the emerging field of networked control systems (NCSs) ([Antsaklis & Baillieul, 2007]). As one of the pioneering works for NCSs, Gao, Chen, and Lam (2008) proposed a new mathematical model for describing NCSs with network-induced delay, packet dropouts/disorder and coding quantization, and proposed efficient synthesis methods for networked control considering various communication constraints in a unified framework. Recently, the (complete) consensus/synchronization problem for networked multi-agent systems has attracted great attention in systems and control theory ([Abdessameud & Tayebi, 2010; Cao & Ren, 2012; Yu, Chen, & Cao, 2010]). However, a real-world complex network may be composed of multiple smaller subnetworks, e.g., communities of natural oscillators are usually composed of interacting sub-populations (Belykh, Belykh, & Mosekilde, 2001; Biyik & Arcak, 2007; Bragagnolo, Morarescu, Daafouz, & Riedinger, 2014; Morarescu, Martion, & Girard, 2014), and such a network in general exhibits richer scenarios than just consensus or synchronization. Very recently, increasing attention has been switching from complete consensus to cluster/group consensus/synchronization (Qin & Yu, 2013; Xia & Cao, 2011; Yu, Qin, & Gao, 2014; Yu & Wang, 2010). This phenomenon is observed when the agents in a network fall into several subgroups, called clusters throughout this paper, for which agents from the same cluster asymptotically reach state agreement in the presence of both intra- and inter-cluster couplings among agents.

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In this paper, we focus our attention on the framework that agents within a cluster are cooperative, but are competitive with those in another cluster. This is different from the consensus problem considered in Biyik and Arcak (2007), Bragagnolo et al. (2014), Morarescu et al. (2014), where the network is also composed of several clusters but all the couplings among agents are cooperative. The positively weighted couplings among the nodes function as a synchronizing scheme while the couplings among nodes from different clusters which are negatively weighted serve as an inhibitory mechanism to desynchronize such nodes. A basic question is to state under what conditions with respect to the coupling strengths and the coupling topology of the network each cluster of agents can converge to or maintain their consensus/synchronization behavior in the presence of interaction between different clusters. Some results obtained in this line of research can be found in Liu and Chen (2011), Lu, Liu, and Chen (2010), Yu and Wang (2010), Yu et al. (2014), most of which are based on the condition that the inter-cluster couplings sum to zeros (termed also the in-degree balanced condition in this paper).

For example, Yu and Wang (2010), Xia and Cao (2011) considered the leaderless group consensus problem for interacting clusters of single-integrator agents; Liu and Chen (2011) analyzed the cluster synchronization behavior for interacting clusters of nonlinear oscillators via pinning control technique. Similar framework was developed recently in Yu et al. (2014), Qin and Yu (2013) to deal with the cluster consensus for network of agents with generic linear system dynamics also with usage of pinning control technique (which is equivalent to involving a leader with the same self-dynamics as that for the follower agents for each cluster of agents).

In this work, we will follow the setup in Qin and Yu (2013), Yu et al. (2014) and investigate the group consensus problem for multiple interacting clusters of double-integrator agents under same/different position and velocity interaction topologies in the framework of static interaction topology by expanding significantly on our preliminary work reported in Qin, Yu, and Anderson (2013). The contribution of the work is threefold. (a) First, for the leaderless case where the position and velocity topologies are modeled by the same graph, we work on the general directed topology and further relax the in-degree balanced condition as imposed in most of the relevant works (Qin & Yu, 2013; Qin et al., 2013; Yu et al., 2014; Yu & Wang, 2010). (b) Second, for the leader-following case, we extend the work in Qin et al. (2013) to the framework which allows not only the general directed topology but also the leaders of time-varying velocities. (c) Finally, in the framework of undirected coupling topology, we extend the result concerning the leaders of time-varying velocities by removing the conditions that require the measurement of the leaders’ accelerations (Abdessameud & Tayebi, 2010).

Differently from the algebraic conditions obtained in Xia and Cao (2011), Yu and Wang (2010) for ensuring group consensus from which it is unclear how the coupling topology relates to the group consensus behavior, this paper focuses on analyzing the structural conditions, viz., ones involving the coupling topology as well as the coupling strength, that guarantee the group consensus. Similarly to that observed in Qin and Yu (2013), Yu et al. (2014), it will be shown that for most system models, group consensus can be achieved if the interaction topologies satisfy certain connectivity assumptions while at the same time, the intra-cluster couplings are sufficiently strong. Some lower bounds on such strengths, in terms of the weights on the couplings, are explicitly provided. Note that the double-integrator models in this paper cannot be included as a special case of the linear system model in Qin and Yu (2013), Yu et al. (2014) due to the existence of nonlinear terms (incurred by the sgn function) in some system dynamics and also the fact that the system dynamics governing the leader and follower agents are different.

It is worth mentioning that there is another line of research investigating the group synchronization (termed also concurrent synchronization) from a different viewpoint based on the nonlinear contraction theory (Pham & Slotine, 2007; Russo, Bernardo, & Slotine, 2013; Russo & Slotine, 2011). The main idea developed therein is to find a flow-invariant subspace by taking advantage of the symmetries in the networks first, and then drive all the agents toward the invariant subspace. A sufficient condition provided in these works to guarantee the convergence to the invariant subspace is that the network system (or its appropriately constructed auxiliary system) is contracting in a manner allowing application of the nonlinear contraction theory. Although this condition is applicable to more general systems with either nonlinear self-dynamics and/or couplings among agents, it is usually too general to provide us with important information on the relation between contraction of the network system and the structure and strength of the network topology, which is a problem of central interest in the MAS consensus community.

In this paper, instead of employing the contraction theory, we work directly on the error system dynamics and then investigate different models.

The rest of the paper is organized as follows. A brief summary of the relevant results in graph and matrix theory is provided Section 2. Leaderless group consensus given the same position and velocity interaction topology is considered in Section 3. Section 4 investigates the case with leaders of time-varying velocities under two different settings, i.e., the one with same position and velocity interactions and the one with different position and velocity interactions. Finally, some concluding remarks are made in Section 5.

Notations: Let diag(\(\mathbf{Z}_1, \ldots, \mathbf{Z}_p\)) denote the block diagonal matrix with the \(i\)th main diagonal block being the square matrix \(\mathbf{Z}_i\), \(\lambda_{\min}(M)\) and \(\lambda_{\max}(M)\) denote respectively the smallest and largest eigenvalues of a symmetric matrix \(M\), and \(\lambda_2(M)\) denote the second smallest eigenvalue of \(M\) if \(M\) is a symmetric matrix. \(\otimes\) denotes the Kronecker product. All other notations are standard.

2. Background and preliminaries

The couplings (or interactions) among agents can be modeled by a weighted directed graph (digraph). Let \(G = (\mathcal{V}, \mathcal{E}, \lambda)\) be a weighted graph consisting of a node set \(\mathcal{V} = \{1, 2, \ldots, N\}\), a set of edges \(E \subseteq \mathcal{V} \times \mathcal{V}\), and a weighted adjacency matrix \(\mathbf{A} = [\alpha_{ij}] \in \mathbb{R}^{N \times N}\) in which \(\alpha_{ij} \neq 0\) if \((j, i) \in \mathcal{E}\) and \(\alpha_{ij} = 0\) otherwise. The set of neighbors of node \(i\) is denoted by \(\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}\). The Laplacian matrix \(\mathbf{L} = [l_{ij}]\) of a weighted graph is defined by \(l_{ij} = \sum_{k=1}^{N} \mathbf{A}_{ijk} \mathbf{s}_{jk} \mathbf{x}_k\) where \(\mathbf{s}_{jk}\) is the all-one vector. Note that since there may exist negative elements in \(\mathcal{A}\), the properties for the standard Laplacian matrix \(\mathbf{L}\) associated with a non-negatively weighted graph do not all apply here. However, there still holds the fact that \(\mathcal{L}_N = 0\). To reflect the interactions among different clusters, the adjacency matrix \(\mathcal{A}\) of \(\mathcal{G}\) can be represented as follows \(\mathcal{A} = [\mathbf{A}_{11} : \ldots : \mathbf{A}_{1q}; \mathbf{A}_{21} : \ldots : \mathbf{A}_{2q}; \ldots; \mathbf{A}_{q1} : \ldots : \mathbf{A}_{qq}]\), where submatrix \(\mathbf{A}_{ik}, \ell, k = 1, \ldots, q, \ell \neq k\), specifies the couplings from cluster \(\mathcal{V}_i\) to cluster \(\mathcal{V}_j\), while \(\mathbf{A}_{ii}\) specifies how the agents within cluster \(\mathcal{V}_i\) interact with each other.

A directed path is a sequence of edges in a directed graph of the form \((i_1, i_2), (i_2, i_3), \ldots, (i_{q-1}, i_q)\) with all \(i_j\) distinct. An undirected path in an undirected graph is defined analogously. A digraph has a directed spanning tree if there exists at least one node, called the root, having a directed path to all of the other nodes. An undirected graph is connected if there is an undirected path between every pair of distinct nodes. A digraph is called weakly connected if replacing all of its directed edges with undirected edges produces a connected (undirected) graph.
As is standard \( \{V_1, \ldots, V_q\} \) is called a partition of the set \( V = \{1, 2, \ldots, N\} \) if \( V_\ell \neq \emptyset \), \( \bigcup_{\ell=1}^{q} V_\ell = V \), and \( V_\ell \cap V_\kappa = \emptyset \) for \( \ell \neq k \), \( \ell, k = 1, \ldots, q \), but in this work each set \( V_\ell, \ell = 1, \ldots, q \) of the partition is known as a cluster. For \( i \in V \), let \( i \) denote the subscript of the subset to which the integer \( i \) belongs, i.e., \( i \in V_\ell \). We say that agents \( i \) and \( j \) are in the same cluster if \( i = j \). Let \( G_i \) denote the underlying topology of cluster \( V_\ell \), \( \ell = 1, \ldots, q \), i.e., \( V_\ell = V(G_i) \). Further, without loss of generality, assume the number of agents in a cluster, say \( V(G_i) \), is \( n_i, 1 \leq \ell \leq q \), and the \( n_i \) agents in \( V(G_i) \) are respectively indexed as \( \sum_{j=1}^{\ell-1} n_j + 1, \ldots, \sum_{j=0}^{\ell-1} n_j \), where \( n_0 = 0 \) \((N = n_1 + \cdots + n_q)\). An edge between agents from the same cluster is called intra-cluster coupling and it is called an inter-cluster coupling if it is between agents from different clusters.

**Remark 1.** There is a special way to generate the clusters for a group of interacting agents according to the associated Laplacian matrix \( \{\text{Montenbruck, Bürger, \& Allgöwer, 2015}\} \). First, the agents are rearranged so that the Laplacian matrix assumes an upper block-triangular form. The clusters are then generated by joining the nodes in the strongly connected components corresponding to neighboring diagonal blocks in the graph Laplacian. As such, the inter-cluster couplings in the resulted clusters are collectively acyclic and nonnegatively weighted. This means that the setup in \( \{\text{Montenbruck et al. (2015)}\} \) is totally different from what is considered in our work. And further, due to such limitations as well as the differences in the models, the techniques developed in \( \{\text{Montenbruck et al. (2015)}\} \) do not apply to our models.

**3. Leaderless consensus: same position and velocity graph topology**

Assume that agent \( i \) evolves according to the following system dynamics

\[
\begin{align*}
\dot{x}_i &= v_i, \\
\dot{v}_i &= \sum_{j=1}^{N} c_{ij} a_j \left( (x_j - x_i) + \gamma (v_j - v_i) \right),
\end{align*}
\]

where \( x_i \in \mathbb{R}^d \) is the position state and \( v_i \in \mathbb{R}^d \) is the velocity state of agent \( i \); \( c_{ij} = c_i \) if \( i = j \) and \( \ell \) and \( c_{ij} = c_0 \) if \( i \neq j, \gamma > 0 \) is the velocity coupling gain; \( a_j \geq 0 \) if \( i = j \) and \( a_j \in \mathbb{R} \) otherwise. Note that \( c_i, \ell, 1, \ldots, q, \) is the overall coupling strength for nodes within cluster \( V_\ell \). As will be further illustrated below, the use in Eq. (1) of couplings \( c_{ij} a_j \) with \( c_i \) obeying the constraints just stated is a notational device that makes it straightforward to consider the effect of strong versus weak intra-cluster couplings.

**Group consensus for system (1) is defined as follows:** The group consensus is said to be achieved for system (1) if, for any initial states of the nodes, there holds \( \lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0 \) and \( \lim_{t \to \infty} \|v_i(t) - v_j(t)\| = 0 \), \( \forall i, j \). To investigate the group consensus, a prerequisite requirement is that the group synchronization manifold

\( \mathcal{S}(n) = \left\{ [x_1^T, x_2^T, \ldots, x_N^T] : x_i = x_j \text{ and } v_i = v_j \text{, } \forall i \neq j \right\} \)

should be invariant for the coupled systems (1). As elaborated in \( \{\text{Lu et al. (2010)}\} \), a necessary and sufficient condition for \( \mathcal{S}(n) \) to be invariant is that the following common inter-cluster coupling condition holds

\[ \sum_{j \in V_\ell \cap V_k} a_j = \sum_{j \in V_\ell \cap V_k} a_j, \quad \forall i, i' \in V_\ell, \quad k, \ell = 1, \ldots, q, \quad k \neq \ell. \]

This means that within the same cluster, the sum of the weights of the incoming couplings from any other single cluster to each of the agents is the same. Given any \( V_\ell \) and \( V_k \), let \( r_{ik} = \sum_{j \in V_\ell \cap V_k} a_j, \forall i \in V_\ell \). A special case of condition (2) is that \( r_{ik} = 0, \forall \ell, k = 1, \ldots, q \), corresponding to the following in-degree balanced condition as that required in most of the existing works concerning group/cluster consensus \( \{\text{Liu \& Chen, 2011; Qin \& Yu, 2013; Yu \& Wang, 2010}\} \).

**Assumption 1.**

\[ \sum_{j \in V(G_i)} a_j = 0, \quad \forall i = 1, \ldots, N, \quad i \in V \setminus V(G_i), \quad \ell = 1, \ldots, q. \]

Condition (2) is equivalent to saying that each \( A_{ik} \in \mathcal{A} \) is an equal-row-sum matrix, and all such sums are zero if condition in Assumption 1 is satisfied. Let \( L = \left[ L_{11} \cdots L_{1q}; L_{21} \cdots L_{2q}; \cdots ; L_{q1} \cdots L_{qq} \right] \), denote the Laplacian matrix associated with \( A \), where \( L_{ik} = -A_{ik}, \forall \ell \neq k \). It is easy to obtain that \( L_{ik} \) is the Laplacian matrix of the non-negatively weighted graph \( G_i \) under Assumption 1. However, if condition (2) holds but not Assumption 1, then this observation is not true. On the other hand, condition (2) and Assumption 1 allow the inter-cluster couplings to be either positively or negatively weighted.

**Lemma 1.** Given any Laplacian matrix \( L = \left[ L_{ij} \right] \) associated with a non-negatively weighted digraph of order \( m \), let \( \bar{L} \) be the submatrix of \( L \) obtained by removing the first row and the first column, where \( E = \left[ 1 \quad 0 \quad 1_{m-1} \quad -1_{m-1} \right] \). Then, \( \left[ 0_{m-1} \quad I_{m-1} ; \bar{L} \quad -\bar{L} \right] \) is a Hurwitz matrix if \( G \) has a directed spanning tree and further \( \gamma \) satisfies the following condition

\[ \gamma > \max_{\mu_k \neq 0} \frac{\text{Im}^2(\mu_k)}{\text{Re}^2(\mu_k) + \text{Im}^2(\mu_k)} \]

where \( \mu_k \) denotes an eigenvalue of \( L \).

**Proof.** Denote by \( \zeta(t) = x_i(t) - x_j(t), \zeta(t) = \gamma v_i(t) - v_j(t) \), \( i, j = 2, \ldots, m \). Then, the complete consensus for system (1) is transformed equivalently to the asymptotical stability of the following system

\[
\begin{align*}
\dot{\zeta}(t) &= 0_{m-1} I_{m-1} - \bar{L} \quad \gamma \bar{L} \\
\zeta(t) &= \zeta(t) + \gamma \bar{L} \\
\end{align*}
\]

The result then follows directly from Theorem 1 in \( \{\text{Yu et al. (2010)}\} \) by noting that system (4) is asymptotically stable if \( G \) and \( \gamma \) satisfy the given conditions.

**3.1. Group consensus under general inter-cluster couplings**

Let \( x_i \) and \( v_i \) denote respectively the stack of position and velocity states of the agents in cluster \( V_\ell, \ell = 1, \ldots, q \); then systems (1) can be rewritten in the following compact form

\[
\begin{pmatrix}
\dot{x}_1 \\
\vdots \\
\dot{x}_q
\end{pmatrix} =
\begin{pmatrix}
M_{11} & M_{12} & \cdots & M_{1q} \\
M_{21} & M_{22} & \cdots & M_{2q} \\
\vdots & \vdots & \ddots & \vdots \\
M_{q1} & M_{q2} & \cdots & M_{qq}
\end{pmatrix} \otimes \bar{L} \times
\begin{pmatrix}
x_1 \\
\vdots \\
x_q
\end{pmatrix} +
\begin{pmatrix}
\delta_{1} \\
\vdots \\
\delta_{q}
\end{pmatrix}
\]

where \( M_{\ell k} = \left[ \begin{array}{ccc}
\delta_{\ell k} & -\gamma_{\ell k} & 0 \\
-\delta_{\ell k} & \delta_{\ell k} - \gamma_{\ell k} & 0 \\
0 & 0 & 0
\end{array} \right] \), \( \delta_{\ell k} \in \mathbb{R} \), \( \delta_{\ell k} \in \mathbb{R} \), \( \ell \neq k \), \( k \neq \ell \), (see condition (2) for the notation of \( r_{\ell k} \)).
Further, let $\Delta_{\ell} = \begin{bmatrix} I_{n_{\ell}-1} & -\mathbf{1}_{n_{\ell}-1} \end{bmatrix}$, $\ell = 1, \ldots, q$, and $\Delta = \text{diag}(\Delta_1, \Delta_2, \ldots, \Delta_q)$. Obviously, there holds $\Delta^2 = \mathbf{1}$. Denote by $\mathbf{c}_\ell = \begin{bmatrix} I_{n_{\ell}-1} & -\mathbf{1}_{n_{\ell}-1} \end{bmatrix} \mathbf{x}$ and $\mathbf{c}_\ell = \begin{bmatrix} I_{n_{\ell}-1} & -\mathbf{1}_{n_{\ell}-1} \end{bmatrix} \mathbf{v}_\ell$. Then, multiplying both sides of Eq. (5) by $\Delta \otimes I_n$ gives
\[
\dot{\mathbf{v}}(t) = (\mathbf{M} \otimes I_n) \mathbf{v}(t),
\]
where $\mathbf{v}(t) = [\mathbf{c}_1^T, \ldots, \mathbf{c}_q^T]^T$, 
\[
\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{D}_1 & \mathbf{M}_2 & \ldots & \mathbf{M}_q \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\mathbf{M}_q & \mathbf{D}_q & \mathbf{M}_1 & \ldots & \mathbf{M}_q \\
\end{bmatrix},
\]
\[
\dot{\mathbf{M}}_{\ell\ell} = \begin{bmatrix} 0 & -\mathbf{L}_{\ell\ell} & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 \\
\end{bmatrix},
\]
\[
\dot{\mathbf{M}}_{\ell k} = \begin{bmatrix} 0 & -\mathbf{D}_{\ell}(0) & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 \\
\end{bmatrix},
\]
\[
\mathbf{D}_\ell = \begin{bmatrix} 0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0 \\
\end{bmatrix}, \quad \forall k \neq \ell,
\]
and $\mathbf{L}_{\ell k}$ is the submatrix of $\Delta_{\ell} \mathbf{L}_{\ell k} \Delta_{\ell}^T$.
If $\mathbf{G}_\ell$ has a directed spanning tree and further,
\[
\gamma > \max_{\mu_{\ell}^2 \neq 0} \sqrt{\frac{\text{det}(\mathbf{M}_{\ell\ell})}{\text{det}(\mathbf{M}_{\ell\ell}^2 + \text{det}(\mathbf{M}_{\ell\ell}^2))}},
\]
where $\mu_{\ell}^2, k = 1, \ldots, n_{\ell}$, are the eigenvalues of $\mathbf{M}_{\ell\ell}$, then there exists a Hurwitz matrix $\mathbf{G}_\ell$ satisfying Lemma 1 from which, in turn, implies that there exists a symmetric positive-definite matrix, say $\mathbf{L}_{\ell\ell} > 0$, such that $\mathbf{M}_{\ell\ell}^2 \mathbf{L}_{\ell\ell} + \mathbf{E}_\ell \mathbf{M}_{\ell\ell}^2 < 0$. Let $\mathbf{M} = \mathbf{M} - \text{diag}(\mathbf{c}_1 \mathbf{M}_1, \ldots, \mathbf{c}_q \mathbf{M}_q)$, thus $\mathbf{M}$ is a matrix independent of the intra-cluster coupling strengths $c_{\ell}$, $\ell = 1, \ldots, q$. Now consider the following Lyapunov function candidate for system (6)
\[
V(t) = \mathbf{v}(t)^T \left( \text{diag}(\mathbf{E}_1, \ldots, \mathbf{E}_q) \otimes I_n \right) \mathbf{v}(t).
\]
Differentiating $V(t)$ along the trajectory of (6) gives
\[
\dot{V}(t) = \mathbf{v}(t)^T \left( \left[ \text{diag} \left( \mathbf{c} \left( \mathbf{E}_1 \mathbf{M}_1 + \mathbf{M}_{\ell\ell} \right) \mathbf{E}_1 \right) \right] \otimes I_n \right) \mathbf{v}(t)
\]
where $\dot{\mathbf{M}} = \text{diag} [\mathbf{c}_1 (\mathbf{E}_1 \mathbf{M}_1 + \mathbf{M}_{\ell\ell} \mathbf{E}_1), \ldots, \mathbf{c}_q (\mathbf{E}_q \mathbf{M}_q + \mathbf{M}_{\ell\ell} \mathbf{E}_q)]$.

Evidently, to guarantee $\dot{V}(t) < 0$, it suffices to have
\[
c_{\ell} \lambda_{\min} \left( -\mathbf{E}_\ell \dot{\mathbf{M}}_{\ell\ell} - \mathbf{M}_{\ell\ell}^2 \mathbf{E}_\ell \right) + \lambda_{\min} \left( -\mathbf{M}_{\ell\ell}^2 \right) > 0, \quad \ell = 1, \ldots, q.
\]
Clearly, this inequality holds for any $c_{\ell} > \frac{\lambda_{\min}(\mathbf{M}_{\ell\ell})}{\lambda_{\max}(\mathbf{E}_\ell \mathbf{M}_{\ell\ell} + \mathbf{M}_{\ell\ell} \mathbf{E}_\ell)}$, $\ell = 1, \ldots, q$. That is to say, $\dot{V}(t) < 0$ if $c_{\ell} > \max_{\ell \leq \ell \leq q, \mu_{\ell}^2 \neq 0} 0$, $\ell = 1, \ldots, q$. Finally, group consensus follows straightforwardly by noting that system (6) is asymptotically stable.

Summarizing the above analysis gives the first of our main results, which can be compared with Lemma 1.

**Theorem 1.** Under condition (2), the interacting clusters of agents (1) achieve group consensus, if (i) the underlying topology for each cluster has a directed spanning tree; (ii) $\gamma > \max_{\ell \leq \ell \leq q, \mu_{\ell}^2 \neq 0} 0$, $\ell = 1, \ldots, q$.

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**Remark 2.** The condition on $\gamma$ in Theorem 1 can be relaxed to $\gamma > 0$ for undirected topology. Note that the common inter-cluster coupling condition (2) is more general than the in-degree balanced condition as required in Qin and Yu (2013), Qin et al. (2013), Yu and Wang (2010), Yu et al. (2014).

Note that in Theorem 1 there may or may not be consensus between agents from different clusters, depending on the initial states of the agents. To achieve the desired group consensus behavior, it is often necessary to include a leader for each cluster of agents. This is what will be considered in the following sections.

4. Leader-following consensus: with multiple leaders of time-varying velocities

Assume that there is a single leader of time-varying velocity for each cluster of agents; and the leader $s_\ell$ for cluster $\mathcal{V}_\ell$ evolves according to the following dynamics
\[
\dot{x}_s(t) = v_s(t), \quad \ell = 1, \ldots, q,
\]
where $v_s(t)$ is the velocity state of leader $s_\ell$.

Note that all other agents within cluster $\mathcal{V}_\ell$ will also be called the follower agents of leader $s_\ell$. Group consensus in the presence of $q$ leaders is said to be achieved if, for any initial states of the agents, there hold $\lim_{t \to \infty} ||x_{\ell}(t) - x_s(t)|| = 0$ $\lim_{t \to \infty} ||v_{\ell}(t) - v_s(t)|| = 0$, $\ell = 1, \ldots, q, i \in \mathcal{V}$.

4.1. Same position and velocity graph topologies

Assume in this framework that agent $i$ evolves according to the following dynamics
\[
\begin{aligned}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= \frac{1}{d_i} \sum_{j=1}^{N} C_{ij} a_{ij} (x_j - x_i) + d_i c_i (v_j - v_i) \\
&\quad + \gamma \sum_{j=1}^{N} C_{ij} a_{ij} (v_j - v_i) + d_i c_i (v_j - v_i) \\
&\quad + \frac{1}{d_i} \sum_{j=1}^{N} C_{ij} a_{ij} \dot{v}_j + d_i c_i \dot{v}_j,
\end{aligned}
\]
where $\xi_i = \sum_{j=1}^{N} a_{ij} + d_i$, $\gamma > 0$; $c_0$ and $c_i$ are as that defined for systems (1); $d_i > 0$ if agent $i$ can receive the state information of leader $s_\ell$ and $d_i = 0$ otherwise.

Let $\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 & \cdots & \mathcal{L}_q \end{bmatrix}$, where $\mathcal{L}_\ell = L_{\ell\ell} + D_{\ell\ell}$, $\ell = 1, \ldots, q$, and $D_{\ell\ell} = \text{diag}(d_1, \ldots, d_q) = \text{diag}(D_1, \ldots, D_q)$. Note that in fact $D_{\ell\ell}$ specifies how the agents in cluster $\mathcal{V}_\ell$ have access to the state information of their leader $s_\ell$.

**Theorem 2.** Under Assumption 1, if det (\mathcal{L}) \neq 0, then for any $\gamma > 0$, multi-agent systems (8) achieve group consensus. In particular, if each $\mathcal{G}_\ell$ has a directed spanning tree and further
\[
c_{\ell} > \max_{\ell \leq \ell \leq q, \mu_{\ell}^2 \neq 0} 0, \quad \ell = 1, \ldots, q.
\]
where $\Sigma$ is the positive diagonal matrix satisfying $\Sigma \bar{L}_{\ell t} + \Sigma \epsilon = 0$
and $\Sigma = \Sigma (L - \text{diag}([1, \ldots, L]\{1\})) + (L - \text{diag}([11, \ldots, \bar{L}1]\{1\})) \Sigma$, then all the eigenvalues of $L$ have positive real parts, and group consensus is achieved for interacting clusters of agents (8) for any $\gamma > 0$.

**Proof.** Let $\bar{x}_i(t) = x_i(t) - x_i(t), \bar{v}_i(t) = v_i(t) - v_i(t), i = 1, \ldots, N$, and $\bar{x}(t) = [\bar{x}^T_1(t), \ldots, \bar{x}^T_N(t)]^T, \bar{v}(t) = [\bar{v}^T_1(t), \ldots, \bar{v}^T_N(t)]^T$. Note that $t$ may be dropped in what follows for notational simplicity. If $j \in \mathcal{V}_i$, then $j = \epsilon$ and thus $\bar{x}(t) = x_i(t) - x_i(t)$. Similarly to that derived in Qin and Yu (2013), one obtains
$$
\sum_{j=1}^N c_{ij}(x_i(t) - x_j(t)) = \sum_{j=1}^N c_{ij} \bar{x}_j(t) - \bar{x}_i(t))
$$
and
$$
\sum_{j \in \mathcal{V}_i} c_{ij}(v_j(t) - v_i(t)) = \sum_{j=1}^N c_{ij} (\bar{v}_j(t) - \bar{v}_i(t)).
$$
This together with (8) yields that
$$(L \otimes I_n) \bar{x} = -(L \otimes I_n) \bar{x} - \gamma (L \otimes I_n) \bar{v}.$$ 
If matrix $L$ is nonsingular, then one has
$$
\begin{bmatrix}
\bar{x}(t) \\
\bar{v}(t)
\end{bmatrix} = - \begin{bmatrix}
0 & -I_n \\
I_n & \gamma I_n
\end{bmatrix} \begin{bmatrix}
\bar{x}(t) \\
\bar{v}(t)
\end{bmatrix}.
$$
(10)

Noting that the characteristic polynomial of the matrix $\begin{bmatrix} 0 & -I_n \\ I_n & \gamma I_n \end{bmatrix}$,
denoted by $P(\lambda)$, is $P(\lambda) = (\lambda^2 - \lambda \gamma + 1)^N$, it follows straightforwardly that all the eigenvalues of the system for system (10) are located in the open left-half plane, therefore leading to the conclusion that $\bar{x}(t) \to 0$ and $\bar{v}(t) \to 0$ as $t \to \infty$. This completes the proof for the first argument.

For the second argument, instead of proving that $L$ is a nonsingular matrix, we prove a stronger result, i.e., all eigenvalues of $L$ have positive real parts if each $c_{ij}$ has a directed spanning tree and $c_{ii} \ell = 1, \ldots, q$, is large enough. Let $\Sigma = \text{diag}([\Sigma_1, \ldots, \Sigma_q])$. Evidently, to prove the eigenvalue argument, it suffices to prove that $\Sigma L + L^T \Sigma > 0$. This inequality holds if $\lambda_{\min}(\text{diag}[c_{ii}\Sigma_1, \ldots, \Sigma_1]), c_{ii}\Sigma_1') + \lambda_{\min}(\text{diag}[c_{ii}\Sigma_q, \ldots, \Sigma_q]) + \lambda_{\min}(\text{diag}[c_{ii}]) > 0$, which can be guaranteed if condition (9) applies.

If the interaction topology among the follower agents is undirected, agents evolving according to the following system dynamics
$$
\begin{align*}
\dot{\bar{x}}_i &= v_i \\
\dot{\bar{v}}_i &= \sum_{j=1}^N c_{ij} \alpha (x_i - x_j) + \gamma (v_i - v_j) \\
&\quad + \alpha c_{ij} \beta (\bar{x}_j - x_i) + (v_i - v_j) \\
&\quad + \alpha c_{ij} \beta (\bar{x}_j - x_i) + (v_i - v_j)
\end{align*}
$$
(11)

where $\alpha > 0$, $\beta > 0$, $\gamma > 0$, achieve group consensus as well, where no measurements of the leaders’ and the neighbors’ accelerations are required. Throughout the following part of this subsection, it is assumed that the leaders’ acceleration satisfies $|\bar{v}_i(t)| \leq \psi_i$, where $\psi_i$ is a positive number.

**Theorem 3. Under Assumption 1, group consensus is achieved for the interacting clusters of agents (11) if (i) $G_{\ell} = (\bar{L}_{\ell t})$ is strongly connected; (ii) $c_{ij} > \max \left\{0, -\frac{\lambda_{\min}(L - \text{diag}([11, \ldots, \bar{L}1]\{1\}))}{\lambda_{\min}(\bar{L}_{\ell t})} \right\}$, $i = 1, \ldots, N$, $\ell = 1, \ldots, q$, and (iii) $\psi_i < \alpha$, $\ell = 1, \ldots, q$, and $0 < \beta < \min \left\{\sqrt{\lambda_{\min}(\bar{L})}, \frac{4 \sqrt{\lambda_{\min}(L)}}{4 \gamma^2 \lambda_{\min}(L)} \right\}$.**

**Proof.** For notational simplicity, assume that $n = 1$; note however, all the results and the technical analysis are still valid for the higher dimension case through introduction of the Kronecker product. By observing the proof for the previous theorems, it is easy to derive the following compact system dynamics
$$
\begin{align*}
\dot{\bar{x}}(t) &= \bar{v}(t) \\
\dot{\bar{v}}(t) &= -(\alpha \bar{x}) L \bar{x} - \alpha \psi_i |L(\beta \bar{x} + \bar{v}) - \bar{v}_0|
\end{align*}
$$
(12)

where $\bar{v}_0 = [v_{i1}, v_{i1}^T, \ldots, v_{i1}^T, v_{i1}^T, \ldots, v_{i1}^T, v_{i1}^T, \ldots, v_{i1}^T, v_{i1}^T, \ldots, v_{i1}^T]$, $\bar{x}(t)$ and $\bar{v}(t)$ are as those defined in the proof of Theorem 2. Note that $L$ is a symmetric positive-definite matrix under conditions (i) and (ii) in the theorem statement.

Next, consider the following Lyapunov function candidate for system (12):
$$
V(t) = \frac{1}{2} \begin{bmatrix} \bar{x}^T \bar{x} \\
\bar{v}^T \bar{v} \end{bmatrix} \begin{bmatrix} \beta \bar{L}^2 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \bar{x} \\
\bar{v} \end{bmatrix}
$$

Since $L > 0$, it follows from the Schur complement lemma (Boyd, Ghaoui, Feron, & Balakrishnan, 1994) that $V(t)$ is positive definite if $\frac{1}{2} \beta \bar{L}^2 - \frac{1}{2} L (\frac{1}{2} \beta^2 L)\frac{1}{2} \bar{L} > 0 \iff \beta < \sqrt{\lambda_{\min}(\bar{L})}$. On the other hand, differentiating $V(t)$ yields
$$
\begin{align*}
\dot{V}(t) &= -\left[\bar{x}^T \bar{x} + \bar{v}^T \bar{v} \right] Q \begin{bmatrix} \bar{x} \\
\bar{v} \end{bmatrix} \\
&\quad - (\alpha \psi_i \bar{x}) L [\bar{x}(\beta \bar{x} + \bar{v}) - \bar{v}_0] \\
&\quad - [\bar{x}^T \bar{x}] Q \begin{bmatrix} \bar{x} \\
\bar{v} \end{bmatrix} - (\alpha - \psi_i) \|L(\beta \bar{x} + \bar{v})\|_1
\end{align*}
$$

where $Q = \begin{bmatrix} \beta \bar{L}^2 & \frac{1}{2} \beta \bar{L}^2 \\
\frac{1}{2} \beta \bar{L}^2 & \gamma^2 \bar{L}^2 \end{bmatrix}$. Evidently, to guarantee that $\dot{V}(t)$ is negative definite, it suffices to have $\psi_i < \alpha$ and also $Q$ positive definite, the latter of which can be guaranteed if $0 < \beta < \sqrt{\lambda_{\min}(\bar{L})} - \frac{\lambda_{\min}(\bar{L})}{4 \gamma^2 \lambda_{\min}(\bar{L})}$ (cf. Lemma 4.1 Cao & Ren, 2012). Then, it follows from Theorem 3.1 in Shevitz and Paden (1994) that $\bar{x}(t) \to 0$ and $\bar{v}(t) \to 0$ as $t \to \infty$, therefore completing the proof.

4.2. Independent position and velocity topologies

In the previous sections, it is assumed that the topologies modeling the position and velocity interactions are the same. However, one can envisage that in some applications, the position and velocity may be measured in different ways, e.g., using different sensors; and further, even if they are measured in the same way, information loss may lead to the heterogeneity of position and velocity interaction topology. To capture this idea, we now assume that the graphs modeling respectively the position and velocity interactions among agents are totally independent. Such a scenario was also considered in Goldin and Raisch (2014), which studies complete consensus for double-integrator agents over networks modeled by undirected graphs. In this subsection, we move beyond their contribution by investigating the cluster consensus problem over directed network topologies.

It is assumed in this framework that there are multiple leaders of the same time-varying velocity, i.e., $v_{i1} = v_i(t)$, $\forall i = 1, \ldots, q$. In what follows, $G^p = (V, E^p, A^p = \{a_{ij}\})$ and $G^v = (V, E^v, A^v = \{b_{ij}\})$ are employed to represent respectively the position and velocity interaction topologies of all the $N$ follower
agents, where $G$ is non-negatively weighted while $G'$ may have negatively weighted edges between agents from different clusters, i.e., $b_{ij} \geq 0$; $a_{ij} \geq 0$ if $i = j$ while $a_{ij} \in \mathbb{R}$ if $i \neq j$. The Laplacian matrices of $G$ and $G'$ are denoted by $L$ and $L'$ respectively. Further, $\mathcal{N}_i^G = \{ j \in \mathcal{V} : a_{ij} \neq 0 \}$ and $\mathcal{N}_i^{G'} = \{ j \in \mathcal{V} : b_{ij} > 0 \}$ denote the position neighbor set and velocity neighbor set of agent $i$ in $G$ and $G'$, respectively. Denote by $G^p_i$, $\ell = 1, \ldots , q$, the underlying position topology of cluster $V_i$.

Let $G$ denote the digraph consisting of $G'$, the $q$ leader agents and the directed edges from these leader agents to the follower agents in $\mathcal{V}$ which have access to their velocity information $v_i$. Suppose that the initially given $G$ has a united directed spanning tree (i.e., for each of the $N$ followers, there exists at least one leader agent that has a directed path in $G$ to the follower agent).

Assume the system dynamics for agent $i$ is described by

$$\begin{align*}
\dot{x}_i = & \ v_i \\
\dot{v}_i = & \ \frac{1}{\zeta_i} \left[ \sum_{j \in \mathcal{N}_i^G} \Gamma_i d_i (x_j - x_i) + C_i d_i (x_i - x_i) \right] \\
& + \frac{\gamma}{\zeta_i} \left[ \sum_{j \in \mathcal{N}_i^{G'}} b_{ij}(v_j(t) - v_i) + \epsilon_i (v_i(t) - v_i) \right] \\
& + \frac{1}{\zeta_i} \left[ \sum_{j \in \mathcal{N}_i^{G'}} b_{ij} + \epsilon_i \dot{v}_i \right],
\end{align*}$$

(13)

where $\gamma > 0$, $\zeta_i = \epsilon_i + \sum_{j=1}^N b_{ij}$; $d_i > 0$ if agent $i$ can receive the position state information of leader $\bar{i}$ and otherwise $d_i = 0$; $\epsilon_i > 0$ if agent $i$ can receive the velocity state information of leader $\bar{i}$ and $\epsilon_i = 0$ otherwise; $c_i$ and $\zeta_i$ have exactly the same meaning as that defined for systems (1). Note that $\zeta_i > 0$ since $G$ has a united directed spanning tree.

Let $D' = \text{diag} (\epsilon_1, \ldots , \epsilon_N)$ and $D'' = \text{diag} (d_1, \ldots , d_N)$, where $D' \in \mathbb{R}^{N \times N}$, $\ell = 1, \ldots , q$. Further, denote by $G^{p}, \ell = 1, \ldots , q$, the graph consisting of $G^p_i$, the leader agent $\ell$, and the edges from this leader agent to those follower agents in cluster $V_i$ which have access to its position information $x_i^{\ell}(t)$. Recall that $L$, the Laplacian matrix associated with the position interaction topology $G^p$, is $L^p = [\alpha_{ij}] = [L_{11} \quad L_{12} \quad L_{1q}; \quad L_{q1} \quad \cdots \quad \cdots \quad L_{qq}]$ denote by

$${\mathcal{L}}^p = \begin{bmatrix} c_1 & 1 & \cdots & 1 \\ 1 & c_2 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots \\ 1 & \cdots & \cdots & c_q \end{bmatrix},$$

where $L_{\ell\ell} = L_{\ell \ell} + D'_{\ell}$, $\ell = 1, \ldots , q$. Then we have the following result.

**Theorem 4.** Assume that the position inter-cluster couplings satisfy Assumption 1 and $G$ has a united directed spanning tree. Then, group consensus is achievable for system (13) with multiple dynamic leaders, i.e., there exists a $\gamma > 0$ such that $\lim_{{t \to \infty}} \|x_i(t) - x_j(t)\| = 0$ and $\lim_{{t \to \infty}} \|v_i(t) - v_j(t)\| = 0$, $\forall i, j \in \mathcal{V}$, if and only if all the eigenvalues of the matrix $L_{\text{off}} = (L'' + D'')^{-1} \mathcal{L}^p$ have positive real parts. In particular, in the framework of undirected position and velocity interaction topologies, group consensus is achievable for any $\gamma > 0$ if (i) $G_{\ell}$, $\ell = 1, \ldots , q$, is weakly connected and $G$ has a united spanning tree; and (ii) $\epsilon_i > \max (0, \frac{-\lambda_{\text{min}}(\mathcal{L}^p)}{\lambda_{\text{min}}(\mathcal{L}^p + d_i)})$, $\ell = 1, \ldots , q$, where $L_{\text{off}} = L^p - \text{diag} (L_{11}, \ldots , L_{qq})$.

**Proof.** Let $\bar{x}_i(t) = x_i(t) - x_\ell(t)$, $\bar{v}_i(t) = v_i(t) - v_\ell(t)$, $i = 1, \ldots , N$, and $\bar{x}_i(t) = [\bar{x}_i^{1}(t), \ldots , \bar{x}_i^{q}(t)]$, $\bar{v}_i(t) = [\bar{v}_i^{1}(t), \ldots , \bar{v}_i^{q}(t)]$.

Note that all the eigenvalues of matrix $L'' + D''$ have positive real parts and thus $L'' + D''$ is nonsingular if $G$ has a united directed spanning tree. Then, similarly to that derived in the previous proof, it is not difficult to obtain that

$$\begin{align*}
\dot{\bar{x}}_i(t) &= \bar{v}_i(t) \\
\dot{\bar{v}}_i(t) &= - \left[ (L'' + D'')^{-1} \mathcal{L}^p \right] \bar{x}_i(t) - \gamma \bar{v}_i(t)
\end{align*}$$

which can be presented equivalently as

$$\begin{align*}
\dot{\bar{x}}_i(t) &= \bar{v}_i(t) \\
\dot{\bar{v}}_i(t) &= - \left[ L_{\text{off}} - \beta b_{ij} \right] \bar{x}_i(t) - \gamma \bar{v}_i(t)
\end{align*}$$

(14)

Evidently, the problem of establishing group consensus is equivalent to the problem of establishing the asymptotic stability of system (14). Denote by $\lambda_1, \ldots , \lambda_N$ the $N$ eigenvalues (repetition may exist) of matrix $L_{\text{off}}$, it follows that the characteristic polynomial of matrix $\left[ \begin{bmatrix} 0 & -\beta \end{bmatrix} \right]$, which we denote by $P(\lambda)$, is $P(\lambda) = \prod_{i=1}^{N} (\lambda^2 - \lambda \gamma + \lambda_i)$. For each $i$, the two roots of equation $\lambda^2 - \lambda \gamma + \lambda_i = 0$ are

$$\lambda_{i,1,2} = \frac{\gamma + \sqrt{\gamma^2 - 4\gamma \lambda_i}}{2} \quad \text{and} \quad \lambda_i = \frac{\lambda_i^2 - \gamma^2}{2\gamma} < \frac{\lambda_i^2}{2\gamma} < \frac{\lambda_i^2}{\gamma} \text{, i.e.,}$$

$$16|\lambda_i|^2 < 8\gamma^2 Re(\lambda_i) \quad \Rightarrow \quad \lambda_i > 2|\lambda_i|^2 \quad \Rightarrow \quad \lambda_i > \frac{2|\lambda_i|^2}{Re(\lambda_i)} \quad \Rightarrow \quad \lambda_i > \frac{2|\lambda_i|}{\sqrt{Re(\lambda_i)}}$$

thereby completing the proof for the first statement.

The second statement follows directly by observing that both $L''$ and $L'' + D''$ are symmetric positive-definite matrices and thus all the eigenvalues of $L_{\text{off}}$ are positive numbers (Horn & Johnson, 1991).

Like that considered in Section 4.1, in the framework of undirected interaction topology one can expect more interesting result where no measurement of leaders’ acceleration is required to achieve the group consensus. The result is detailed as follows.

**Theorem 5.** Assume that both the position and velocity interaction topologies are undirected and the position interaction topology satisfies the in-degree balanced condition Assumption 1. If $G_{\ell}$, $\ell = 1, \ldots , q$, is weakly connected, $G$ has a united directed spanning tree, and further (i) $\epsilon_i > \max (0, \frac{-\lambda_{\text{min}}(\mathcal{L}^p)}{\lambda_{\text{min}}(\mathcal{L}^p + d_i)})$; (ii) $\psi < \alpha$ and $0 < \beta < \min \left\{ \frac{\lambda_{\text{min}}(\mathcal{L}^p)}{2\lambda_{\text{min}}(\mathcal{L}^p + d_i)} , \frac{\lambda_{\text{min}}(\mathcal{L}^p + d_i)}{2\lambda_{\text{min}}(\mathcal{L}^p)} \right\}$; and (iii) $\lambda_{\text{min}}(\mathcal{L}^p) > (2 + \sqrt{2})/2$, where $\mathcal{L}^p = L'' + D''$, then group consensus is achieved for interacting clusters of agents evolving according to the following dynamics

$$\begin{align*}
\dot{x}_i &= \ v_i \\
\dot{v}_i &= \ \sum_{j \in \mathcal{N}_i^G} \Gamma_i d_i (x_j - x_i) + C_i d_i (x_i - x_i) \\
& + \frac{1}{\beta} \left[ \sum_{j \in \mathcal{N}_i^{G'}} b_{ij}(v_j(t) - v_i) \right] \\
& + \alpha \text{sgn} \left\{ \beta \left[ \sum_{j \in \mathcal{N}_i^{G'}} C_i d_i (x_j - x_i) \right] \right\} \\
& + \left[ \sum_{j \in \mathcal{N}_i^{G'}} b_{ij}(v_j(t) - v_i) \right] \\
& + \epsilon_i (v_i(t) - v_i) \right].
\end{align*}$$
of wheeled mobile robots with underlying topology as shown in Fig. 1. We consider two interacting clusters where the leaders have the same time-varying velocity, and therefore completing the proof.

**Proof.** Similarly to the proof for Theorem 3, one obtains the following compact system dynamics

\[
\begin{align*}
\dot{\mathbf{x}} &= \mathbf{v} \\
\dot{\mathbf{v}} &= -L^p \mathbf{x} - \gamma L^v \mathbf{v} - \alpha \text{sgn} \left[ \beta L^p \mathbf{x} + L^v \mathbf{v} \right] - \mathbf{v}_0,
\end{align*}
\]

where \( \mathbf{v}_0 = 1_N \otimes \dot{v}_i \). Note that \( L^p \) is symmetric positive-definite under condition (i) and the condition that each \( G_i^p \) is weakly connected; further, the condition that \( G^q \) has a united directed spanning tree yields the fact that \( L^q \) is also symmetric positive-definite.

Now consider the following Lyapunov function candidate:

\[
V(t) = \mathbf{x}^T(t) L^p \mathbf{x}(t) + 2 \beta \mathbf{x}^T(t) L^v \mathbf{v}(t) + \mathbf{v}^T(t) L^v \mathbf{v}(t).
\]

It follows from the Schur Complement lemma (Boyd et al., 1994) that \( V(t) \) is positive definite if \( \beta < \sqrt{\frac{\lambda_{\text{max}}(L^v)}{\lambda_{\text{max}}(L^2)}} \). Differentiating \( V(t) \) yields

\[
\dot{V} \leq - \left[ \mathbf{x}^T(t) \mathbf{v}(t) \right] Q \left[ \mathbf{x}^T(t) \mathbf{v}(t) \right]^T - 2(\alpha - \psi) \| \beta L^p \mathbf{x} + L^v \mathbf{v} \|_2,
\]

where

\[
Q = \begin{bmatrix}
2\beta(L^p)^2 & 2L^p L^v - L^p \\
2L^p L^v - L^p & 2\beta(L^v)^2 - 2L^p
\end{bmatrix}.
\]

Evidently, to guarantee that \( \dot{V}(t) \) is negative definite, it suffices to have \( \psi < \alpha \) and also \( Q \) positive definite. Furthermore, \( Q > 0 \), according to the Schur complement lemma (Boyd et al., 1994), can be guaranteed if (a) \( \frac{\lambda_{\text{min}}(L^p)}{\lambda_{\text{min}}(L^v)} > 0 \), (b) \( 4(L^p)^2 - (2L^p L^v - L^p)(L^v)^2 - \beta^2 L^p \geq 0 \), and (c) \( 4L^p L^v - L^p > 0 \). Now (a) holds if \( \beta < \sqrt{\frac{\lambda_{\text{min}}(L^v)}{\lambda_{\text{max}}(L^v)}} \), while (b) holds if \( (\lambda_{\text{min}}(L^v) - \beta^2 L^p) > 0 \), which can be guaranteed if \( \beta < \sqrt{\frac{\lambda_{\text{min}}(L^v)}{\lambda_{\text{max}}(L^v)}} \), and further,

\[
4(L^p)^2 - 2(2L^p L^v - L^p)(L^v)^2 - (2L^p L^v - L^p) > 0,
\]

which is equivalent to \( 2 \gamma - \left[ 4\gamma - 4(L^v)^{-1} + (L^v)^{-2} \right] > 0 \), therefore yielding the condition that \( \lambda_{\text{min}}(L^v) > (2 - \sqrt{2}) \). Then, it follows from Theorem 3.1 in Shevitz and Paden (1994) that \( V(t) \to 0 \) and \( \mathbf{v}(t) \to 0 \) as \( t \to \infty \), therefore completing the proof. \( \blacksquare \)

**Example 1.** This example illustrates the case of a directed \( G^p \), where the leaders have the same time-varying velocity, and \( G^q \) is allowed to take any structure. We consider two interacting clusters of wheeled mobile robots with underlying topology as shown in Fig. 2. Different position and velocity interaction topologies: varying velocity.

![Fig. 1. Different position and velocity interaction topologies: varying velocity.](image1)

**Fig. 1.** The system dynamics of the wheeled robots are described by

\[
\begin{align*}
\dot{x}_i &= v_i \cos(\theta_i), \quad \dot{y}_i = v_i \sin(\theta_i), \quad \dot{\theta}_i = \omega_i, \quad i = 1, \ldots, 5, \quad (15)
\end{align*}
\]

where \((x_i, y_i)\) is the position of the center of the \( i \)-th robot, \( \theta_i \) is the orientation of the \( i \)-th robot, and \( v_i \) and \( \omega_i \) are the linear and rotational velocities of the \( i \)-th robot. To avoid using the nonlinear double-integrator dynamics system model in (16), one then obtains a double-integrator dynamics system model

\[
\begin{bmatrix}
\dot{x}_{hi} \\
\dot{y}_{hi}
\end{bmatrix} = \begin{bmatrix}
u_{hi} \\

\end{bmatrix}, \quad \begin{bmatrix}
\dot{u}_{hi} \\
\dot{v}_{hi}
\end{bmatrix} = \begin{bmatrix}
\tau_{hi} \\
\tau_{vi}
\end{bmatrix},
\]

where \( \nu_{hi} \) and \( \tau_{hi} \) are the control inputs.

Furthermore, the two leaders are assumed to move with the same time-varying velocity, i.e., \( v_1^i = v_2^i = 1 \), \( \omega_1^i = \omega_2^i = 0.8 \sin(t) \). By choosing different initial states, the two leaders can have different trajectories. If \( c_1 = c_2 = 1 \), then all eigenvalues of \( L_{y_1} \) have positive real parts. Applying control strategy (13) to the double-integrator dynamics (16), it is easily observed from Fig. 2 and Fig. 3, which plot respectively the position and velocity trajectories of the 7 robots, that group consensus is reached for \( \gamma = 2 \) if \( c_1 = c_2 = 1 \).
5. Conclusions

We have analyzed various leaderless and leader-following group consensus for agents with double-integrator dynamics under two different frameworks: in one, the underlying position and velocity interaction topology are the same and in the other, the position and velocity topologies are modeled by totally independent graphs. For both frameworks, we have explored the conditions with respect to the coupling strengths and the coupling topology of the network to guarantee the group consensus. Such conditions show that for most of the group consensus algorithms, there holds a consistent structural result that the complete consensus for agents within the same cluster can be achieved if the underlying topology for each cluster of agents satisfies certain connectivity assumptions and further, the intra-cluster couplings are sufficiently strong.

For the future work, an interesting direction is to investigate the case with switching topologies by integrating the novel notion of stage dwell time switching (Zhang, Zhou, & Braatz, 2016) and effective techniques for finite-time stability analysis of switched nonlinear systems (Fu, Ma, & Chai, 2015). Inspired by the interesting work in Qiu, Ding, Gao, and Yin (2016), another direction for future research would be to investigate the reliable control for multi-agent systems within the distributed parameter framework.

References


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