Distributed formation control with relaxed motion requirements

Adrian N. Bishop\textsuperscript{1,2,3,*}, Mohammad Deghat\textsuperscript{1,2}, Brian D. O. Anderson\textsuperscript{1,2} and Yiguang Hong\textsuperscript{4}

\textsuperscript{1}National ICT Australia, Canberra, Australia  
\textsuperscript{2}Australian National University, Canberra, Australia  
\textsuperscript{3}University of Technology, Sydney, Australia  
\textsuperscript{4}Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China

SUMMARY

Heterogeneous formation shape control with interagent bearing and distance constraints involves the design of a distributed control law that ensures the formation moves such that these interagent constraints are achieved and maintained. This paper looks at the design of a distributed control scheme to solve different formation shape control problems in an ambient two-dimensional space with bearing, distance and mixed bearing and distance constraints. The proposed control law allows the agents in the formation to move in any direction on a half-plane and guarantees that despite this freedom, the proposed shape control algorithm ensures convergence to a formation shape meeting the prescribed constraints. This work provides an interesting and novel contrast to much of the existing work in formation control where distance-only constraints are typically maintained and where each agent’s motion is typically restricted to follow a very particular path. A stability analysis is sketched, and a number of illustrative examples are also given. Copyright © 2014 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The general distributed formation control problem involves a group of agents that are tasked with maintaining a prescribed geometrical formation described in terms of distance and/or angular constraints.

There are two common aspects of each formation control scheme that precede the controller design. Firstly, the sensing technology and sensing graph should be formed. The sensing technology describes what kind of measurements are taken, and the sensing graph describes for each agent what aspects of what other agents in the formation are measured. The sensing technology for formation control typically consists of either bearing measurements [1–6] or distance measurements [7], but typically, both kinds of measurements are taken, which amounts to a relative position measurement [8–18]. Bearing measurements are normally taken locally in each agent’s individual coordinate basis, rather than a global coordinate basis. Thus, each agent is only required to have a local equivalent of, say, a ’north’ direction to which bearings are measured.

Secondly, albeit not independently, the control graph and the controlled parameters are defined [8]. It is typical for the topology of the control and sensing graph to be equivalent meaning that agents control some geometrical relationship to those agents concerning which some measure-
ments are taken. However, it is typical that the control constraints be either distance [12–15] or bearing-only constraints [1–5]. Formations with mixed control constraints have also been considered [19, 20].

The control graph together with the particular controlled parameters determines what desired formation shapes/scales are feasible along with their uniqueness. Obviously, defining a complete distance constraint graph between a group of agents will suffice to define a unique formation shape. However, defining a certain (well-chosen) subset of these distance constraints can often (generically) define a unique formation shape; for example, see the notion of graph rigidity as it applies to formation control in [5, 8, 12–15, 20–23].

Given a sensing and a control architecture, one then seeks to design the control laws that, actively and in a distributed fashion, seek to establish and maintain the desired parameters using the locally sensed information at each agent. There now exists a large literature on formation control; for example, see the related work in [8–16, 18], but the problem remains interesting because of the various problem formulations, the distributed nature of the problem itself and the existence of undesired equilibria in many of the existing systems [10, 11, 24].

In this work, we propose a set of control laws to control the shape of a formation, in a distributed fashion, using relative position measurements. We allow the geometric constraints of the formation shape control at each agent to be (i) interagent distance-only constraints, (ii) interagent bearing-only constraints or (iii) a mixture of interagent distance and bearing constraints. Because the agents can restrict their control to any of the aforementioned geometric constraints, the formation control problems considered here and the control laws outlined differ from much of the existing work on formation control.

The main contribution of this work is the design and analysis of a novel, relaxed formation control law for distributed formation control with an arbitrary number of agents and the noted heterogeneous set of constraints. Unlike most shape control algorithms in the literature, the controllers proposed in this paper are not gradient-based controllers. The distributed control law introduced here is relaxed in the sense that each agent is free to choose its own heading within a relatively large region of values. That is, the controller is not a typical feedback controller that maps, for example, a control error to a distinct control input. Rather, the control law defines a large (half-plane) sector of the plane towards which each agent must steer. The agent is then free to, in a relaxed way, pick a particular heading from this large set. This is a considerable amount of freedom given to each agent. For example, an agent may choose a specific heading that avoids an external obstacle. This controller may also be robust to various forms of error such as that obtained from a vision system. Nevertheless, even given such a relaxed motion control law, a strong convergence result is established, which guarantees exponential convergence of the formation to the desired shape.

The idea of a relaxed formation control law that provides individual agents with a good deal of freedom in choosing their headings has been previously considered in [2, 25, 26]. The novelty of this work stems from the properties that the relaxed formation control law is distributed and works in formation control problems with an arbitrary number of agents and given a heterogeneous set of control constraints (consisting of interagent distances, bearings or a mixture of the two). The fact that exponential stability of the desired formation shape is achieved given such a relaxed motion constraint on each agent is a significant contribution to the field of distributed formation control.

Finally, it is noted that this work is a generalisation of [13, 20, 21] and corrects descriptions, notation and result statements in the author’s prior work [20, 21]. The work in [5], which extends [21], could also be adapted in a manner similar to that proposed in this paper (gaining the associated benefits).

2. COMMON SETUP AND NOTATION

Consider $n$ agents indexed by $V = \{1, 2, \ldots, n\}$ and with positions $p_i \in \mathbb{R}^2$. We use an undirected graph $G(V, E)$ to describe the configuration of the formation where $E \subseteq V \times V$ denotes the edge set of the graph and describes the interaction between the agents in the formation with $m = |E|$. We assume an unordered pair of agents $(i, j)$ is in $E$ if (a) $i$ and $j$ can measure the relative position of each other and (b) $i$ is in some way constrained in relation to $j$ and $j$ is in some way constrained in
relation to $i$. The constraint between two agents might be a distance constraint, a bearing constraint or both. We denote the neighbourhood set of agent $i$ as $\mathcal{N}_i = \{ j \in \mathcal{V} : (i, j) \in \mathcal{E} \}$.

**Definition 1 (Formal point formation)**

A point formation $\mathcal{F}_\mu(\mathcal{G})$ is defined by a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and a map $p : \mathcal{V} \to \mathbb{R}^2$, which takes agent $i$ in $\mathcal{V}$ to its respective position $\mathbf{p}_i$ in ambient space $\mathbb{R}^2$.

Let $t \in [0, \infty)$ denote time. The motion of agent $i$ is governed by

$$\frac{d}{dt} \mathbf{p}_i = \dot{\mathbf{p}}_i = \mathbf{u}_i$$

where $\mathbf{u}_i$ is a control vector to be determined. The combined motion of the formation is $\dot{\mathbf{p}} = \mathbf{u}$ where $\mathbf{p} = [\mathbf{p}_1^\top \mathbf{p}_2^\top \ldots \mathbf{p}_n^\top]^\top$.

Suppose agent $i$ can measure the relative position to agent $j$ iff $j \in \mathcal{N}_i \iff i \in \mathcal{N}_j$ where $\mathcal{N}_i$ is the set of neighbours of $i$.

**Remark**

To simplify the problem formulation and analysis, we make the assumption here that all agents have a common coordinate basis and in particular have a common sense of ‘north’ from which bearing measurements are taken. However, after we introduce the proposed formation control law, we highlight clearly the fact that this control law is locally independent of a global coordinate basis at each agent. In other words, to implement the local control at agent $i$, only a relative position measurement of agent $j, \forall j \in \mathcal{N}_i$ in a local coordinate basis known only to agent $i$ is actually required.

Let $\phi_{ij} = \phi_{ji} + \pi \mod (2\pi)$ denote the bearing to agent $j$ at agent $i$ taken with respect to a common (global) coordinate basis known to all agents. We introduce the following set of bearings $B$ at $\mathbf{p}$

$$B(\mathbf{p}) = \{ \phi_{ij} \in [0, 2\pi) : i < j, (i, j) \in \mathcal{E} \}$$

where $|B| = m$. Here, $B(\mathbf{p})$ represents the actual bearing measurements taken by the agents in some configuration. The set of ranges at $\mathbf{p}$ are

$$D(\mathbf{p}) = \{ d_{ij}^2 \in \mathbb{R}^+ : i < j, (i, j) \in \mathcal{E} \}$$

where $|D| = m$ and $d_{ij}^2 = \| \mathbf{p}_i - \mathbf{p}_j \|^2 = d_{ji}^2$. Here, $D(\mathbf{p})$ represents the actual distance measurements taken by the agents in some configuration. If an agent actually measures $d_{ij}$, then it also knows $d_{ji}^2$.

Define two (not necessarily, but possibly, disjoint) sets $\mathcal{E}_B \subseteq \mathcal{E}$ and $\mathcal{E}_D \subseteq \mathcal{E}$ such that $\mathcal{E}_B \cup \mathcal{E}_D = \mathcal{E}$. Define a possibly nonsimple graph $\mathcal{G}(\mathcal{V}, \mathcal{E}_B, \mathcal{E}_D)$ comprising the vertices $\mathcal{V}$ and the edge set $\mathcal{E}_B \cup \mathcal{E}_D$ and such that when $(i, j) \in \mathcal{E}_B$ and $(i, j) \in \mathcal{E}_D$, then $\mathcal{G}(\mathcal{V}, \mathcal{E}_B, \mathcal{E}_D)$ has two edges between agents $i$ and $j$.

Define a set of constant bearings

$$B_c = \{ \phi_{ij}^* \in [0, 2\pi) : i < j, (i, j) \in \mathcal{E}_B \}$$

where $\phi_{ij}^* = \phi_{ji}^* + \pi$. Similarly, define a set of constant squared distances between agent pairs,

$$D_c = \{ d_{ij}^* = [0, \infty) : i < j, (i, j) \in \mathcal{E}_D \}$$

where $d_{ij}^* = \| \mathbf{p}_i - \mathbf{p}_j \|^2 = d_{ji}^*$. Here, $d_{ij}^* \geq 0$. One could say that $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with $B(\mathbf{p})$ and $D(\mathbf{p})$ defined by $\mathcal{E}$ represents the sensing graph, whilst $\mathcal{G}(\mathcal{V}, \mathcal{E}_B, \mathcal{E}_D)$ with $B_c$ and $D_c$ defined by $\mathcal{E}_B$ and $\mathcal{E}_D$ respectively represents the control graph.

In particular, the sets $B_c$ and $D_c$ represent the desired values for the interagent bearings and ranges in the sense that the high-level formation control problem is really one of steering the set of agents such that $B(\mathbf{p})$ restricted to $\mathcal{E}_B$ converges to $B_c$ and similarly such that $D(\mathbf{p})$ restricted to $\mathcal{E}_D$ converges to $D_c$. 

Assume a formation $\mathcal{F}_p$ is given. Then, a pair of sets $\mathcal{B}_e$ and $\mathcal{D}_e$ of bearings and distances are realisable if and only if each $\phi_{ij}^* \in \mathcal{B}_e$ and $d_{ij}^* \in \mathcal{D}_e$ can exist between the respective $p_i$ and $p_j$ simultaneously.

For example, consider a triangular formation with three agents and suppose three desired distance constraints are given between each pair of agents in the formation. Obviously, these three desired distances must satisfy the triangle inequality. If in addition, bearing constraints are given for which the interior angles of the triangle can be deduced, these must be consistent with the distance constraints. By definition, $\mathcal{B}(p)$ and $\mathcal{D}(p)$ are automatically realisable.

**Assumption 1**
The set of desired bearing values $\mathcal{B}_e$ and desired distance values $\mathcal{D}_e$ that define the desired formation are realisable.

### 3. Infinitesimal Rigidity and Formations: Distance-Constrained Formation Control

Rigidity theory has been considered previously in formation control in [8, 11–15] and derives itself from the theory of rigid graphs [27].

We assume in this section that the constraints in formation shape control are distance-only constraints. Thus, $\mathcal{E}_B = \emptyset$ and $\mathcal{E}_D = \mathcal{E}$.

Two formations $\mathcal{F}_q$ and $\mathcal{F}_p$ are said to be *equivalent* if their underlying graphs $\mathcal{G}(\mathcal{V}, \mathcal{E})$ are identical and the set of interagent distances in $\mathcal{D}(p)$ are equal to the distances in $\mathcal{D}(q)$.

Consider a formation $\mathcal{F}_p$ and a continuously parameterised formation trajectory defined by a time-varying $\mathbf{q}_i(t)$ for all $i \in \mathcal{V}$ such that $\mathcal{F}_q(t)$ is defined by a time-varying map $q(t) : \mathcal{V} \rightarrow \mathbb{R}^2$. Both $\mathcal{F}_p$ and $\mathcal{F}_q(t)$ are defined by the same underlying $\mathcal{G}(\mathcal{V}, \mathcal{E})$. Suppose $\mathbf{q}_i(0) = p_i$ for all $i$. Then, for each $(i, j) \in \mathcal{E}$, consider the constraint

$$ (p_i - p_j) \cdot (\mathbf{q}_i - \mathbf{q}_j) = d_{ij}^2, \quad (i, j) \in \mathcal{E}, \quad t \geq 0 $$(6)

The time-derivative of this constraint is then

$$ (p_i - p_j) \cdot (\dot{\mathbf{q}}_i - \dot{\mathbf{q}}_j) = 0, \quad (i, j) \in \mathcal{E}, \quad t \geq 0 $$

(7)

If such a constraint holds for each $(i, j) \in \mathcal{E}$, then the solutions $\dot{\mathbf{q}}_i$ of the corresponding $|\mathcal{E}| = m$ homogenous linear equations define an infinitesimal formation motion with respect to $\mathcal{F}_p$.

Assume $\mathcal{F}_p$ is given and $\mathcal{F}_q(t)$ is a continuously parameterised trajectory defined on the same underlying $\mathcal{G}(\mathcal{V}, \mathcal{E})$. Then, $\mathcal{F}_q$ is said to be a flex with respect to $\mathcal{F}_p$ if and only if $\mathcal{F}_q(0) = \mathcal{F}_p$ and (7) is satisfied for all $(i, j) \in \mathcal{E}$.

Thus, $\mathcal{F}_q(t)$ is a flex with respect to $\mathcal{F}_p$ if and only if $\mathcal{F}_q(0) = \mathcal{F}_p$ and

$$ (p_i - p_j) \cdot (\dot{\mathbf{q}}_i(t) - \dot{\mathbf{q}}_j(t)) = 0, \quad (i, j) \in \mathcal{E}, \quad t \geq 0 $$

(8)

which can be written in matrix form as

$$ R_D(p)\dot{q} = 0 $$

(9)

where $p = [\mathbf{p}_1 \mathbf{p}_2 \ldots \mathbf{p}_n]^\top$ and similarly for $q$. $R_D(p) \in \mathbb{R}^{m \times 2n}$ is called the rigidity matrix for formations [28].

**Definition 2 (Rigid formations)**

A point formation $\mathcal{F}_p$ is said to be an infinitesimally rigid formation if the only solutions to $R_D(p)\dot{q} = 0$ correspond to derivatives of rotations and/or translations.

This definition results in the following algebraic condition for rigidity [13]. The formation $\mathcal{F}_p$ of $n$ agents is infinitesimally rigid if rank $(R_D(p)) = 2n - 3$, which means that a formation that starts off equivalent will remain so for all time.
A rigid formation has only trivial flexes. A necessary condition for the formation to be rigid is that the embedding of the agents in $\mathbb{R}^{2n}$ corresponds to a so-called regular point, defined as one where $R(p)$ has maximal rank with respect to $p$ given a fixed number of agents and a fixed formation topology.

**Example 1**
Consider four agents indexed by 1, 2, 3 and 4. An example of a rigid formation is illustrated in Figure 1. Conditions for testing and confirming rigidity are given subsequently.

The graph $G(V, E)$ associated with the formation has five edges. The edges are $\{(1, 2), (1, 3), (1, 4), (2, 3), (3, 4)\}$ arranged in lexicographical order. The rigidity matrix for the formation is given by (10). The rigidity matrix is a $5 \times 8$ matrix in this example.

\[
\begin{bmatrix}
\begin{array}{cccc}
\text{agent 1} & \text{agent 2} & \text{agent 3} & \text{agent 4} \\
\text{edge (1, 2)} & (p_1 - p_2)^T & (p_2 - p_1)^T & 0 & 0 \\
\text{edge (1, 3)} & (p_1 - p_3)^T & 0 & (p_3 - p_1)^T & 0 \\
\text{edge (1, 4)} & (p_1 - p_4)^T & 0 & 0 & (p_4 - p_1)^T \\
\text{edge (2, 3)} & 0 & (p_2 - p_3)^T & (p_3 - p_2)^T & 0 \\
\text{edge (3, 4)} & 0 & 0 & (p_3 - p_4)^T & (p_4 - p_3)^T \\
\end{array}
\end{bmatrix} = R_D(p) \tag{10}
\]

The rows correspond to the independent constraints in the graph associated with the formation, and the columns correspond to the agents.

When $F_p$ is rigid for any (and thus all) regular points $p$, then we say the graph $G(V, E)$ associated with $F_p$ is regularly rigid. We often refer also to the formation $F_p$ whose graph $G(V, E)$ is regularly rigid as a regularly rigid formation or more formally as a regularly infinitesimally rigid formation.

**4. PARALLEL RIGIDITY AND FORMATIONS: BEARING-CONSTRAINED FORMATION CONTROL**

The theory of parallel rigid formations was introduced in [22, 29] again based on graph theoretical origins [27].

We assume in this section that the formation shape control objective is formed by bearing-only constraints, and thus, $\mathcal{E}_D = \emptyset$ and $\mathcal{E}_B = \mathcal{E}$.

Two formations $F_q$ and $F_p$ are said to be equivalent if their underlying graphs are identical and the set of bearing measurements in $B(p)$ are equal to the bearings in $B(q)$.
Consider two formations \( F_p \) and \( F_q \) defined by the same graph \( G(\mathcal{V}, \mathcal{E}) \) and respective mappings 

\[ p : \mathcal{V} \rightarrow \mathbb{R}^2 \quad \text{and} \quad q : \mathcal{V} \rightarrow \mathbb{R}^2. \]

For each \((i, j) \in \mathcal{E}\), consider the constraint

\[
(p_i - p_j)_{\perp} \cdot (q_i - q_j) = 0
\]

where the operator \((\cdot)_{\perp}\) rotates a plane vector by \(\pi/2\) counterclockwise. Then, it follows that \( F_p \) and \( F_q \) are parallel drawings \([22, 29]\) of each other in the sense that for each \((i, j) \in \mathcal{E}\), the vectors \((p_i - p_j)_{\perp}\) and \((q_i - q_j)_{\perp}\) are parallel. The system of equations (11) for all \((i, j) \in \mathcal{E}\) is a system of \(|\mathcal{E}| = m\) homogenous linear equations in the \( q_i \) and \( q_j \) when the \( p_i \) and \( p_j \) are treated as known parameters.

Assume that \( F_p \) is given and \( F_q \) is defined on the same underlying graph \( G(\mathcal{V}, \mathcal{E}) \) as \( F_p \). Then, \( F_q \) is said to be a parallel point formation with respect to \( F_p \) if and only if (11) is satisfied for all \((i, j) \in \mathcal{E}\). A parallel point formation \( F_q \) is trivial with respect to \( F_p \) if it is equivalent to \( F_p \) and if \( F_q \) can be obtained from \( F_p \) via a translation, then a dilation \(^1\) (or vice versa) on \( \mathbb{R}^2 \), and then, \( F_p \) is also trivial with respect to \( F_q \). All other pairs of parallel point formations are non-trivial with respect to each other.

Consider a formation trajectory defined by a time-varying \( q_i(t) \) for all \( i \in \mathcal{V} \) such that \( F_q(t) \) is defined by the same \( G(\mathcal{V}, \mathcal{E}) \) and the time-varying map \( q(t) : \mathcal{V} \rightarrow \mathbb{R}^2 \). Suppose \( q_i(0) = p_i \) for all \( i \). Then, \( F_q(t) \) is a parallel point formation to \( F_p \) if

\[
(p_i - p_j)_{\perp} \cdot (q_i(t) - q_j(t)) = 0, \quad (i, j) \in \mathcal{E}, \quad t \geq 0
\]

Conversely, a solution to the resulting linear system of equations defines a parallel point formation \( F_q(t) \). Differentiating (12) with respect to time, we have

\[
(p_i - p_j)_{\perp} \cdot (\dot{q}_i(t) - \dot{q}_j(t)) = 0, \quad (i, j) \in \mathcal{E}, \quad t \geq 0
\]

which can be written in matrix form as

\[
R_B(p)q = 0
\]

where \( R_B(p) \in \mathbb{R}^{m \times 2n} \) is called the parallel rigidity matrix \([22, 29]\).

**Definition 3 (Parallel rigid formations)**

A point formation \( F_p \) is said to be a parallel rigid formation if all parallel point formations of \( F_p \) are trivial with respect to \( F_p \).

A formation that is parallel rigid is one in which the shape and orientation, albeit not the scale, is uniquely defined in \( \mathbb{R}^2 \). The novelty of this characterisation is that it allows one to consider only the graphical topology of the formation and by appropriately choosing the agent interactions, for example, the links \((i, j) \in \mathcal{E}\), one can define a unique formation shape.

**Example 2**

Consider four agents indexed by 1, 2, 3 and 4. In this case, each edge in the formation represents a bearing constraint. Conditions for testing and confirming parallel rigidity are given subsequently.

The graph associated with the formation has five edges, and there are thus a total of 10 bearing constraints. The edges, arranged in lexicographical order, are \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}. The bearing-constrained rigidity matrix for the formation is given by (15). The bearing-constrained rigidity matrix is a \( 5 \times 8 \) matrix in this example.

\(^1\)For a formation \( F_p \), a dilation changes the size but not the shape or orientation in \( \mathbb{R}^2 \) of the formation. That is, for each pair \( p_i, p_j \in \mathbb{R}^2, i, j \in \mathcal{V} \) a dilation of the object \( F_p \) preserves the bearing \( \phi_{ij} \) and thus \( \phi_{ji} \) but scales all \( d_{ij} = d_{ji} \) by the same positive constant.
is said to be a shake with respect to $F$ as introduced previously in the case of distance-only or bearing-only constraints respectively.

When $\mathbf{q}$ takes on and it corresponds to the fact that the trajectories of the formation at $\mathbf{q}$ and parallel rigid formations [22, 29] and their graph origins [27].

The idea of stiff point formations discussed herein follows [28] and is related to rigid formations [8] and parallel rigid formations [22, 29] and their graph origins [27].

Now, suppose $\mathcal{E}_B \subseteq \mathcal{E}$ and $\mathcal{E}_D \subseteq \mathcal{E}$ are such that $\mathcal{E}_B \cup \mathcal{E}_D = \mathcal{E}$ and $\mathcal{E}_B \neq \emptyset$ and $\mathcal{E}_D \neq \emptyset$.

Two formations $\mathcal{F}_q$ and $\mathcal{F}_p$ are said to be equivalent if their underlying control graphs $\mathcal{G}(\mathcal{V}, \mathcal{E}_B, \mathcal{E}_D)$ are identical and the sets of bearing measurements in $\mathcal{B}(\mathbf{p})$ and distance measurements in $\mathcal{D}(\mathbf{p})$ in one of the formations are equal to those in the bearing and distance measurements sets defined by the other formation.

Consider a formation $\mathcal{F}_p$ and a continuously parameterised formation trajectory defined by a time-varying $\mathbf{q}_i(t)$ for all $i \in \mathcal{V}$ such that $\mathcal{F}_q(t)$ is defined by a time-varying map $q(t) : \mathcal{V} \rightarrow \mathbb{R}^2$.

Both $\mathcal{F}_p$ and $\mathcal{F}_q(t)$ are defined by the same underlying $\mathcal{G}(\mathcal{V}, \mathcal{E})$. Suppose $\mathbf{q}_i(0) = \mathbf{p}_i$ for all $i$. Then, for some $(i, j) \in \mathcal{E}$, consider the constraint

$$ (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{q}_i - \mathbf{q}_j) = 0, \quad t \geq 0 \tag{16} $$

and the constraint

$$ (\mathbf{p}_i - \mathbf{p}_j) \perp \cdot (\dot{\mathbf{q}}_i - \dot{\mathbf{q}}_j) = 0, \quad t \geq 0 \tag{17} $$

as introduced previously in the case of distance-only or bearing-only constraints respectively.

Assume that $\mathcal{F}_p$ is given and $\mathcal{F}_q(t)$ is defined on the same underlying graph $\mathcal{G}(\mathcal{V}, \mathcal{E}_B, \mathcal{E}_D)$. Then, $\mathcal{F}_q$ is said to be a shake with respect to $\mathcal{F}_p$ if and only if (17) is satisfied for all $(i, j) \in \mathcal{E}_B$ and (16) is satisfied for all $(i, j) \in \mathcal{E}_D$ and all $t \geq 0$.

Thus, $\mathcal{F}_q(t)$ is a shake with respect to $\mathcal{F}_p$ if

$$ (\mathbf{p}_i - \mathbf{p}_j) \cdot (\dot{\mathbf{q}}_i(t) - \dot{\mathbf{q}}_j(t)) = 0, \quad (i, j) \in \mathcal{E}_D, \quad t \geq 0 $$

$$ (\mathbf{p}_i - \mathbf{p}_j) \perp \cdot (\dot{\mathbf{q}}_i(t) - \dot{\mathbf{q}}_j(t)) = 0, \quad (i, j) \in \mathcal{E}_B, \quad t \geq 0 \tag{18} $$

5. STIFFNESS THEORY AND FORMATIONS: CONTROL OF FORMATIONS WITH MIXED BEARING AND DISTANCE CONSTRAINTS

The idea of stiff point formations discussed herein follows [28] and is related to rigid formations [8] and parallel rigid formations [22, 29] and their graph origins [27].
which can be written in matrix form as

\[ \mathbf{R}(\mathbf{p})\dot{\mathbf{q}} = 0 \quad (19) \]

where \( \mathbf{R}(\mathbf{p}) \in \mathbb{R}^{m \times 2n} \) is called the constraint matrix for formations with distance and bearing constraints [28].

**Definition 4 (Stiff formations)**

A point formation \( \mathcal{F}_p \) is said to be a stiff formation if all shakes of \( \mathcal{F}_p \) can be obtained via translations.

**Example 3**

Consider again the four agents indexed by 1, 2, 3 and 4 illustrated in Figure 1. Conditions for testing and confirming stiffness are given subsequently.

Suppose that edges (1,2) and (1,3) correspond to distance constraints, edges (1,4) and (3,4) correspond to bearing constraints and edge (2,3) corresponds to both a distance and bearing constraint. The edges, arranged in lexicographical order, are \{(1, 2), (1, 3), (1, 4), (2, 3), (3, 4)\}. Then, the graph \( G(\mathcal{V}, \mathcal{E}) \) associated with the formation has five edges whilst the graph \( G(\mathcal{V}, \mathcal{E}_B, \mathcal{E}_D) \) associated with the formation has six edges (two between agents 2 and 3). The constraint matrix for the formation is given by (20). The constraint matrix is a \( 6 \times 8 \) matrix in this example.

\[
\begin{bmatrix}
    \mathbf{R}(\mathbf{p}) \\
    \end{bmatrix} = \begin{bmatrix}
    (p_1 - p_4)^T & 0 & 0 & 0 \\
    0 & (p_2 - p_3)^T & 0 & 0 \\
    0 & 0 & (p_3 - p_4)^T & 0 \\
    (p_1 - p_2)^T & (p_2 - p_1)^T & (p_3 - p_1)^T & (p_3 - p_2)^T \\
    \end{bmatrix}
\]

The rows correspond to the independent constraints in the graph associated with the formation, and the columns correspond to the agents.

The convention for ordering the rows of \( \mathbf{R}(\mathbf{p}) \) is as follows. We suppose that the constraints \( \mathbf{R}(\mathbf{p})\dot{\mathbf{q}} = 0 \) in (19) are written such that the rows corresponding to the bearing constraints are written on top of those corresponding to the distance constraints and that within this partitioning, the rows are ordered lexicographically with respect to the edge labelling in the graph \( G(\mathcal{V}, \mathcal{E}_B, \mathcal{E}_D) \); see (20).

A formation \( \mathcal{F}_p \) of \( n \) agents is stiff if \( \text{rank}(\mathbf{R}(\mathbf{p})) = 2n - 2 \). Refer to (19) and note the condition \( \text{rank}(\mathbf{R}(\mathbf{p})) = 2n - 2 \) implies the kernel of \( \mathbf{R}(\mathbf{p}) \) is of dimension 2. It is easily shown that this is the lowest dimension the kernel can take on and it corresponds to the fact that the trajectories of the formation at \( \mathbf{q} \) in (19) are free up to translations (accounting for two linearly independent solutions \( \dot{\mathbf{q}} \) to (19)).

A formation is said to be in regular position \( \mathbf{p} \) in \( \mathbb{R}^{2n} \) if \( \text{rank}(\mathbf{R}(\mathbf{p})) = 2n - 2 \). Consider two formations \( \mathcal{F}_p \) and \( \mathcal{F}_q \) in regular positions defined on the same underlying graph \( G(\mathcal{V}, \mathcal{E}) \). Then, \( \mathcal{F}_p \) is stiff if and only if \( \mathcal{F}_q \) is stiff. Thus, when \( \mathcal{F}_p \) is stiff for all regular points \( \mathbf{p} \), then we say the graph \( G(\mathcal{V}, \mathcal{E}) \) associated with \( \mathcal{F}_p \) is regularly stiff.

We often refer also to the formation \( \mathcal{F}_p \) whose graph \( G(\mathcal{V}, \mathcal{E}) \) is regularly stiff as a regularly stiff formation.

### 6. THE FORMATION CONTROL PROBLEM

Define the following vector of bearing measurements

\[ \mathbf{b}(\mathbf{p}) = \text{column}(\mathcal{B}(\mathbf{p}); \mathcal{E}_B) \quad (21) \]
where column \( (B(p); E_B) \) defines a \( |E_B| \times 1 \) column vector by stacking the bearings in \( B(p) \) that are indexed by edges in \( E_B \). The bearings are stacked according to a lexicographical ordering such that \( \phi_{ij} \) is above \( \phi_{kl} \) if \( j < l \) and \( \phi_{ij} \) is above \( \phi_{kl} \) if \( i < k \). Define

\[
d(p) = \text{column}(D(p); E_D)
\]

(22)
to be a \( |E_D| \times 1 \) column vector by stacking the squared distances in \( D(p) \) corresponding to indices in \( E_D \) according to a lexicographical ordering such that \( d_{ij}^2 \) is above \( d_{il}^2 \) if \( j < l \) and \( d_{il}^2 \) is above \( d_{ik}^2 \) if \( i < k \). Similarly, define \( b_c = \text{column}(E_c; E_B) \) and \( d_c = \text{column}(D_c; E_D) \).

Obviously, if \( E_D = \emptyset \), then \( d(p) \) is a null vector, and if \( E_B = \emptyset \), then \( b(p) \) is a null vector. Such cases occur when considering only infinitesimally rigid or parallel rigid formation control problems.

Both \( b_c \) and \( d_c \) are formed by stacking all the constraints in (4) and (5) respectively into column vectors. On the other hand, \( b(p) \) and \( d(p) \) are formed by stacking (typically, in the case of stiff formations, a subset of) measurements such that they correspond row-wise with \( b_c \) and \( d_c \) in terms of their respective indexing. Note \( b(p) \) is determined by the bearing measurements and is a function of \( p \), whereas \( b_c \) is a vector of desired bearing constraints and is constant, similarly for \( d(p) \) and \( d_c \).

Now, it is possible to define an error vector as

\[
e = \left[ (b(p)^T d(p)^T)^T - [b_c^T d_c^T]^T \right]
\]

(23)
and note \( e \to 0 \) for some formation \( F_p \) implies the formation is equivalent to the desired formation. If either \( E_D = \emptyset \) or \( E_B = \emptyset \), then the error vector is obviously \( e = b(p) - b_c \) or \( e = d(p) - d_c \) respectively.

**Assumption 2**

The formation \( F_p (G(V, E)) \) is rigid, parallel rigid or stiff and at a corresponding regular point with \( p_i \neq p_j \) at \( t = 0 \), \( \forall i, j \in V \).

A regular point has been previously defined as a point at which the rigidity matrix, parallel rigidity matrix or the stiffness constraint matrix is of maximal rank. Depending on the scenario considered, such regular points may or may not be equivalent.

We now outline the three formation control problems considered in this paper.

**Problem 1**
The distributed rigid formation control problem is to design a control input \( u_i, \forall i \in V \), as a function of at most \( \phi_{ij}, d_{ij} \) and \( d_{ij}^* \), for all \( j \in N_i \), such that \( e = (d(p) - d_c) \to 0 \).

**Problem 2**
The distributed parallel rigid formation control problem is to design a control input \( u_i, \forall i \in V \), as a function of at most \( \phi_{ij}, d_{ij} \) and \( \phi_{ij}^* \), for all \( j \in N_i \), such that \( e = (b(p) - b_c) \to 0 \).

**Problem 3**
The distributed stiff formation control problem is to design a control input \( u_i, \forall i \in V \), as a function of at most \( \phi_{ij}, d_{ij} \) and \( d_{ij}^* \) or \( \phi_{ij}^* \) or both depending on the type of desired constraints given between \( i \) and \( j \), for all \( j \in N_i \), such that \( e \to 0 \).

Before outlining the control law proposed to solve each of these problems, we note that the Jacobian of \( e \) evaluated at a point \( p \in \mathbb{R}^{2n} \) is given by

\[
J_e(p) = \nabla e = \frac{\partial}{\partial p} \left( \begin{bmatrix} b(p) \\ d(p) \end{bmatrix} - \begin{bmatrix} b_c \\ d_c \end{bmatrix} \right)
\]

(24)
where \( J_e(p) \in \mathbb{R}^{m \times 2n} \).
When $E_B = \emptyset$, we then have

$$J_e(p) = R_D(p)$$

(25)

where $R(p)$ is the rigidity matrix for the formation $F_p$.

When $E_D = \emptyset$, we then have

$$J_e(p) = D^{-1}R_B(p)$$

(26)

where $D = \text{diag}(D(p))$ and $R_B(p)$ is the parallel rigidity matrix for the formation $F_p$.

When $E_B \neq \emptyset$ and $E_D \neq \emptyset$ with $E_B \cup E_D = E$, we then have

$$J_e(p) = \begin{bmatrix} D^{-1} & 0 \\ 0 & 1 \end{bmatrix} R(p)$$

(27)

where $R(p)$ is the constraint matrix for the formation $F_p$.

Example 4

Consider four agents indexed by 1, 2, 3 and 4 and the rigid formation illustrated in Figure 1 of Example 1. Again, the edges, arranged in lexicographical order, are $\{(1, 2), (1, 3), (1, 4), (2, 3), (3, 4)\}$. The constraints associated with each edge are as in Example 1. We write

$$d(p) = \begin{bmatrix} d_{12}^2 & d_{13}^2 & d_{14}^2 & d_{23}^2 & d_{34}^2 \end{bmatrix}^T$$

(28)

where $d_{ij}$ is a function of $p_i$ and $p_j$. The rigidity matrix for the formation is given by (10). The Jacobian $J_e(p)$ of the error vector $e$ is given by (29) and is of the same dimension as (10).

$$J_e(p) = \begin{bmatrix} -d_{12}\cos\phi_{12} & -d_{12}\sin\phi_{12} & 0 & 0 & 0 \\ d_{12}\cos\phi_{12} & d_{12}\sin\phi_{12} & 0 & 0 & 0 \\ 0 & 0 & -d_{23}\cos\phi_{23} & -d_{23}\sin\phi_{23} & 0 \\ 0 & 0 & -d_{23}\sin\phi_{23} & d_{23}\cos\phi_{23} & 0 \\ d_{23}\cos\phi_{23} & d_{23}\sin\phi_{23} & 0 & 0 & 0 \\ -d_{24}\cos\phi_{24} & d_{24}\sin\phi_{24} & 0 & 0 & 0 \\ -d_{24}\sin\phi_{24} & -d_{24}\cos\phi_{24} & 0 & 0 & 0 \end{bmatrix} = J_e(p)$$

The rows correspond to the edges in the graph associated with the formation, and the columns correspond to the agents. We note that agent $i$ knows locally the columns of $J_e(p)$ corresponding to it. In a global coordinate system, this can be seen via inspection of $J_e(p)$, and the realization that an agent $i$ that knows $\phi_{ij}$ and $d_{ij}$ also knows $\phi_{ji}$ and $d_{ji}$. (We relax, following the derivation of the control law, the requirement for a global coordinate system. In particular, the control law requires only that agent $i$ measure the relative position of its neighbours in a locally defined coordinate basis at agent $i$. At this point, however, such a relaxation would significantly complicate the analysis.)

Example 5

Consider four agents indexed by 1, 2, 3 and 4 and the rigid network of Example 2. Again, the edges, arranged in lexicographical order, are $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}$. There are a total of 10 bearing measurements and constraints. We write

$$b(p) = [\phi_{12} \phi_{13} \phi_{14} \phi_{23} \phi_{24}]^T$$

(30)

where each measured $\phi_{ij}$ is a function of $p_i$ and $p_j$. The bearing-constrained rigidity matrix for the formation is given by (15). The bearing-constrained rigidity matrix is a $5 \times 8$ matrix in this example. The Jacobian $J_e(p)$ of the error vector $e$ is given by (31) and is of the same dimension as (15).

$$J_e(p) = \begin{bmatrix} \sin\phi_{12} & -\cos\phi_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ \cos\phi_{12} & \sin\phi_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\cos\phi_{23} & -\sin\phi_{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sin\phi_{23} & \cos\phi_{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\cos\phi_{24} & -\sin\phi_{24} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\phi_{24} & \cos\phi_{24} & 0 & 0 \end{bmatrix} = J_e(p)$$

(31)
The rows correspond to the edges in the graph associated with the formation, and the columns correspond to the agents. We note that agent $i$ knows locally the columns of $\mathbf{J}_e(\mathbf{p})$ corresponding to it. In a global coordinate system, this again follows from the fact that an agent $i$ that knows $\phi_{ij}$ and $d_{ij}$ also knows $\phi_{ji}$ and $d_{ji}$.

**Example 6**

Consider four agents indexed by 1, 2, 3 and 4 and the stiff formation illustrated in Figure 1 of Example 3. Again, the edges, arranged in lexicographical order, are \{(1, 2), (1, 3), (1, 4), (2, 3), (3, 4)\}. The constraints associated with each edge are as in Example 3. We write

$$
\begin{align*}
\mathbf{b}(\mathbf{p})^T \mathbf{d}(\mathbf{p})^T &= \begin{bmatrix}
\phi_{14} & \phi_{23} & \phi_{34} & d_{12}^2 & d_{13}^2 & d_{23}^2
\end{bmatrix}^T
\end{align*}
$$

where each measured $\phi_{ij}$ or $d_{ij}$ is a function of $\mathbf{p}_i$ and $\mathbf{p}_j$. The constraint matrix for the formation is given by (20). The Jacobian $\mathbf{J}_e(\mathbf{p})$ of the error vector $\mathbf{e}$ is given by (33) and is of the same dimension as (20).

$$
\begin{bmatrix}
\mathbf{e}(1, 4) \\
\mathbf{e}(1, 3) \\
\mathbf{e}(2, 3) \\
\mathbf{e}(1, 2)
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{e}(1, 4) \\
\mathbf{e}(1, 3) \\
\mathbf{e}(2, 3) \\
\mathbf{e}(1, 2)
\end{bmatrix}
\begin{bmatrix}
\phi_{14} & \phi_{23} & \phi_{34} & d_{12}^2 & d_{13}^2 & d_{23}^2
\end{bmatrix}^T
$$

The rows correspond to the edges in the graph associated with the formation, and the columns correspond to the agents. We note again that agent $i$ knows locally the columns of $\mathbf{J}_e(\mathbf{p})$ corresponding to it. In a global coordinate system, this can be seen via inspection of $\mathbf{J}_e(\mathbf{p})$, and the realization that an agent $i$ that knows $\phi_{ij}$ and $d_{ij}$ also knows $\phi_{ji}$ and $d_{ji}$. (Again, we show later that the control law requires only that agent $i$ measures the relative position of its neighbours in a locally defined coordinate basis at agent $i$.)

### 7. THE PROPOSED CONTROL LAW

To this point, we still assume the existence of a global coordinate frame within which each agent measures the relative positions of its neighbours; that is, each agent measures the bearing and distance to its neighbours with respect to a common ‘north’. The control law will be first derived under this assumption for simplicity. Given a local coordinate frame at each agent $i$, and measurements of the bearing and distance to the neighbours of agent $i$ in this local frame, we show that a similarly defined control law at agent $i$ is equivalent to that proposed for agent $i$ under a global coordinate basis; that is, we show that the control law at agent $i$ is invariant to a local rotation of agent $i$’s coordinate frame of reference.

The control law proposed is a modification of the typical gradient-type control law, associated with the function $\frac{1}{2} \mathbf{e}^T \mathbf{e}$, that is commonly employed in formation control problems [11, 13, 20, 21]. Define a $2 \times 2$ rotation matrix by

$$
\Omega(\theta_i) = \begin{bmatrix}
\cos \theta_i & -\sin \theta_i \\
\sin \theta_i & \cos \theta_i
\end{bmatrix}
$$

and the corresponding diagonal matrix of rotation matrices

$$
\Omega(\theta) = \begin{bmatrix}
\Omega(\theta_1) & 0 & \ldots & 0 \\
0 & \Omega(\theta_2) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \Omega(\theta_n)
\end{bmatrix}
$$

which is a \(2n \times 2n\) square matrix, and \(\theta = [\theta_1 \ldots \theta_n]^T\). Then, define the control law according to

\[
\dot{p} = u = -\Omega(\theta(t))(\nabla e)^T e = -\Omega(\theta(t))J_e(p)^T e
\]

(36)

where \(\theta(t)\) is a time-varying continuous signal that can be arbitrarily chosen subject to the constraint \(\theta(t) \in [-\pi/2 + \epsilon, \pi/2 - \epsilon]\) for some sufficiently small \(\epsilon > 0\). We consider the following three different scenarios.

**Case 1.** \(\mathcal{E}_B = \emptyset\): In this case, we just have distance constraints, and the control law can be written as

\[
\dot{p} = u = -\Omega(\theta(t))J_e(p)^T e = -\Omega(\theta(t))R_e^T(p)e
\]

(37)

More specifically, the control law for an individual agent when \(\mathcal{E}_B = \emptyset\) is given by

\[
\dot{p}_i = u_i = \Omega(\theta_i(t)) \sum_{j \in \mathcal{N}_i} \left[ \frac{d_{ij} \cos \phi_{ij}}{d_{ij} \sin \phi_{ij}} \right] (d_{ij}^2 - d_{ij}^2)
\]

(38)

The summation term is a superposition of \(|\mathcal{N}_i|\) vectors pointing towards the neighbours of agent \(i\).

**Case 2.** \(\mathcal{E}_D = \emptyset\): In this case, we just have bearing constraints, and the control law can be written as

\[
\dot{p} = u = -\Omega(\theta(t))J_e(p)^T e = -\Omega(\theta(t))R_e^T(p)D^{-1}e
\]

(39)

More specifically, the control law for an individual agent when \(\mathcal{E}_D = \emptyset\) is given by

\[
\dot{p}_i = u_i = \Omega(\theta_i(t)) \sum_{j \in \mathcal{N}_i} \frac{1}{d_{ij}} \left[ \begin{array}{c} -\sin \phi_{ij} \\ \cos \phi_{ij} \end{array} \right] (\phi_{ij} - \phi_{ij}^*)
\]

(40)

where the summation amounts to a superposition of \(|\mathcal{N}_i|\) vectors pointing perpendicularly to the respective \(|\mathcal{N}_i|\) links in the formation \(\mathcal{F}_p\) leaving agent \(i\) and where each vector is scaled by the length of the link in the formation and an appropriate error term (which may be negatively signed). The form of the summation (40) can be easily intuited using Example 5.

**Case 3.** \(\mathcal{E}_B \neq \emptyset\) and \(\mathcal{E}_D \neq \emptyset\) with \(\mathcal{E}_B \cup \mathcal{E}_D = \mathcal{E}\): In this case, we have both distance and bearing constraints, and the control law can be written as

\[
\dot{p} = u = -\Omega(\theta(t))J_e(p)^T e = -\Omega(\theta(t))R_e^T(p)D^{-1}e
\]

(41)

More specifically, the control law for an individual agent when \(\mathcal{E}_B \neq \emptyset\) and \(\mathcal{E}_D \neq \emptyset\) is

\[
\dot{p}_i = u_i = \Omega(\theta_i(t)) \sum_{j : (i,j) \in \mathcal{E}_D} \left[ \frac{d_{ij} \cos \phi_{ij}}{d_{ij} \sin \phi_{ij}} \right] (d_{ij}^2 - d_{ij}^2) + \Omega(\theta_i(t)) \sum_{j : (i,j) \in \mathcal{E}_B} \frac{1}{d_{ij}} \left[ \begin{array}{c} -\sin \phi_{ij} \\ \cos \phi_{ij} \end{array} \right] (\phi_{ij} - \phi_{ij}^*)
\]

(42)
The first summation is a superposition of \(| j : (i, j) \in \mathcal{E}_D | \) vectors pointing away from the neighbours of agent \(i\) and with which there is a distance constraint between agent \(i\) and that neighbour. The second summation is a superposition of \(| j : (i, j) \in \mathcal{E}_B | \) vectors pointing perpendicular to those links leaving agent \(i\) and corresponding to a bearing constraint. Each vector is scaled by an appropriate error term (which may be negatively signed), and those corresponding to bearing-only constraints are also scaled by the inverse range between the agents.

The existence and uniqueness of a solution to the coupled system of differential equations (38), (40) and (42) are guaranteed using standard arguments [13, 30] if the trajectories over \(t \in [0, \infty)\) are such that \(d_{ij} > 0\) for the subset of interagent distances \((i, j) \in \mathcal{E}\).

### 7.1. Locally implementing the controller with measurements in a local coordinate frame

In this subsection, we show the following: suppose that agent \(i\) derives its trajectory through application of one of the previously defined controllers, implemented using local measurements, that is, measurements with respect to a local coordinate frame known only to agent \(i\). Then, each agent’s trajectory when viewed in some arbitrary global coordinate frame is equivalent to that derived by the individual agent’s using measurements taken with respect to this global coordinate basis.

To show this formally, let \(\mathbf{p}_j\) and \(\mathbf{i}_j\) be respectively the position of agent \(j\) in a global coordinate basis and in agent \(i\)’s local coordinate basis. Also, let \(\phi_{ij}\) and \(\mathbf{i}_j\) be respectively the bearing to agent \(j\) at agent \(i\) taken with respect to the global coordinate basis and agent \(i\)’s local coordinate basis.

Then, the trajectory of agent \(i\) in the distance-only constraint case when local measurements are used is defined by

\[
\mathbf{i}_t = -\Omega(\theta_i) \sum_{j \in \mathcal{N}_i} (\mathbf{p}_i - \mathbf{i}_j) (d_{ij}^2 - d_{ij}^{*2})
\]

\[
= \Omega(\theta_i) \sum_{j \in \mathcal{N}_i} \begin{bmatrix} d_{ij} \cos \phi_{ij} \\ d_{ij} \sin \phi_{ij} \end{bmatrix} (d_{ij}^2 - d_{ij}^{*2})
\]

\( (43) \)

Let \(\mathbf{p}_j = \Gamma_i \mathbf{i}_j + \mathbf{w}_i\) where \(\Gamma_i\) is a \(2 \times 2\) rotation matrix that rotates vectors in agent \(i\)’s local coordinate system back to the arbitrary global system, and \(\mathbf{w}_i\) translates the origin of agent \(i\)’s frame to the origin of the global frame.

Then, it is easy to see that \(\mathbf{p}_i - \mathbf{p}_j = \Gamma_i (\mathbf{i}_j - \mathbf{i}_j) \) and \(\Omega(\theta_i) \Gamma_i = \Gamma_i \Omega(\theta_i)\) as \(\Omega(\theta_i)\) and \(\Gamma_i\) are both rotation matrices. So (43) can be written as

\[
\Gamma_i \mathbf{i}_t = -\Gamma_i \Omega(\theta_i) \sum_{j \in \mathcal{N}_i} (\mathbf{i}_i - \mathbf{i}_j) (d_{ij}^2 - d_{ij}^{*2})
\]

\[
= -\Omega(\theta_i) \sum_{j \in \mathcal{N}_i} \Gamma_i (\mathbf{i}_i - \mathbf{i}_j) (d_{ij}^2 - d_{ij}^{*2})
\]

\( (44) \)

and therefore,

\[
\Gamma_i \mathbf{i}_t = -\Omega(\theta_i) \sum_{j \in \mathcal{N}_i} (\mathbf{p}_i - \mathbf{p}_j) (d_{ij}^2 - d_{ij}^{*2})
\]

\[
= \Omega(\theta_i) \sum_{j \in \mathcal{N}_i} \begin{bmatrix} d_{ij} \cos \phi_{ij} \\ d_{ij} \sin \phi_{ij} \end{bmatrix} (d_{ij}^2 - d_{ij}^{*2})
\]

\( (45) \)

with \(\dot{\mathbf{i}}_i = \Gamma_i \mathbf{i}_t\). In other words, the motion of agent \(i\) when viewed in the arbitrary global coordinate frame is invariant to the situation in which the relative position measurements from agent \(i\) to \(j \in \mathcal{N}_i\) are taken in agent \(i\)’s local frame or the situation in which the same relative measurements are taken in the stated global frame. The action of the control law is independent of the coordinate basis used by the agents to derive the control.

Another way to think of this is to consider the controller (38) or (45) for agent \(i\) in the global frame. The summation term defining this controller is a superposition of \(|\mathcal{N}_i|\) vectors pointing
towards the neighbours of agent \(i\). Now, if agent \(i\) measures the bearings, for example, to its neighbours with respect to a local frame, then it is clearly still possible for agent \(i\) to define the state summation (agent \(i\) still knows the direction to its neighbours) and hence an equivalent control action.

The same argument applies to the bearing-only constraint case or the case in which both bearing and distance constraints are considered. We outline the case with bearing-only constraints for completeness.

Let \(\phi_{ij}^*\) and \(i\phi_{ij}^*\) be the desired values for \(\phi_{ij}\) and \(i\phi_{ij}\). We must assume that the set of \(\phi_{ij}^*, i < j\), is a realisable set of bearings. This is a natural assumption and requires a centralised constraint design using a global coordinate frame; note centralised control design is implicit in all formation control laws. If the position of agent \(j\) in a global frame is given by \(p_j = \Gamma_j i\ p_j + w_j\) as before and \(\Gamma_j\) is defined by \(\gamma \in [0, 2\pi]\), then we define \(i\phi_{ij}^* = \phi_{ij}^* + \gamma_i\) assuming \(\Gamma_j\) rotates vectors counterclockwise. This requires, at least, that during the design of the formation shape control law (and the formation shape itself), the local coordinate frames be known with respect to a global frame and these frames are time-invariant. Centralised control design is somehow implicit in (arguably) all formation control schemes even though the implementation is distributed.

Then, the trajectory of agent \(i\) in the bearing-only constraint case when local measurements are used is defined by

\[
\begin{align*}
i \dot{p}_i &= -\Omega(\theta_i) \sum_{j \in N_i} \left( i \ p_j^+ - i \ p_j^- \right) \frac{i \phi_{ij} - i \phi_{ij}^*}{d_{ij}} \\
&= \Omega(\theta_i) \sum_{j \in N_i} \left[ -\sin i \phi_{ij} \cos i \phi_{ij} \right] \frac{i \phi_{ij} - i \phi_{ij}^*}{d_{ij}}
\end{align*}
\]

(46)

where the operator \((\cdot)^\perp\) rotates a plane vector by \(\pi/2\) counterclockwise as defined earlier.

As before, it is easy to see that \(p_j^+ = p_j^- = \Gamma_j \left( i \ p_j^+ - i \ p_j^- \right)\), and \(\Omega(\theta_j) \Gamma_j = \Gamma_j \Omega(\theta_j)\) as \(\Omega(\theta_j)\) and \(\Gamma_j\) are both rotation matrices. Further, \(i \phi_{ij} - i \phi_{ij}^* = \phi_{ij} - \phi_{ij}^*\). So, (46) can be written as

\[
\begin{align*}
\Gamma_j i \dot{p}_i &= -\Gamma_j \Omega(\theta_j) \sum_{j \in N_i} \left( i \ p_j^+ - i \ p_j^- \right) \frac{i \phi_{ij} - i \phi_{ij}^*}{d_{ij}} \\
&= -\Omega(\theta_j) \sum_{j \in N_i} \Gamma_j \left( i \ p_j^+ - i \ p_j^- \right) \frac{i \phi_{ij} - i \phi_{ij}^*}{d_{ij}}
\end{align*}
\]

(47)

and therefore,

\[
\begin{align*}
\Gamma_j i \dot{p}_i &= -\Omega(\theta_j) \sum_{j \in N_i} \left( p_j^+ - p_j^- \right) \frac{\phi_{ij} - \phi_{ij}^*}{d_{ij}} \\
&= \Omega(\theta_j) \sum_{j \in N_i} \left[ -\sin \phi_{ij} \cos \phi_{ij} \right] \frac{\phi_{ij} - \phi_{ij}^*}{d_{ij}}
\end{align*}
\]

(48)

with \(\dot{p}_i = \Gamma_j i \dot{p}_i\). As before, the motion of agent \(i\) when viewed in the arbitrary global coordinate frame is invariant to the situation in which the relative position measurements from agent \(i\) to \(j \in N_i\) are taken in agent \(i\)’s local frame or the situation in which the same relative measurements are taken in the stated global frame.

Another way to think of this is to consider the controller (40) or (48) for agent \(i\) in the global frame. The summation term defining this controller is a superposition of \(|N_i|\) vectors pointing perpendicular to the neighbours of agent \(i\). Now, if agent \(i\) measures the bearings, for example, to their neighbours with respect to a local frame, then it is clearly still possible for agent \(i\) to define the state summation (agent \(i\) still knows the direction to its neighbours) and hence an equivalent control action. The set of all locally desired bearing values must be realisable when rotated back to the global
frame. The same is true in the distance-only constraint case, although interagent distances in a local frame in this case are equal to the interagent distances in a global frame. Both scenarios typically require a centralised control design (even though they are implemented in a distributed fashion).

Summarising, in all of the control laws previously derived, each agent has the freedom of measuring the relative position to its neighbours in a local coordinate frame and then applying these local measurements when deriving the local control action.

8. STABILITY RESULTS

In this section, we will outline stability results for all three of the preceding formation control scenarios. Firstly, we note formally that the control law is independent of a chosen global coordinate system.

**Lemma 1**
For all \( \mathbf{w} \in \mathbb{R}^2 \) it follows that \( \mathbf{J}_e(p) = \mathbf{J}_e(p + (\mathbf{1} \otimes \mathbf{w})) \) where \( \mathbf{1} \) is an \( n \)-dimensional column vector of all 1’s. Moreover, for every orthogonal matrix \( \mathbf{X} \in \mathbb{R}^{2 \times 2} \) it follows that \( \mathbf{J}_e(p)(\mathbf{I}_n \otimes \mathbf{X})^\top = \mathbf{J}_e((\mathbf{I}_n \otimes \mathbf{X})p) \) where \( \mathbf{I}_n \) is a \( n \times n \) identity matrix.

The preceding lemma appears in [13] for distance-only formation control but applies to all three formation control scenarios. Consider the set

\[
Z^* = \{ p \in \mathbb{R}^{2n} : e = 0 \}
\]  

(49)

of equilibrium points corresponding to the formation \( \mathcal{F}_p \) reaching the desired target defined by \( \mathbf{d}_e \) and/or \( \mathbf{b}_e \). Each formation that lives in \( Z^* \) is regularly rigid because of Assumptions 2 and 1. Unfortunately, the set \( Z^* \) may not be the only equilibrium set for the differential system (37). Consider the set

\[
Z_* = \{ p \in \mathbb{R}^{2n} : \mathbf{J}_e(p)^\top e = 0 \}
\]  

(50)

and note that it is trivial to conclude that \( \dot{p} = 0 \) if and only if \( p \in Z_* \). A question remains as to when \( Z^* = Z_* \).

Consider (36) and note that \( \dot{e} = \mathbf{J}_e(p)\dot{p} \). Then, the error dynamics can be written as

\[
\dot{e} = -\mathbf{J}_e(p)\dot{\theta}(t)\mathbf{J}_e(p)^\top e
\]  

(51)

Let

\[
v(e) = \frac{1}{2}e^\top e
\]  

(52)

denote a Lyapunov-like function with the following properties: \( v \geq 0 \) with \( v = 0 \) if and only if \( e = 0 \); \( v \) is smooth and

\[
\dot{v} = -e^\top \mathbf{J}_e(p)\dot{\theta}(t)\mathbf{J}_e(p)^\top e
\]  

(53)

where we restrict \( \dot{\theta}(t) \in [-\pi/2 + \epsilon, \pi/2 - \epsilon] \) according to the design of the controller.

**Theorem 1**
Suppose Assumptions 1 and 2 hold. Then, for all continuous functions \( \dot{\theta}(t) \in [-\pi/2 + \epsilon, \pi/2 - \epsilon] \), the formation converges to \( Z_* \) asymptotically.
Proof Define $\Omega_d(\theta) \in \mathbb{R}^{2n \times 2n}$ as

$$
\Omega_d(\theta) = \begin{bmatrix}
\cos \theta_1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & \cos \theta_1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \cos \theta_2 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \cos \theta_2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & \cos \theta_n & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & \cos \theta_n
\end{bmatrix}
$$

and note that $x^T \Omega_d(\theta)x = x^T \Omega(\theta)x$ for all $x \in \mathbb{R}^{2n}$. Consider $v(e)$ in (52) and its derivative in (53). Then, $\dot{v}(e)$ can be written as

$$
\dot{v} = -e^T J_e(p) \Omega_d(\theta(t)) J_e(p)^T e.
$$

Because $\cos \theta_i$ is positive for all $i$ when $\theta_i \in (-\pi/2, \pi/2)$, $\Omega_d(\theta(t))$ is positive definite, and thus $\dot{v}$ is zero iff $J_e(p)^T e = 0$. So, we can conclude that the formation converges to $Z_\ast$.

We showed in the aforementioned theorem that if the graph is rigid, parallel rigid or stiff, then the formation converges to $Z_\ast$.

Corollary 1
Suppose Assumptions 1 and 2 hold. Then, for some continuous functions $\theta_i(t) \in [-\pi/2 + \epsilon, \pi/2 - \epsilon]$ and for some initial formation positions, the formation converges asymptotically to $Z_\ast$.

We will show in the succeeding text that under further conditions on the framework $(G, p)$, the formation converges to $Z_\ast$ for all continuous functions $\theta_i(t) \in [-\pi/2 + \epsilon, \pi/2 - \epsilon]$. To do this, let us first define regularly minimally rigid/stiff graphs.

Definition 5 (Regularly minimally rigid/stiff)
A formation is regularly minimally rigid/stiff if and only if for any regular point, $J_e(p)$ has full row rank.

Proposition 1
Suppose Assumption 1 holds and the formation is regularly minimally rigid/stiff. Then, for all continuous functions $\theta_i(t) \in [-\pi/2 + \epsilon, \pi/2 - \epsilon]$, there exists a neighbourhood of $Z_\ast$, within which formations converge to $Z_\ast$ at an exponential rate.

Proof Consider (52) and (55). Because $J_e(p)$ has full row rank for all $p$ within a neighbourhood of $Z_\ast$ and $\Omega_d(\theta(t))$ is positive definite, $J_e(p) \Omega_d(\theta(t)) J_e(p)^T$ is also positive definite, and thus,

$$
\dot{v} \leq -\lambda_{\inf}(J_e(p) \Omega_d(\theta(t)) J_e(p)^T) \|e\|^2.
$$

where $\lambda_{\inf}(\cdot)$ denotes the minimum (infimum) eigenvalue with respect to the eigenvalue set, $t$ and $p$ in the relevant neighbourhood of $Z_\ast$. Then, according to Theorem 4.10 of [31], $e = 0$ is exponentially stable.
9. ILLUSTRATIVE EXAMPLES

In this section, we demonstrate some examples for the developed distributed formation control algorithms for bearing-only constraints, distance-only constraints and mixed bearing and distance constraints.

9.1. Distance-only constraints

We first consider a formation of three agents. The desired shape of the formation is an equilateral triangle with sides equal to 2. The trajectories of the agents when $\theta_i(t), i = 1, 2, 3$ are zero for $t \geq 0$ are shown in the left-hand side figure of Figure 2. We now assume there are some obstacles in the path of the agents such that collision will occur if the agents do not change their directions of motion. We further assume that each agent can detect obstacles if it gets close to an obstacle and can find some $\theta_i(t) \in [-\pi/2 + \epsilon, \epsilon\pi/2]$ to move away from the obstacle. The trajectories of the agents for this scenario are shown in the right-hand side figure of Figure 2. It can be seen that the agents achieve the desired formation shape and avoid collision with obstacles using time-varying $\theta_i(t)$.

Consider another formation with four agents indexed by $\mathcal{V} = \{1, 2, 3, 4\}$ and suppose the interaction topology is defined by the links $\mathcal{E} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}$. Then, the control error for this formation has the form

$$e = d(p) - d_c = \begin{bmatrix} d_{12}^2 & d_{13}^2 & d_{14}^2 & d_{23}^2 & d_{24}^2 \end{bmatrix}^T \begin{bmatrix} 3^2 & 3^2 & 3^2 & 3^2 \end{bmatrix}^T.$$ (57)

The formation is defined as in Examples 1 and 4, and the rigidity matrix and error Jacobian take the form of (10) and (29), respectively. We first assume that $\theta_i(t) = 0 \forall i = 1, \cdots, n$. The formation motion along with the convergence of $d_{ij} - d_{ij}^\ast \to 0$ for $(i, j) \in \mathcal{E}, i < j$ is illustrated in Figure 3. Then, we consider the case where the angles $\theta_i(t)$ for $i = 1, \cdots, 4$ change randomly at each 0.05 s such that $\theta_i(t)$ are in $(-\pi/2, \pi/2)$ for all $t > 0$ and are discontinuous. The results are shown in Figure 4.

9.2. Bearing-only constraints

Suppose the formation has four agents and assume the edge set of the graph is the same as the edge set in the distance-only case and the control error for the formation has the form

$$e = b(p) - b_c = \begin{bmatrix} \phi_{12} \phi_{13} \phi_{14} \phi_{23} \phi_{24} \end{bmatrix}^T \begin{bmatrix} \arctan(3/4) & \pi / 2 & 3\pi / 2 \end{bmatrix}^T.$$ (58)

Figure 2. The left-hand side figure shows the trajectory of the agents when all $\theta_i(t), i = 1, 2, 3$ are zero for $t \geq 0$. The right-hand side figure shows the trajectory of the agents when the agents are in danger of collision with obstacles and each agent changes its $\theta_i(t)$ to avoid collision. The initial position of the agents is shown by ‘O’, and the final position is shown by ‘□’. The dotted and dashed lines show respectively the distance between the neighbour agents at $t = 0$ and when time goes to infinity.
Figure 3. The motion of a formation consisting of four mobile agents with distance-only constraints. The initial position of the agents is shown in the left-hand side figure by ‘○’, and the final position is shown by ‘□’.

Figure 4. The motion of a formation consisting of four mobile agents with distance-only constraints when the angles $\theta_i$ for $i = 1, \ldots, 4$ change randomly (and discontinuously) after each 0.05 s. The initial position of the agents is shown in the left-hand side figure by ‘○’, and the final position is shown by ‘□’.

Figure 5. The motion of a formation consisting of four mobile agents with bearing-only constraints. The initial position of the agents is shown in the left-hand side figure by ‘○’, and the final position is shown by ‘□’.
Thus, the formation is defined as in Examples 2 and 5, and the parallel rigidity matrix and error Jacobian take the form of (15) and (31), respectively. Simulation results for the case where $\theta_i = 0 \ \forall i = 1, \ldots, 4$ are shown in Figure 5.

### 9.3. Mixed bearing and distance constraints

We finally consider the case where we have both bearing and distance constraints. Suppose the graph edges are $\mathcal{E} = \{(1, 2), (1, 3), (1, 4), (2, 3), (3, 4)\}$ where edges $(1, 2)$ and $(1, 3)$ correspond to distance constraint, edges $(1, 4)$ and $(3, 4)$ correspond to bearing constraint and edge $(2, 3)$ corresponds to both bearing and distance constraint. The formation is defined as in Examples 3 and 6, and $\mathbf{R}(\mathbf{p})$ and the error Jacobian take the form of (20) and (33). The control error in this case would be

$$
\mathbf{e} = \left[\mathbf{b}(\mathbf{p})^T \ \mathbf{d}(\mathbf{p})^T\right]^T - \left[\mathbf{c}_1^T \ \mathbf{c}_2^T\right]^T \\
= \left[\phi_{14} \ \phi_{13} \ d_{12}^2 \ d_{13}^2 \ d_{23}^2\right]^T - \left[0 \ \pi \left(2\pi - \arctan\left(\frac{3}{4}\right)\right) \ 5^2 \ 3^2 \ 4^2\right]^T.
$$

We consider two different cases where $\theta_i = 0 \ \forall i = 1, \ldots, 4$ and $\theta_i$ change randomly and discontinuously after each 0.05 s. The results are illustrated in Figures 6 and 7.

---

**Figure 6.** The motion of a formation consisting of four mobile agents with mix bearing and distance constraints. The initial position of the agents is shown in the left-hand side figure by ‘○’, and the final position is shown by ‘□’.

**Figure 7.** The motion of a formation consisting of four mobile agents with mix bearing and distance constraints when the $\theta_i$ for $i = 1, \ldots, 4$ change randomly (and discontinuously) after each 0.05 s. The initial position of the agents is shown in the left-hand side figure by ‘○’, and the final position is shown by ‘□’.
This paper looks at the design of a distributed control scheme to solve the formation shape control problem in the case of distance-only, bearing-only or a mix of distance and bearing constraints. The control law introduced is relaxed in the sense that each agent is free to choose its direction of motion with a large set of headings defined by the controller. Indeed, the agents can move in any direction towards (almost) an entire half-plane, giving each agent a considerable amount of freedom in planning its motion. Given only this relaxed motion constraint, a strong local exponential convergence result is provided, which proves the formation converges to the desired shape.

In this section, we will consider some discussion points and directions for potential future research. Firstly, we have assumed in this paper that the agents’ motion is governed by a single integrator model. There are however some results in the literature on formation shape control under a gradient control law for systems with non-holonomic agents (see, e.g. [32]) or agents with double-integrator dynamics (see [33]). A possible direction for future work includes extending the formation control laws for these more general agent dynamical models in such a way so as to allow a relaxed motion requirement as proposed in this work. Such extensions seem initially to be possible.

A further possible direction of future research is to consider relaxed motion extensions to more sophisticated formation control laws (beyond simple gradient methods) such as those proposed in [34] based on backstepping, [35] based on sliding-mode controllers or those in [36], which involve discontinuous perturbation signals and so on.

We also note that the convergence rate of the distance or bearing error might conceivably decrease when the agents’ motions are changed though application of some time-varying $\theta_i(t)$. The effect of this parameter on convergence speeds is a topic that may be of interest in future research.

Finally, it would also be possible to consider a generalisation of the use of $\Omega(\theta_i(t))$ to $k_i(t)\Omega(\theta_i(t))$ for some (continuously) time-varying, bounded and strictly positive $k_i(t)$. If one notes that $\Omega(\theta_i(t))$ provides robustness to poor bearing sensing, then the addition of $k_i(t)$ may provide a form of robustness to erroneous distance sensing. Generalisations of the provided control law to three-dimensional space may also be considered.

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