Optimality analysis of sensor-target localization geometries

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\textbf{A B S T R A C T}

The problem of target localization involves estimating the position of a target from multiple noisy sensor measurements. It is well known that the relative sensor-target geometry can significantly affect the performance of any particular localization algorithm. The localization performance can be explicitly characterized by certain measures, for example, by the Cramer–Rao lower bound (which is equal to the inverse Fisher information matrix) on the estimator variance. In addition, the Cramer–Rao lower bound is commonly used to generate a so-called uncertainty ellipse which characterizes the spatial variance distribution of an efficient estimate, i.e., an estimate which achieves the lower bound. The aim of this work is to identify those relative sensor-target geometries which result in a measure of the uncertainty ellipse being minimized. Deeming such sensor-target geometries to be optimal with respect to the chosen measure, the optimal sensor-target geometries for range-only, time-of-arrival-based and bearing-only localization are identified and studied in this work. The optimal geometries for an arbitrary number of sensors are identified and it is shown that an optimal sensor-target configuration is not, in general, unique. The importance of understanding the influence of the sensor-target geometry on the potential localization performance is highlighted via formal analytical results and a number of illustrative examples.

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1. Introduction

It is well known that the relative sensor-target geometry can significantly affect the potential performance of any particular localization algorithm (Gustafsson & Gunnarsson, 2005; Nardone, Lindgren, & Gong, 1984). Noting that the Cramer–Rao lower bound is a function of the relative sensor-target geometry, a number of authors have attempted to identify those geometric configurations which minimize some measure of this variance lower bound, see Bishop, Fidan, Anderson, Dogancay, and Pathirana (2007), Bishop, Fidan, Anderson, Pathirana, and Dogancay (2007), Bishop and Jensfelt (2009), Dempster (2006), Dogancay and Himam (2008), Kadar (1998), Martinez and Bullo (2006) and Nardone et al. (1984). The idea is that such geometric configurations are likely to result in accurate localization (at least for efficient estimation algorithms).

In Dempster (2006), Kadar (1998) and Nardone et al. (1984) a partial characterization of the optimal sensor-target geometry for bearing-only localization is given using various scalar measures of the Cramer–Rao lower bound or the corresponding Fisher information matrix. Typically, assumptions on the sensor-target range are made; e.g. equal sensor-target ranges for all sensors. The bearing-only localization geometry was analyzed more generally in Bishop et al. (2007) and Dogancay and Himam (2008) for arbitrary sensor-target ranges. The optimal geometry for time-of-arrival localization was introduced in Bishop et al. (2007). In Ash and Moses (2008), the assumption of uniform angular sensor spacing on a perimeter of a sensor network is examined and the
resulting range-only localization performance is explored in detail. In Martinez and Bullo (2006) an important result concerning the localization geometry for sensors which measure certain functions of the sensor-target range is provided in both $\mathbb{R}^2$ and $\mathbb{R}^3$. A special case of Martinez and Bullo (2006) is highlighted and analyzed in detail here for the typical range-only localization problem in $\mathbb{R}^2$.

In Martinez and Bullo (2006) the problem of moving the sensors in order to track moving targets while maintaining an optimal localization geometry is also examined. In fact, for mobile sensor-based localization problems, a similar measure of localization performance (as a function of the dynamic sensor-target geometry) can be used to identify optimal sensor trajectories and to derive control laws for driving sensors along such trajectories, e.g. see Bishop and Pathirana (2008), Dogancay (2007), Fogel and Gavish (1988), Jauffret and Pillon (1996), Martinez and Bullo (2006), Nardone et al. (1984), Oshman and Davidson (1999) and Song (1996).

In this paper we consider only the static localization problem involving a single (stationary) target and multiple (stationary) sensors located in two-dimensions. In this case, the Cramer–Rao lower bound can be used to generate a so-called uncertainty ellipse which characterizes the spatial variance distribution of an efficient target estimate. The Cramer–Rao bound, and thus the uncertainty ellipse, is inherently a function of the sensor-target geometry along with the specific measurement technology employed by the sensors. As such, the specific aim of this paper is to rigorously identify those relative sensor-target geometries which result in a volume (or area) measure of the uncertainty ellipse being minimized. An estimation algorithm which achieves the Cramer–Rao lower bound will generate a target estimate with the smallest area uncertainty ellipse in these geometries and with the particular measurement technology employed by the sensors.

Specifically, we analyze in depth the geometry of the optimal sensor-target angular geometries for range-only and time-of-arrival-based localization. It is shown that the optimal sensor-target angular geometries for range-only and time-of-arrival-based localization are not unique and, indeed, an infinite number of optimal sensor-target geometric configurations exist if the number of sensors exceeds a certain small number. Additionally, a rigorous characterization of the optimal sensor-target angular geometries for bearing-only localization is provided in this paper. For bearing-only localization, we identify the optimal angular sensor-target configurations for an arbitrary number of sensors, and for fixed, but arbitrary, sensor-target ranges. We deviate from the conventional wisdom purported in the literature (Ash & Moses, 2008) which considers it desirable to position sensors uniformly around the entire perimeter of the target. Instead, we show that the true optimal geometric configurations for bearing-only localization are explicitly dependent on the sensor-target ranges and that the optimal geometry can change significantly as a result of varying (even a single) sensor-target range value. Again the sensor-target angular geometries for bearing-only localization are not unique and an infinite number of optimal configurations can be identified when the number of sensors exceeds a small number.

2. Notation and related conventions

A single stationary target and multiple sensors are considered and located in $\mathbb{R}^2$. The target’s location is $p = [x_p \ y_p \ T]^T$. The sensors are labeled $1, \ldots, N \geq 2$ with the location of the $i$th sensor denoted by $s_i = [x_{si} \ y_{si}]^T$. The range between sensor $s_i$ and the target $p$ is denoted by $r_i = \|p - s_i\|$. The true azimuth bearing $\phi_i$ from sensor $i$ to the target is measured positive clockwise from the $y$-axis, i.e. positive clockwise from north. The angle subtended at the target by two sensors $i$ and $j$ is denoted by $\phi_{ij} = \phi_i - \phi_j \mod \pi \in [0, \pi]$. If $\phi_{ij} = 0$ or $\phi_{ij} = \pi$, then the sensors $i$ and $j$ are said to be collinear with the target. A typical scenario depicting the relevant geometrical parameters is illustrated in Fig. 1.

2.1. Bearing-only localization

Consider the target position $p = [x_p \ y_p \ T]^T$ and sensor locations $s_i = [x_{si} \ y_{si}]^T$, for all $i \in \{1 \ldots, N\}$. The measured azimuth bearing $\phi_i$ from sensor $i$ to the target can be expressed by

$$\phi_i = \phi(p) + \epsilon_i = \tan^{-1}(x_p - x_{si} \ \ y_p - y_{si}) + \epsilon_i \ (1)$$

where $\tan^{-1}(x/y)$ is the inverse tangent of $x/y$ where the signs of both $x$ and $y$ are used to determine the correct quadrant of the result. The measurement errors $\epsilon_i$, $\forall i \in \{1 \ldots, N\}$, are assumed to be mutually independent and Gaussian distributed with zero mean and the same variance $\sigma_{\epsilon_i}^2$.

The set of bearing measurements from $N$ sensors can be written in vector format as

$$\hat{p} = \Phi(p) + \epsilon = [\phi_1(p) \ \ldots \ \phi_N(p)]^T + [\epsilon_1 \ \ldots \ \epsilon_N]^T \ (2)$$

such that the covariance of $\epsilon$ is given by $R_\epsilon = \sigma_\epsilon^2 I_N$ where $I_N$ is an $N$-dimensional identity matrix; i.e. $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2 I_N)$.

The following is a formal statement of the bearing-only localization problem using the notation defined in this subsection.

**Problem 1.** Assume there exists a set of $N \geq 2$ bearing measurements from $N$ spatially distinct sensors. Using the observable set of random bearing measurements $\hat{p} \sim \mathcal{N}(\Phi(p), R_\epsilon)$, find an estimate $\hat{p}$ of the true target location $p$.

2.2. Range-only localization

As previously stated, the true range between the $i$th sensor $s_i$ and the target $p$ is given by $r_i = \|p - s_i\|$. The measured range at the $i$th sensor obeys the following model $r_i = r_i + \epsilon_i$ where $\epsilon_i$ is the measurement error. The errors $\epsilon_i$, $\forall i \in \{1 \ldots, N\}$ are assumed to be mutually independent and Gaussian distributed with zero mean and the same variance $\sigma_{\epsilon_i}^2$, i.e. $\epsilon_i \sim \mathcal{N}(0, \sigma_{\epsilon_i}^2)$. The range measurements from $N$ range sensors can be stacked as follows

$$\hat{r} = \gamma(p) + \epsilon = [r_1 \ \ldots \ r_N]^T + [\epsilon_1 \ \ldots \ \epsilon_N]^T \ (3)$$

such that $\hat{r} \sim \mathcal{N}(\gamma(p), R_\gamma)$ where $R_\gamma = \sigma_{\gamma_i}^2 I_N$. The following range-only localization problem can now be stated formally using the notation defined in this subsection.

**Problem 2.** Assume there exists a set of $N \geq 2$ range measurements from $N$ spatially distinct sensors. Using the observable set of random range measurements $\hat{r} \sim \mathcal{N}(\gamma(p), I_N)$, find an estimate $\hat{p}$ of the true target location $p$.

2.3. Time-of-arrival and time-difference-of-arrival localization

Consider a target emitter $p = [x_p \ y_p \ T]^T \in \mathbb{R}^2$ which transmits a signal at a specific time $\tau$. Let the location of the event characterized by $p$ and $\tau$ be denoted by $x = [x_p \ y_p \ \tau]^T \in \mathbb{R}^3$. **Fig. 1.** The relative geometry between two sensors and a single stationary target in $\mathbb{R}^2$. 

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Suppose that each sensor can measure the time of arrival of the transmitted signal at the sensor. This time of arrival at the ith sensor is denoted by \( t_i \). Then \( t_i \) obeys the following relationship:

\[
 t_i(x) = \frac{\| p - s_i \|}{v} + \tau
\]

where \( v \) is the signal propagation speed. We normalize such that \( v \equiv 1 \). Generally the measurement is assumed to be noisy, so that \( \tilde{t}_i = t_i(x) + e_i \) where \( t_i(x) \) is the true time of signal arrival and \( e_i \) is the measurement error. The errors \( e_i, \forall i \in \{1, \ldots, N\} \) are assumed to be mutually independent and Gaussian distributed with zero mean and the same variance \( \sigma^2_i \). Stacking the measurements from \( N \) sensors leads to the following measurement vector

\[
 \tilde{y}(x) = y(x) + e = \left[ t_1 - t_0 \right] + e_1 + e_2 + \cdots + e_N
\]

where now we assume that \( \tilde{y}(x) \sim N(y(x), R_e) \) where \( R_e = \sigma^2_1 I \) is the covariance of \( \tilde{y} \). The problem of estimating \( p \) from the given noisy measurements \( \tilde{y} \) is known as the time-of-arrival localization problem. The time-of-arrival localization problem also results in an estimate of the time of signal transmission \( \tau \) (although this parameter is not always required).

An alternative approach to estimate the location \( p \) from the given timing measurements \( \tilde{t}_i(x) = t_i(x) + e_i, \forall i \in \{1, \ldots, N\} \) involves taking the time differences. The true time-difference \( d_{ij} = (t_j - t_i) v \) between sensor \( i \) and \( j \) where \( i \neq j \) results in the following range-difference equations

\[
 d_{ij}(p) = \| p - s_i \| - \| p - s_j \|, \quad \forall i, j \in \{1, \ldots, N\}
\]

with \( v \equiv 1 \). Note that there are only \((N - 1)\) independent range-difference equations that can actually be formed. In this formulation we have eliminated the unknown \( \tau \). Without loss of generality we only consider range-difference equations between sensor 1 and sensor \( i \). If we take the time difference \( \tilde{t}_i - t_0 \) then we obtain the following range-difference measurements \( d_{ij} = d_{ij}(p) + e_{i-1}, \forall i \in \{2, \ldots, N\} \) where \( e_{i-1} \) is the range-difference measurement error. Writing the range difference equations in vector form gives

\[
 \tilde{d} = d(p) + e = [d_{12}, d_{13}, \ldots, d_{1N}] + [e_1, e_2, \ldots, e_{N-1}]^T
\]

Note that the covariance matrix of \( e \) is now given by

\[
 R_e = 2\sigma^2 I \text{circulant}
\]

where \( \text{circulant} (\cdot) \) generates a square circulant matrix, and we have \( d \sim N(d(p), R_e) \). The problem of estimating \( p \) from the given noisy measurements \( \tilde{d} \) is known as the time-of-arrival localization problem (or the range-difference based localization problem).

Clearly the information available for solving the two localization problems is equivalent (Shin & Sung, 2002), and generally both require at least four sensors in order to uniquely solve for an estimate of \( p \) (although three sensors can lead to a possibly ambiguous location estimate). Of course, different algorithms of varying quality can be designed separately for the two formulation problems. In any case, we can choose which formulation yields the simplest analysis with the geometrical results being equally applicable to both problems. In this paper, only the time-of-arrival based formulation is examined due to its simplicity of formulation.

Problem 3. Assume there exists a set of \( N \geq 3 \) time-of-arrival measurements from \( N \) spatially distinct sensors. Unless the observable set of random timing measurements \( \tilde{y} \sim N(y(x), \sigma^2_{\text{circulant}}(\tilde{y})) \), find an estimate \( \hat{x} \) of the true event location \( x \) which includes both the event location \( p \) and the time of the event \( \tau \).

Remark 1. In each of Problems 1–3, if the expected value of the estimate \( \mathbb{E}[\hat{p}(\tilde{y}(x))] \) is constrained to be \( p(x) \), then the corresponding problem is one of unbiased localization.

2.4. Equivalence of relative angular configurations

Without loss of generality we will always restrict the sensor indexing such that the true bearings obey \( \phi_i \geq \phi_j \) when \( j > i \) and \( \forall i, j \in \{1, \ldots, N\} \). We can re-index the sensor locations (and their corresponding parameters such as target-range etc.) such that this restriction holds.

Definition 1. Two relative sensor-target angular configurations, one with sensors \( \{s_1, \ldots, s_9\} \) and the other with sensors \( \{s_5, \ldots, s_9\} \) are said to be: (a) equivalent if there exists a permutation \( p_i : \{1, \ldots, N\} \rightarrow \{1, \ldots, N\} \) such that the value of \( \theta_i = [0, \pi] \) for each sensor pair \( (s_i, s_j) \) in the first configuration is equal to \( \theta_{p_i(i)p_i(j)} \in [0, \pi] \) in the second configuration; (b) differ by a collinear reflection if they are not equivalent and there exists a permutation \( p_i : \{1, \ldots, N\} \rightarrow \{1, \ldots, N\} \) such that the value of \( \theta_i \in [0, \pi] \) for each sensor pair \( (s_i, s_j) \) in the first configuration is equal to \( \theta_{p_i(i)p_i(j)} \in [0, \pi] \) in the second configuration in modulo \( \pi \), i.e. when the value of each \( \theta_i \) or \( \theta_{p_i(i)p_i(j)} \) is restricted to \( \in [0, \pi] \).

For example, a configuration of three sensors \( \{1, 2, 3\} \) with \( \phi_1 = \frac{\pi}{2}, \phi_2 = \frac{\pi}{4} \) and \( \phi_3 = \frac{\pi}{8} \) is equivalent to a configuration of sensors \( \{1', 2', 3'\} \) with \( \phi_1' = \frac{\pi}{2}, \phi_2' = \frac{11\pi}{12} \) and \( \phi_3' = 2\pi \). To see this, note that for the first configuration we have \( \theta_{12} = \frac{\pi}{2} \) and \( \theta_{13} = \frac{2\pi}{3} \) for the second configuration we have \( \theta_{12}' = \theta_{13}' = \frac{\pi}{2} \) and \( \theta_{12}' = \frac{\pi}{2} \). The two configurations are different only up to a permutation of the indices and a global rotation of the coordinate system.

Consider now a configuration of four sensors with \( \phi_1 = 0, \phi_2 = \frac{\pi}{2}, \phi_3 = \frac{\pi}{4} \) and \( \phi_4 = \frac{\pi}{8} \) and a configuration of sensors with \( \phi_1' = 0, \phi_2' = \frac{\pi}{2}, \phi_3' = 0 \) and \( \phi_4' = \frac{\pi}{4} \). Clearly, the two configurations are not equivalent. However, if for the second configuration we reflected sensor 3 and sensor 4 about the target position, then the two configurations would be identical. In this case, the two configurations are said to differ by a collinear reflection.

3. A metric for optimal sensor placement

A metric is defined in this section which can be used to determine the optimal sensor placement for localization with different measurement technologies. The metric is defined for a general measurement vector \( \tilde{z} = x(w) + n \) and a general vector \( w \in \mathbb{R}^k \) which is to be estimated from the observable measurements \( \tilde{z} \in \mathbb{R}^q \). Here, \( n \in \mathbb{R}^k \) is a vector of Gaussian random variables with zero mean and a constant covariance matrix \( \Sigma \).

Under the standard (and adopted) assumption of Gaussian measurement errors, the likelihood function of \( w \) given the measurement vector \( \tilde{z} \sim N(z(w), \Sigma) \) is given by

\[
 f_z(\tilde{z}; w) = \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \times \exp \left( -\frac{1}{2} (\tilde{z} - z(w))^T \Sigma^{-1} (\tilde{z} - z(w)) \right)
\]

where \( |\Sigma| \) is the determinant of \( \Sigma \) and \( z(w) \) is the mean value of \( \tilde{z} \). In general, the Cramer–Rao inequality lower bounds the covariance achievable by an unbiased estimator (under two mild regularity conditions). For an unbiased estimate \( \hat{w} \) of \( w \), the Cramer–Rao bound states that

\[
 \mathbb{E}[ (\hat{w} - w)(\hat{w} - w)^T ) \geq J(w)^{-1} \equiv C(w)
\]

where \( J(w) \), defined below, is called the Fisher information matrix. In general, if \( J(w) \) is singular then no unbiased estimator for \( x \) exists with a finite variance (Stoica & Marzetta, 2001; Van Trees, 1968). If (10) holds with equality then the estimator is called efficient and the parameter estimate \( \hat{w} \) is unique (Van Trees, 1968).
The matrix \( I(\mathbf{w}) \) quantifies the amount of information that the observable random measurement vector \( \tilde{z} \) carries about the unobservable parameter \( \mathbf{w} \). The \((i,j)\)th element of \( I(\mathbf{w}) \) is given by

\[
(I(\mathbf{w}))_{ij} = \mathbb{E} \left[ \frac{\partial}{\partial w_i} \ln \left( f_2(\tilde{z}; \mathbf{w}) \right) \frac{\partial}{\partial w_j} \ln \left( f_2(\tilde{z}; \mathbf{w}) \right) \right]
\]

(11)

where again \( \mathbf{w} \) is the parameter to be estimated. Under the assumption of Gaussian measurement errors and when the error covariance is independent of the parameter, the entire Fisher information matrix is simply given by

\[
I(\mathbf{w}) = \nabla_{\mathbf{w}} z(\mathbf{w})^T \Sigma^{-1} \nabla_{\mathbf{w}} z(\mathbf{w})
\]

(12)

where \( \nabla_{\mathbf{w}} z(\mathbf{w}) \) is the jacobian of the measurement vector with respect to the parameter \( \mathbf{w} \). The Fisher information metric characterizes the nature of the likelihood function \( f(\mathbf{z} | \mathbf{w}) \). If the likelihood function is sharply peaked then the true value \( \mathbf{w} \) is easier to estimate from the measurement \( \mathbf{z} \) than if the likelihood function is flatter. Independent measurements from additional sensors in general positions cannot decrease the total information.

Note that \( C(\mathbf{w}) = I(\mathbf{w})^{-1} \) is symmetric positive definite (so long as \( I(\mathbf{w}) \) is invertible) and defines a so-called uncertainty ellipsoid. Denote the eigenvalues of \( C(\mathbf{w}) \) by \( \lambda_i \), and note that \( \sqrt{\lambda_i} \), \( \forall i \in \{1, \ldots, M\} \) is the length of the ith axis of the ellipsoid. Note also that axes lie along the corresponding eigenvectors of \( C(\mathbf{w}) \). A scalar functional measure of the ‘size’ of the uncertainty ellipsoid provides a useful characterization of the potential performance of an unbiased estimator. In this paper, the volume of the uncertainty ellipsoid is used as an intuitively meaningful measure of the total uncertainty in an estimate \( \mathbf{w} \) of \( \mathbf{w} \). For computational simplicity we consider the Fisher information determinant \( \det(I(\mathbf{w})) \) as a computable measure of the volume of the ellipsoid generated by \( C(\mathbf{w}) \).

Considering the determinant \( \det(I(\mathbf{w})) \) as a function of the design parameters \( (\mathbf{w}, \mathbf{w}) \) in the nonlinear case has a long history in experimental design and is known as the D-criterion (Goodwin & Payne, 1977; Ucinski, 2004). A D-optimum (i.e. maximum) design for the system parameters will minimize the volume of the uncertainty ellipsoid generated by \( C(\mathbf{w}) \), is invariant under scale changes in the parameters and is invariant to linear transformations of the output (Ucinski, 2004).

3.1. The sensor-target geometry and the metric of estimator performance

Imagine that \( \mathbf{w} \) is defined to be a target or event location and \( \tilde{z} = z(\mathbf{w}) + \mathbf{n} \) is the vector of \( N \) geometrical and nonlinear measurements of \( \mathbf{w} \) taken from \( M \) spatially distinct sensor locations. It is easy to conclude that \( z(\mathbf{w}) \) and hence \( I(\mathbf{w}) \) are explicit functions of the relative sensor-target geometry.

**Definition 2.** For range-only and time-of-arrival-based localization, a relative sensor-target angular configuration is said to be optimal if it maximizes the Fisher information determinant over the space of all angle positions \( \phi_i \), \( \forall i \in \{1, \ldots, N\} \). An optimal relative sensor-target angular configuration for range-only or time-of-arrival-based localization is said to be unique and if only if that optimal configuration is equivalent to, or differs only by a collinear reflection from, every other optimal configuration in the sense of Definition 1.

**Definition 3.** For bearing-only localization, a relative sensor-target angular configuration is said to be optimal for a specified set of fixed but arbitrary sensor-target ranges if it maximizes the Fisher information determinant over the space of all angle positions \( \phi_i \), \( \forall i \in \{1, \ldots, N\} \) given those fixed sensor-target ranges. An optimal relative sensor-target angular configuration for bearing-only localization is said to be unique if and only if that optimal configuration is equivalent to, or differs only by a collinear reflection from every other optimal configuration in the sense of Definition 1.

The purpose of the analysis in this paper is not to construct estimators, but rather to characterize the effect of the localization geometry on the performance of a generic unbiased and efficient estimator.

4. The geometry of range-only based localization

In this section we highlight, and re-derive in a simplified but specific way, a result on the optimal range-only localization geometry (Martinez & Bullo, 2006). We then go further and focus on deriving and understanding certain geometrical and practically significant consequences of the given general optimality conditions. Our aim is to provide easily applicable techniques for optimally placing any number of range-only sensors.

Given the measurement vector \( \mathbf{r}(\mathbf{p}) \), the entire Fisher information matrix for range-only localization is given by (12) with \( \mathbf{r}(\mathbf{p}) = z(\mathbf{w}) \). Correspondingly,

\[
I_r(\mathbf{p}) = \frac{1}{\sigma_t^2} \sum_{i=1}^{N} \left[ \frac{\sin^2(\phi_i) \sin(2\phi_i)}{2} \right] \]

(13)

is the derived Fisher information matrix given \( N \) range-only sensor measurements. When \( N = 1 \), the determinant \( \det(I_r(\mathbf{p})) \) vanishes for all \( \mathbf{p} \). Thus, no unbiased estimator with a finite variance exists for the location \( \mathbf{p} \) when \( N = 1 \). In general, at least \( N \geq 2 \) range-only sensors are required to estimate the value of \( \mathbf{p} \). However, due to the nonlinearity in the equations \( r_i = ||\mathbf{p} - \mathbf{s}_i|| \), a unique solution typically requires \( N > 2 \) sensors in general.

**Theorem 1.** Consider the range-only based localization problem, i.e. Problem 2, with \( \phi_i \), \( \forall i \in \{1, \ldots, N\} \) denoting the angular positions of the sensors. The following are equivalent expressions for the Fisher information determinant \( \det(I_r(\mathbf{p})) \) for range-only based localization:

(i) \( \det(I_r(\mathbf{p})) = \frac{1}{4\sigma_t^2} \left[ N^2 - \left( \sum_{i=1}^{N} \cos(2\phi_i) \right)^2 - \left( \sum_{i=1}^{N} \sin(2\phi_i) \right)^2 \right] \)

(14)

(ii) \( \det(I_r(\mathbf{p})) = \frac{1}{4\sigma_t^2} \sum_{j=1}^{N} \sin^2(\phi_j - \phi_i), \quad j > i \)

(15)

where \( \delta = \{(i,j)\} \) is defined as the set of all combinations of \( i \) and \( j \) with \( i, j \in \{1, \ldots, N\} \) and \( j > i \).

**Proof.** The proof of (14) is straightforward and follows from construction. First, write (13) as

\[
I_r(\mathbf{p}) = \frac{1}{\sigma_t^2} \left[ \sum_{i=1}^{N} \frac{1 - \cos(2\phi_i)}{2} - \sum_{i=1}^{N} \frac{\sin(2\phi_i)}{2} \right] \]

(16)

Now a rearrangement of the the determinant formula leads to the Eq. (14). For part (ii), we refer to Martinez and Bullo (2006). □

An advanced expression for the determinant is provided in Martinez and Bullo (2006) for arbitrary range measurement functions.
Theorem 1

Consider the range-only localization problem, i.e.,

\[ \text{(17)} \]

Given an arbitrary number \( N \) of range-only sensors, the Fisher information determinant for the range-only localization problem in \( \mathbb{R}^3 \) is

\[ \det \mathbf{F} = \frac{2^N}{N!} \]

for all adjacent sensor pairs \( i, j \) with \( |j - i| = 1 \) or \( |j - i| = N - 1 \), and then by a possible application of Proposition 1 on (18).

\[ \text{Proof.} \]

The proof of this proposition is straightforward and involves verifying that (18) satisfies (17) of Theorem 2. Then, sensor reflections as detailed in Proposition 1 do not change the value of the determinant and thus do not change the optimality of the sensor-target configuration. Further details are omitted for brevity.

\[ \square \]

Corollary 1

Consider the angles \( \vartheta = \vartheta_j \) subtended at the target by two adjacent sensors, where adjacency implies \( |j - i| = 1 \) or \( |j - i| = N - 1 \) for range-only localization. The optimal values of \( \vartheta \) are \( \vartheta = \frac{\pi}{N} \) and \( \vartheta = \frac{1}{N} \). These two optimal angular configuration values are two separate optimal angular configurations of the \( N \) range-only sensors.

4.1. The range-only localization geometry with \( N \) sensors

The following is the main result concerning optimal range-only localization geometries which first appeared in Martinez and Bullo (2006). The proof is immediate from Theorem 1 (i).

\[ \text{Theorem 2.} \]

Consider the range-only localization problem, i.e., Problem 2, with \( \phi, \forall i \in \{1, \ldots, N\} \) denoting the angular positions of the sensors. Given an arbitrary number of range-only sensors \( N \), the Fisher information determinant \( \text{(14)} \) for range-only localization is upper-bounded by \( \frac{2^N}{N!} \). This upper-bound is achieved if and only if the angular positions of the sensors are such that

\[ \sum_{i=1}^{N} \cos(2\phi_i(x)) = 0 \quad \text{and} \quad \sum_{i=1}^{N} \sin(2\phi_i(x)) = 0 \]  \quad (17)

are simultaneously satisfied.

Angular sensor positions which maximize the determinant \( \text{(14)} \), i.e., which satisfy \( \text{(17)} \), generate an optimal sensor-target geometry for range-only localization. The remainder of this section focuses on deriving and understanding certain consequences of Theorem 2 for static localization scenarios.

Proposition 1

The following actions do not affect the value of the Fisher information matrix: (1) changing the true individual sensor-target ranges, i.e., moving a sensor from \( s_i \) to \( p + k(s_i - p) \) for some \( k > 0 \); and (2) reflecting a sensor about the ambient target, i.e., moving a sensor from \( s_i \) to \( 2p - s_i \).

The optimal sensor-target geometry is not unique when \( N > 2 \), as implied by the following proposition.

Proposition 2

Consider the range-only localization problem, i.e., Problem 2, where the angle subtended at the target by two sensors \( i \) and \( j \) is denoted by \( \vartheta = \vartheta_{ij} \). Some particular optimal sensor angular geometries for range-only localization with \( N \geq 3 \) sensors can be obtained by first letting

\[ \vartheta = \vartheta = \frac{2}{N} \]  \quad (18)

for all adjacent sensor pairs \( i, j \in \{1, \ldots, N \geq 3\} \) with \( |j - i| = 1 \) or \( |j - i| = N - 1 \), and then by a possible application of Proposition 1 on (18).

Fig. 2. The determinant value as a function of target position with two sensors measuring the target range. The sensors are located at \( s_i = [-1/2 0]^T \) and \( s_j = [1/2 0]^T \).

According to Corollary 3, the optimal geometry occurs when \( \vartheta_{12} = \vartheta_{21} = \frac{1}{2} \pi \). The determinant maximum value in this case is \( \frac{2^N}{N!} = 1 \).

4.2. Range-only localization with two sensors and three sensors

The optimal range-only localization geometry for \( N = 2 \) sensors is unique and is given by the following corollary.

Corollary 3

When \( N = 2 \), the optimal sensor-target geometry is unique and occurs when \( \vartheta_{12} = \vartheta_{21} = \frac{1}{2} \pi \).

In general, range-only localization with \( N = 2 \) sensors will result in a target estimate ambiguity. In the case of a localization ambiguity, it might still be possible to localize given additional (a priori) knowledge of the region containing the target's position. Consider two sensors with positions given by \( s_i = [-1/2 0]^T \) and \( s_j = [1/2 0]^T \). Then, Fig. 2 illustrates the determinant value, with \( \sigma_f = 1 \), and for target coordinates obeying \( x_f \in [-3/2, 3/2] \) and \( y_f \in [-3/2, 3/2] \). Fig. 2 shows the optimal geometry occurs when the target is positioned such that \( \vartheta_{12} = \frac{1}{2} \pi \). The optimal angular positions of the sensors are unique up to rotations and sensor reflections about the target.

The three sensor scenario is particularly important in practice. The optimal range-only localization geometry for \( N = 3 \) sensors...
is completely characterized by the two optimal angular configurations introduced in Corollary 1. The relative determinant (14) value over various arbitrary sensor-target geometries is useful to examine. Let \( A = \phi_0(p) - \phi_1(p) \) and \( B = \phi_2(p) - \phi_1(p) \); then for \( N = 3 \) and \( \sigma_r^2 = 1 \) the determinant (14) can be written as \( \text{det}(I_r(p)) = \sin^2(A) + \sin^2(B) + \sin^2(A - B) \). Then it is possible to plot the value of the determinant with \( \sigma_r^2 = 1 \) directly over the possible values of \( A, B \in [0, 2\pi] \). The surface and contour plots are given in Fig. 3.

Fig. 3 illustrates that for \( A, B \in [0, 2\pi] \), the determinant is maximum at eight points which are easily visualized in the contour plot. However, recalling the earlier standing assumption on sensor indices which implies \( \phi_1 \geq \phi_2 \geq \phi_3 \), then only four maximums are consistent with this assumption, i.e., when \( A > B \).

Now consider a special case where the sensors form a unit equilateral triangle with \( s_1 = [-1/2, 0]^T \), \( s_2 = [1/2, 0]^T \) and \( s_3 = [0, \sqrt{3}/2]^T \). The surface of the determinant (14) is given in Fig. 4 for target coordinates obeying \( x_p \in [-1, 1] \) and \( y_p \in [-1/2, 1] \). The maximum determinant value for the case illustrated in Fig. 4 is \( \frac{9}{4} \). The determinant is maximized when the target is at the center of the triangle or anywhere on the circle defined by the three sensor positions.

5. The geometry of time-of-arrival localization

The goal of this section is to identify properties of the relative time-of-arrival sensor-target geometry which optimize a measure of localization performance (Bishop et al., 2007). No similar results on optimal sensor placement for time-of-arrival based localization exist in the literature. Indeed, the characterization given in this paper is complete for all efficient and unbiased estimation algorithms.

Given the measurement vector \( y(x) \), the entire Fisher information matrix for time-of-arrival-based localization is given by (12) with \( y(x) = z(w) \). Correspondingly,

\[
I_r(x) = \frac{1}{\sigma_t^2} \sum_{i=1}^{N} \begin{bmatrix}
\sin^2(\phi_i) & \frac{\sin(2\phi_i)}{2} & \sin(\phi_i) \\
\frac{\sin(2\phi_i)}{2} & \cos^2(\phi_i) & \cos(\phi_i) \\
\sin(\phi_i) & \cos(\phi_i) & 1
\end{bmatrix}
\]

(19)

where \( i \) indexes the timing measurement from the \( i \)th sensor. It is straightforward to show that when \( N \leq 2 \) the determinant \( \text{det}(I_r(x)) \) vanishes for all \( x \). In general, at least \( N \geq 3 \) sensors are required in order to estimate the value of \( x \), and due to the nonlinearity in the equations for \( \hat{t}_i \), \( N > 3 \) sensors are actually required for the estimate of \( x \) to be uniquely defined (i.e. to eliminate any ambiguity in the solution for \( x \)).

Theorem 3. Consider the time-of-arrival-based localization problem, i.e. Problem 3, with \( \phi_i, \forall i \in \{1, \ldots, N\} \) denoting the angular positions of the sensors. Then, the determinant \( \text{det}(I_r(x)) \) of the Fisher information matrix \( I_r(x) \) for time-of-arrival based localization is given by

\[
\text{det}(I_r(x)) = \frac{1}{\sigma_t^6} \left[ \frac{N^3}{4} - \frac{N}{4} \left( \sum_{i=1}^{N} \cos(2\phi_i) \right)^2 \right]
\]

(20)

where \( \sigma_r^2 \) is the common variance of each timing measurement, i.e. \( \tilde{t}_i \sim \mathcal{N}(t_i, \sigma_r^2) \).

Proof. The proof follows from construction. Firstly, write (19) as

\[
I_r(x) = \frac{1}{\sigma_t^2} \left[ \sum_{i=1}^{N} \begin{bmatrix}
\frac{1}{2} - \cos(2\phi_i) & \frac{\sin(2\phi_i)}{2} & \frac{\sin(\phi_i)}{2}
\end{bmatrix} \begin{bmatrix}
\sin(\phi_i) \\
\sin(2\phi_i) \\
\cos(\phi_i)
\end{bmatrix} \right]
\]

(21)

Taking the determinant of the \( 3 \times 3 \) matrix and rearranging leads to (20).

5.1. Time-of-arrival localization geometry with \( N \) sensors

An unbiased and efficient estimate of \( x \) (or, probably more commonly, \( p \)) will, in general, achieve the smallest error variance.
Horn

Consider the time-of-arrival-based localization problem. One particular optimalsensor-target configuration is simultaneously satisfied with \(N\) positionsof the sensors. Then, the Fisher information determinant is upper-bounded by \(\frac{N^3}{4\sigma_i^2}\) which is achieved if and only if

\[
\begin{align*}
\sum_{i=1}^{N} \sin(\phi_i(x)) &= 0 \quad \text{and} \quad \sum_{i=1}^{N} \sin(2\phi_i(x)) = 0 \\
\sum_{i=1}^{N} \cos(\phi_i(x)) &= 0 \quad \text{and} \quad \sum_{i=1}^{N} \cos(2\phi_i(x)) = 0
\end{align*}
\]

are simultaneously satisfied with \(N \geq 3\).

**Proof.** Firstly, write (19) as

\[
J_i(x) = \frac{1}{\sigma_i^2} \begin{bmatrix}
\sum_{i=1}^{N} \cos(2\phi_i) \\
\sum_{i=1}^{N} \sin(2\phi_i) \\
\sum_{i=1}^{N} \sin(\phi_i) \\
\sum_{i=1}^{N} \cos(\phi_i)
\end{bmatrix}
\]

and bounded according to

\[
0 < \frac{1}{\sigma_i^2} A - \frac{1}{N\sigma_i^2} b b^T \leq \frac{1}{\sigma_i^2} A.
\]

It now follows that

\[
\det (J_i(x)) = \frac{N}{\sigma_i^2} \det \left( \frac{1}{\sigma_i^2} A - \frac{1}{N\sigma_i^2} b b^T \right) \leq \frac{N}{\sigma_i^2} \det \left( \frac{1}{\sigma_i^2} A - \frac{1}{N\sigma_i^2} b b^T \right)
\]

and bounded to

\[
0 < \frac{1}{\sigma_i^2} A - \frac{1}{N\sigma_i^2} b b^T \leq \frac{1}{\sigma_i^2} A.
\]

Angular sensor positions which maximize the determinant, i.e. which satisfy (22), generate an optimal sensor-target geometry for time-of-arrival-based localization.

**Proposition 3.** One particular optimal sensor-target configuration for the unbiased and efficient estimation of the target location \(p\) (or more generally the event location \(x\)) occurs when \(N \geq 3\) and

\[
\theta_{ij} = \theta_{ji} = \frac{2}{N} \pi
\]

for all adjacent sensor pairs \(i, j \in \{1, \ldots, N \geq 3\} \) with \(|j - i| = 1\) or \(|j - i| = N - 1\).

**Proof.** With no loss of generality let \(\phi_1 = 0\) and \(\phi_j \geq \phi_1\) when \(j > i\) such that the condition (27) of Proposition 3 implies \(\phi_j = \frac{j}{2} \pi + \phi_1\) for all \(j \in \{2, \ldots, N\}\) with \(|j - i| = 1\) or \(|j - i| = N - 1\). Now if \(N\) is odd and \(M = (N + 1)/2\) then \(\sin(\phi_k) = -\sin(\phi_i)\) where \(k \in \{2, \ldots, M\}\) and \(l = N + 2 - k\) and \(\sum_{i=1}^{N} \sin(\phi_i(x)) = -1\) which implies \(\sum_{i=1}^{N} \cos(\phi_i(x)) = 0\). If \(N\) is even...
and $M = N/2$ then $\sin(\phi_k) = -\sin(\phi_l)$ and $\cos(\phi_k) = -\cos(\phi_l)$ for all $k, l \in \{1, \ldots, M\}$ which similarly implies $\sum_{i=1}^{N} \sin(\phi_i) = 0$ and $\sum_{i=1}^{N} \cos(\phi_i) = 0$. Similar reasoning, e.g. see Proposition 2, will show that $\phi_l = \frac{2}{\pi} \phi + \phi_i$ for all $i \in \{2, \ldots, N\}$ with $j - i = 1$ or $j = N - 1$ satisfies $\sum_{i=1}^{N} \sin(2\phi_i) = 0$ and $\sum_{i=1}^{N} \cos(2\phi_i) = 0$. This proves the sufficiency of (27) in maximizing the determinant, i.e. in satisfying (22) of Theorem 4. $\Box$

**Theorem 4** and Proposition 3 provide conditions which are independent of the individual sensor-target ranges. Consider $N \geq 3$ time-of-arrival sensors tasked with localizing a single target. Denote the set of $N \geq 3$ sensors by $\mathcal{V}$. Now assume that $\mathcal{V}$ can be partitioned into some arbitrary number $M$ of subsets $\mathcal{B}_i$ such that $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ and $|\mathcal{B}_i| \geq 3, \forall i \in \{1, \ldots, M\}$ with $i \neq j$.

**Corollary 4.** Given an arbitrary number $N \geq 3$ of time-of-arrival sensors, then an optimal $N$ sensor configuration can be obtained by arranging all subsets of sensors $\mathcal{B}_i, \forall i \in \{1, \ldots, M\}$ such that condition (22) of Theorem 4 is satisfied independently for those sensors in $\mathcal{B}_i, \forall i \in \{1, \ldots, M\}$.

For $3 \leq N \leq 5$, the optimal sensor-target angular configuration for time-of-arrival-based localization is unique in the sense of Definition 2 and is given by Proposition 3. For $N > 5$, there exists an infinite number of optimal sensor configurations.

**5.2. Time-of-arrival localization with three sensors and four sensors**

With $N = 3$ sensors and three time-of-arrival measurements (or two time-difference measurements) it is possible that an ambiguity in the estimate of the target location $p$ exists since two hyperbola branches can intersect in more than one location. In the case of a localization ambiguity it might still be possible to localize given additional (a priori) knowledge of a region containing the target’s position.

Let $A = \phi_1(p) - \phi_2(p)$ and $B = \phi_2(p) - \phi_3(p)$; then for $N = 3$ and $\sigma^2 = 1$ the determinant (20) is equivalent to $\det(I_p(x)) = (\sin(A) - \sin(B) - \sin(A - B))^2$. Now it is possible to plot the value of the determinant (20) with $\sigma^2 = 1$ directly over the possible values of $A, B \in [0, 2\pi]$. The surface and contour plots are given in Fig. 5.

Fig. 5 illustrates that for $A, B \in [0, 2\pi]$, the value of the determinant is maximum when $A = \frac{2}{\pi} \pi$ and $B = \frac{3}{\pi} \pi$ or when $A = \frac{3}{\pi} \pi$ and $B = \frac{2}{\pi} \pi$. However, recalling the earlier standing assumption on sensor indices which implies $\phi_1 \geq \phi_2 \geq \phi_3$, then the only consistent maximum is $A = \frac{2}{\pi} \pi$ and $B = \frac{3}{\pi} \pi$.

Now consider a special case where the sensors form a unit equilateral triangle with $s_1 = [-1/2, 0]$, $s_2 = [1/2, 1/2]^T$, $s_3 = [1/2, -1/2]^T$ and $s_4 = [-1/2, -1/2]^T$. The sensors are arranged to form a unit square centered at the origin. The value of the Fisher information determinant is shown in Fig. 7 for target coordinates obeying $x_p \in [-5/4, 5/4]$ and $y_p \in [-5/4, 5/4]$.

From Fig. 7, it is clear that the optimal geometry is achieved when the target is located at the origin (or at the center of the unit square) as expected. In addition, it is reasonable to assume that good localization performance can be achieved when the target is located anywhere inside the unit square.

**6. The geometry of bearing-only based localization**

In this section we examine the optimal angular geometry for bearing-only localization given arbitrary sensor-target ranges. We deviate from a common assumption in the literature which considers a uniform angular spacing around the target to be a desirable configuration. Based on Bishop et al. (2007) and Dogancay and Hnam (2008) we provide a formal analysis of the optimal geometry and show how this geometry changes significantly with changes in (relative) sensor-target range values.

Given the measurement vector $\Phi(p)$, the entire Fisher information matrix for bearing-only localization is given by (12) with $\Phi(p) = z(w)$. Correspondingly,

\[
\mathbf{I}_\theta(p) = \frac{1}{\sigma^2} \sum_{i=1}^{N} \left[ \frac{\cos^2(\phi_i(p))}{\sin^2(\phi_i(p))} - \frac{\sin(2\phi_i(p))}{2} \right]
\]

where $i$ indexes the bearing measurement from the $i$th sensor. Here, $\det(I_\theta(p))$ is used to analyze the relative sensor-emitter geometry for bearing-only localization.
Theorem 5. Consider the bearing-only localization problem, i.e. Problem 1, with \( \phi_i, \forall i \in \{1, \ldots, N\} \) denoting the angular positions of the sensors. Then, the following are equivalent expressions for the Fisher information determinant for bearing-only localization:

(i) \[ \det(I_\phi(p)) = \frac{1}{\sigma_\phi^2} \sum_i \frac{\sin^2(\phi_i - \phi_j)}{r_i^2 r_j^2}, \quad j \neq i \tag{29} \]

(ii) \[ \det(I_\phi(p)) = \frac{1}{4\sigma_\phi^2} \left[ \left( \sum_{i=1}^{N} \frac{1}{r_i^2} \right)^2 - \left( \sum_{i=1}^{N} \frac{\cos(2\phi_i)}{r_i^2} \right)^2 - \left( \sum_{i=1}^{N} \frac{\sin(2\phi_i)}{r_i^2} \right)^2 \right] \tag{30} \]

where \( \delta = \{(i, j)\} \) is the set of all combinations of \( i \) and \( j \) with \( i, j \in \{1, \ldots, N\} \) and \( j > i \), implying that \( |\delta| = \binom{N}{2} \).

Proof. Note that \( R_\phi = \sigma_\phi^2 I \) and thus \( R_\phi^{-1} = 1/\sigma_\phi^2 I \). Let \( G = \nabla_\phi \Phi(p) \) so that from (12) we also find

\[ \det(I_\phi(p)) = \frac{1}{\sigma_\phi^2} \det(G^T G) = \frac{1}{\sigma_\phi^2} \sum_{m=1}^{\binom{N}{2}} \det(G_m)^2 \tag{31} \]

using the Cauchy-Binet formula; see e.g. Horn and Johnson (1985). Here \( G_m \) is a \( 2 \times 2 \) minor of \( G \) taken from the set of minors indexed by \( \delta = \{(i, j)\} \). Thus for some \( i < j \), a particular \( G_m \) is given as

\[ G_m = \begin{bmatrix} \frac{1}{r_i} & \frac{1}{r_j} \\ \frac{1}{r_i} & \frac{1}{r_j} \end{bmatrix} \cos(\phi_i - \phi_j) \]

and the expression for the determinant given by (29) follows easily by trigonometry. For part (ii) we write (28) as

\[ I_\phi(x) = \frac{1}{\sigma_\phi^2} \left[ \sum_{i=1}^{N} \frac{1 + \cos(2\phi_i)}{2r_i^2} - \sum_{i=1}^{N} \frac{\sin(2\phi_i)}{2r_i^2} \right] \]

Taking the determinant of this matrix and rearranging leads easily to (30); see also Dogancay and Hmam (2008) for details.

For arbitrary, but fixed, sensor-target ranges \( r_i > 0 \), the angular sensor positions which maximize the determinant expressions given in Theorem 5 generate an optimal sensor-target geometry for bearing-only localization. The following Theorem concerns the determinant optimization problem associated with \( N \) sensors at fixed but arbitrary sensor-target ranges, but with adjustable angular positions. The proof follows from Theorem 5.

Theorem 6. Consider the bearing-only localization problem, i.e. Problem 1, where \( \phi_i, \forall i \in \{1, \ldots, N\} \) denotes the angular positions of the sensors. Let \( r_i = ||p - s_i|| \) be arbitrary but fixed for all \( i \in \{1, \ldots, N\} \). The following optimization problems are equivalent:
Consider the bearing-only localization problem, i.e. Theorem 6

Fig. 8 Reflecting a sensor about the emitter position, i.e. moving a sensor from s to 2p − s, does not affect the value of the Fisher information determinant for bearing-only localization.

Therefore, an optimal sensor angular configuration is not generally unique for given arbitrary sensor-emitter ranges. Unlike in the previous two sections, we will first examine the optimality of the relative sensor-target geometry for two special cases, i.e. for bearing-only localization with N = 2 and N = 3 sensors.

6.1. The bearing-only localization geometry with two sensors and three sensors

The following result completely characterizes optimal bearing-only sensor-target angular geometry for the special case involving N = 2 sensors.

Proposition 4. Consider the bearing-only localization problem, i.e. Problem 1, with N = 2 sensors and with θi, j denoting the angle subtended at the target by sensors 1 and 2. Let r1 = ∥p − s∥ be arbitrary but fixed for all i ∈ {1, 2}. Then the optimal two-sensor angular geometric arrangement for bearing-only localization occurs when θ12 = θ21 = π/2.

The optimality of the bearing-only localization geometry for N = 2 is special, when compared to those cases involving N > 2 sensors. The reason for this distinction is that the optimal angular configuration of the N = 2 bearing sensors is independent of the individual sensor-target range values. Of course, decreasing r1 and/or r2 will increase the relative optimality.

The following result completely characterizes optimal bearing-only sensor-target angular geometry with N = 3 sensors.

Theorem 7. Consider the bearing-only localization problem, i.e. Problem 1, with N = 3 sensors and where the angle subtended at the target by two sensors i and j is denoted by θi, j. Let r1 be arbitrary but fixed ∀i ∈ {1, 2, 3}. Then, every optimal angular separation θ12, θ13 and θ23 can be obtained by first solving

\[
\begin{align*}
\theta_{12} &= \frac{1}{2} \arccos \left( \frac{r_1^2 + r_2^2 - r_3^2}{2r_1r_2} \right) \\
\theta_{13} &= \frac{1}{2} \arccos \left( \frac{r_1^2 + r_3^2 - r_2^2}{2r_1r_3} \right) \\
\theta_{23} &= 2\pi - \theta_{12} - \theta_{13}
\end{align*}
\]

when the \arccos(·) arguments are not greater than one, and then by an application of Corollary 5 on the derived sensor positions. Now assume that both \arccos(·) arguments are greater than one. Then, the Fisher information determinant for bearing-only localization is maximized when

\[
\begin{align*}
\theta_{12} &= \frac{\pi}{2} & \text{if } r_1 < r_2 \text{ or } r_2 < r_3, \text{ i.e. } i, j \in \{1, 2, 3\} \setminus \{k\} \\
\theta_{13} &= 0 \text{ or } \pi, \text{ otherwise}
\end{align*}
\]

which includes sensor reflections as per Corollary 5.

Proof. The proof of Theorem 7 is long and appears in full in Bishop et al. (2007). □

In general, Corollary 5 implies a maximum of four different optimal angular configurations can be obtained from the initial solution (38) by reflecting sensors about the target.

Consider the following example which shows the change in optimal angular geometry in terms of the relative changes in sensor-target ranges. The illustration is given in Fig. 8.

Fig. 8 shows the variation of the optimal geometry as the range r1 changes from r1 ≪ r2 = r3 to r1 ≫ r2 = r3 while r2 and r3 are held constant. The particular ranges and θ12 (in degrees) are given in the figure titles and the caption provides an explanation of the changes in the geometry illustrated.

6.2. The bearing-only localization geometry with N sensors

Clearly, the optimization problems given in Theorem 6 are difficult to solve directly for an arbitrary number N ≥ 3 of sensors and arbitrary sensor-target ranges. The next two results form the basis for characterizing the optimal bearing-only localization geometries with N ≥ 3 sensors and arbitrary but fixed sensor-target ranges.

Theorem 8. Consider the bearing-only localization problem, i.e. Problem 1, with φi, Vi ∈ {1, . . . , N} denoting the angular positions of the sensors. Let r1 = ∥p − s∥ be arbitrary but fixed for all i ∈ {1, . . . , N > 2}. The Fisher information determinant given in Theorem 5 is upper-bounded by ∑N i=1 1 r2 i. The upper-bound is achieved if and only if the following conditions

\[
\begin{align*}
\sum_{i=1}^{N} \cos (2\phi_i) &= 0 \quad \text{and} \quad \sum_{i=1}^{N} \sin (2\phi_i) = 0
\end{align*}
\]

are satisfied by some φi, ∀i ∈ {1, . . . , N}. Furthermore, values of φi, ∀i ∈ {1, 2, . . . , N}, solving (40) can be found if and only if the following condition

\[
\frac{1}{r_j} \leq \sum_{i=1, i\neq j}^{N} \left( \frac{1}{r_i} \right)
\]

holds for all j ∈ {1, . . . , N > 2}.

Proof. Clearly, by Eq. (30), the upper bound is as stated and satisfaction of (40) is necessary and sufficient for the attainment of the upper bound. It remains to consider when there exist φi, ∀i ∈ {1, 2, . . . , N}, allowing the satisfaction of (40). Eq. (40) can be rewritten as

\[
\sum_{i=1}^{N} \frac{1}{r_i} \begin{bmatrix} \cos (2\phi_i) \\ \sin (2\phi_i) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

and suppose that the inequality condition (41) does not hold. Then, clearly, (42) has no solution since one term on the left hand side will have a norm greater than the summed norms of all the other vectors. Hence, the necessity of (41) is established. The sufficiency of (41) is trivial to check. □

The following result characterizes the optimal geometry for bearing-only localization when the key inequality condition (41) of Theorem 8 does not hold.
Theorem 9. Consider the bearing-only localization problem, i.e. Problem 1, where $\phi_i$, $\forall i \in \{1, \ldots, N\}$ denotes the angular positions of the sensors, let $r_i = \|p - s_i\|$ be arbitrary but fixed for all $i \in \{1, \ldots, N\} > 2$. If

$$\frac{1}{r_j^2} > \sum_{i=1, i \neq j}^{N} \frac{1}{r_i^2} \quad (43)$$

holds for some $j \in \{1, \ldots, N\}$, then the determinant given in Theorem 5 is upper-bounded by $\frac{1}{r_j^2} \sum_{i=1, i \neq j}^{N} \frac{1}{r_i^2}$ and the upper-bound is achieved under (43) if and only if

$$\phi_j(p) = \phi_j(p) \pm \frac{\pi}{2}$$

for all $i \in \{1, \ldots, N\} \setminus \{j\}$. This implies $\theta_j = \frac{\pi}{2}$, for all $i \in \{1, \ldots, N\} \setminus \{j\}$ and $\theta_k = c\pi$, where $c \in \{0, 1\}$ for all $i, k \in \{1, \ldots, N\} \setminus \{j\}$.

Proof. Firstly, the upper-bound will be derived via construction for range values satisfying (43). Refer back to Eq. (42) and note that under the assumption that (43) holds, the vector on the left hand side necessarily has a minimum norm equal to the difference

$$\frac{1}{r_j^2} - \sum_{i=1, i \neq j}^{N} \frac{1}{r_i^2}. \quad (45)$$

Putting this value (45) into the determinant given in Theorem 5 part (ii) leads to

$$\det(I_p(p)) = \frac{1}{4\sin^2 \phi_0} \left[ \sum_{i=1, i \neq j}^{N} \frac{1}{r_i^2} \right]^2 - \left( \frac{1}{r_j^2} - \sum_{i=1, i \neq j}^{N} \frac{1}{r_i^2} \right)^2 \quad (46)$$

for the same $j$ and for all $i \in \{1, \ldots, N\} \setminus \{j\}$. Rearranging the Eq. (46) leads to the upper-bound. Note that the upper-bound under the condition (43) has been explicitly constructed and it remains to show how this upper-bound can be achieved.

With no loss of generality, the condition (44) can be subsumed by the special case where $\phi_j = \frac{\pi}{2}$ for the same $j \in \{1, \ldots, N\}$ in (45) and $\phi_i = 0, \forall i \in \{1, \ldots, N\} \setminus \{j\}$. This can be achieved by a global rotation of the coordinate system. Now note that $\sin(2\phi_j) = \sin(2\phi_i) = 0$ and $\cos(2\phi_j) = 1$ and $\cos(2\phi_i) = -1$ for those values of $i$ and $j$ specified previously. Putting these terms into the determinant (30) given in Theorem 5 part (ii) leads directly to (46) and thus proves the sufficiency of (44). The necessity of (44) follows easily when $N > 2$. □
Theorem 9 illustrates an interesting characteristic of the optimal bearing-only localization geometry. If (41) is not satisfied, i.e., (43) is satisfied for some sensor $j$, then that sensor $j$ is much closer to the target relative to all the other sensors. Moreover, that same sensor $j$ should be placed at a right angle to all other sensors, i.e., the other sensors should be collinear, and that sensor configuration is unique in the sense of Definition 3.

Alternatively, when (43) in Theorem 9 does not hold, then the condition (40) given in Theorem 8 leads to the optimal sensor configuration. Finding values of $\phi_i, \forall i \in \{1, 2, \ldots, N\}$, that solve (40) is not straightforward, in general, for an arbitrary number of sensors $N$ and for fixed but arbitrary ranges $r_i, \forall i \in \{1, 2, \ldots, N\}$. Nevertheless, special cases can be considered explicitly with $N$ sensors (Dogancay & Hmam, 2008). The following result completely characterizes the optimal bearing-only localization geometry for $N$ sensors and equal sensor-target ranges.

**Theorem 10.** Consider the bearing-only localization problem, i.e., Problem 1, with $\phi_i, \forall i \in \{1, \ldots, N\}$ denoting the angular positions of the sensors. Given $r_i = r_j, \forall i, j \in \{1, \ldots, N\}$, then the values for $\phi_i, \forall i \in \{1, \ldots, N\}$ that simultaneously solve the following equations

\[
\sum_{i=1}^{N} \sin(2\phi_i(p)) = 0 \quad \text{and} \quad \sum_{i=1}^{N} \cos(2\phi_i(p)) = 0 \quad (47)
\]

maximize the Fisher information determinant given in Theorem 5 for bearing-only localization.

**Proof.** The proof of Theorem 10 can be derived directly from the expression in Theorem 6 part (iii). However, we consider the following appealing conclusion. With no loss of generality let $r_i = r_j = 1, \forall i, j$ and note that, for equal range values, the Fisher information determinant for bearing-only localization given in Theorem 5 is identical to the same for range-only localization given in Theorem 1. Then, Theorem 10 can be deduced directly from Theorem 2. See also Bishop et al. (2007) and Dogancay and Hmam (2008). □

Theorem 10 implies the following useful proposition which interestingly is analogous to Proposition 2.

**Proposition 5.** Consider the bearing-only localization problem, i.e., Problem 1, where the angle subtended at the target by two sensors $i$ and $j$ is denoted by $\theta_{ij}$. Assume that $r_i = r_j, \forall i, j \in \{1, \ldots, N\}$. Some particular optimal sensor angular geometries for bearing-only localization with $N > 2$ can be obtained by first letting

\[
\theta_{ij} = \frac{2}{N} \pi \quad (48)
\]

for all adjacent sensor pairs $i, j \in \{1, \ldots, N \geq 3\}$ with $|j - i| = 1$ or $|j - i| = N - 1$, and then by a possible application of Corollary 5 on (48).

Proposition 5 implies the following corollary which summarizes the two intuitively appealing special cases.

**Corollary 6.** Assume that $r_i = r_j, \forall i, j \in \{1, \ldots, N\}$ and consider the angles $\theta_{ij} = \theta_{ji}$ subtended at the target by two adjacent sensors where adjacency implies $|j - i| = 1$ or $|j - i| = N - 1$ with $i, j \in \{1, \ldots, N \geq 3\}$. Then $\theta_{ij} = \theta_{ji} = \frac{2}{N} \pi$ and $\theta_{ij} = \theta_{ji} = \frac{1}{N} \pi$ are two separate optimal angular configurations of the $N$ sensors relative to the single target.

Again, if (41) is not satisfied, then one sensor is much closer to the target compared to the other sensors and should be placed at a right angle to all other sensors. The optimal angular geometry that results is unique in the sense of Definition 3.

On the other hand, given arbitrary range values $r_i, \forall i$ which satisfy (41), then the optimal sensor angular positions which solve (40) in Theorem 8 do not generate a unique sensor-target configuration when $N > 2$. Moreover, finding values of $\phi_i, \forall i \in \{1, 2, \ldots, N\}$, that solve (40) is not generally straightforward. We have already explored the special cases involving $N = 3$ sensors and $N = 4$ sensors with equal sensor-target ranges. In addition, we note the following corollary which stems directly from Theorem 8 and which can be used to simplify the optimal sensor placement problem.

**Corollary 7.** Consider $N \geq 4$ bearing-only sensors tasked at localizing a single target and assume that $r_i$ is arbitrary but fixed, $\forall i \in \{1, \ldots, N \geq 4\}$. Denote the set of $N \geq 4$ sensors by $V$ and assume further that (41) is satisfied for the entire set of sensors $i \in V$. Now assume that $V$ can be split into some arbitrary number $M$ of subsets $B_i$ such that $B_i \cap B_j = \emptyset$ and $|B_i| \geq 2, \forall i, j \in \{1, \ldots, M\}$ with $i \neq j$. Moreover, the assumption (41) is satisfied by the subset group of range values for the sensors in $B_i$, then an optimal $N$ sensor configuration can be obtained by placing all subsets of sensors $B_i, \forall i \in \{1, \ldots, M\}$ in optimal angular positions as specified by (40) in Theorem 8.

Note that Corollary 7 can be particularly useful in forming optimal sensor configurations with an arbitrary number $N \geq 4$ of sensors. We have assumed that (41) is satisfied for the entire set of sensors $i \in V$. However, a second key assumption that must be satisfied in order to use Corollary 7 is that (41) must be satisfied by the subset group of range values for the sensors in each set $B_i, \forall i \in \{1, \ldots, M\}$. Apart from these two requirements, the choice of sensor subsets is completely arbitrary so long as the cardinality of each subset is at least two.

If Corollary 7 can be employed, then an infinite number of sensor configurations can be obtained by rotating any sensor subset $B_i$ relative to (and independent of) any other sensor subset $B_i$ with $i \neq j$. Hence, for $N \geq 4$ bearing-only sensors there can exist an infinite number of optimal sensor configurations unless (41) does not hold for all range values.

However, the satisfaction of (41) for all $N$ sensors does not necessarily imply an infinite number of optimal configurations since a partition of the sensors which satisfies all of the conditions stated in Corollary 7 might not be possible. In fact, if there are $N \geq 4$ sensors for which Corollary 7 cannot be used but for which (41) is satisfied for the entire set of sensors $i \in V$, then the maximum number of unique optimal angular configurations is equal to $N + 1$ and is obtained by repeated applications of Corollary 5 on one (initial) particular optimal angular configuration (as defined by (40) in Theorem 8).

Consider now the special case of $N \geq 4$ sensors with equal target ranges. Every mutually exclusive subset of sensors $B_i$ with $|B_i| \geq 2$ will then satisfy the range condition (41). In this special case, it makes sense to partition the sensors into groups of two and three and then optimally place the sensor subsets as illustrated in the previous two subsections, i.e. according Propositions 4 and 5. Any subset $B_i$ with $|B_i| \geq 2$ can be rotated independently and arbitrarily relative to any other subset.

Now imagine that we have $N = 4$ or $N = 5$ sensors with arbitrarily fixed range values satisfying (41) but with $r_i \neq r_j, \forall i, j \in V$. Then clearly Corollary 7 cannot be applied since no partition of the sensors is possible satisfying the required partition conditions stated in Corollary 7. However, in practice (and for $N > 5$), the likelihood that Corollary 7 is applicable increases as the number of sensors increases.

7. Discussion on the optimality analysis of sensor-target localization geometries

The results in this paper assume an unbiased and efficient estimator is used to estimate the target location. However, the
estimation technique used in practice is likely to be biased (Gavish & Weiss, 1992; Nardone et al., 2004). For example, even the well-known maximum likelihood localization techniques are only asymptotically unbiased and efficient, i.e., require the number of sensors to approach infinity. However, inference algorithms are actually designed with unbiasedness in mind and with a goal of achieving the Cramer–Rao lower bound. The Cramer–Rao bound for unbiased estimators (i.e., used in this paper) is therefore an interesting benchmark with which intuitively pleasing results on sensor placement have been derived. However, these results can only be considered a guide for practical sensor placement with an accuracy dependent on the bias and efficiency characteristics of the particular estimator employed. It is well known that the variance (or mean-square-error) of an estimate can sometimes be made smaller at the expense of increasing the bias (Eldar, 2006). The work of Eldar (2004) and Hero, Fessler, and Usman (1996) explores the concept of bias-variance trade-offs in estimation. In Eldar (2004) and Van Trees (1968) a biased Cramer–Rao inequality and in Eldar (2004) and Hero et al. (1996) a uniform Cramer–Rao inequality are developed and can be used to study this so-called bias-variance trade-off. In practice, we are often limited by the choice of estimation algorithm with the maximum likelihood algorithm being statistically optimal and asymptotically unbiased and efficient. Given a specific estimator (or possibly a class of estimator), then the results of Eldar (2004), Eldar et al. (2006) and Van Trees (1968) can also be used to extend the results given here to practical estimation algorithms such as maximum likelihood. Moreover, the results of Eldar (2006) might potentially be useful in designing localization algorithms and relative sensor placement schemes which permit estimation with a variance below the unbiased Cramer–Rao bound. These problems are yet to be rigorously addressed within the localization literature.

Throughout this paper we have assumed the measurement error variance is independent of the true measurement value. In the case of bearing-only localization this is a typical assumption. For range-only localization it is sometimes assumed that the standard deviation is multiplied by a percentage of the true range value. If this is the case, then the Fisher information matrix for range-only localization clearly becomes very similar to the Fisher information matrix for bearing-only localization (in fact the only difference is the percentage term and the symbol associated with the variance, i.e. r instead of \( \phi \)). As such, the optimal geometry for range-only localization geometry changes to resemble that of bearing-only localization (which intuitively makes sense for the case involving range measurement noise conditioned on a percentage of the true range). Moreover, since this is the most natural way to include a dependency on the true range by the variance of the range measurement, this paper actually accounts for this scenario implicitly. For time-of-arrival based localization it is less likely that the measurement noise is dependent on the true measurement value due to the nature of the time measurements involved. However, one might also expect that if the variance of the noise associated with the measurement of \( t_i \) was dependent on \( t_i \), being perhaps a function of the sensor-target range, then the optimal sensor placement would be dependent on the ranges in a similar way to the range-only localization problem; see Bishop et al. (2007).

8. Conclusion

A direct and rigorous characterization of the relative sensor-target geometry was given in this paper for localization with range-only, bearing-only and time-of-arrival measurements. The characterizations are given in terms of the potential localization performance of unbiased and efficient estimators. The contributions provided in this paper have explicit and measurable connection between the sensor-target geometry and the chosen measure of localization performance; i.e. the area of the uncertainty ellipse. A number of necessary and sufficient conditions on the sensor-target angular positions were given, which when satisfied meant in the minimization of an unbiased and efficient target estimate’s uncertainty ellipse area (for the specific measurement technology considered). It is also shown how the optimal sensor configuration is not generally unique.

References


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