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Stability Test for Two-Dimensional Recursive Filters

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Abstract—For deciding the stability of a two-dimensional filter, it is of interest to determine whether or not a prescribed polynomial in the variables z_1 and z_2 is nonzero in the region $|z_1| \leqslant 1 \cap |z_2| \leqslant 1$. A new procedure for testing for this property is given, which does not involve the use of bilinear tranformations. Key parts of the test involve the construction of a Schur–Cohn matrix and the checking for positivity on the unit circle of a set of self-inversive polynomials.

I. Introduction

A two-dimensional digital recursive linear filter can be defined by its two-dimensional z-transform

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$$G(z_1, z_2) = \frac{\sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} z_1^i z_2^j}{\sum_{i=0}^{p} \sum_{j=0}^{q} b_{ij} z_1^i z_2^j} \tag{1}$$

where the quantities a_{ij} and b_{ij} are real constants. The filter is stable if and only if (see [1]–[3])

$$B(z_1, z_2) = \sum_{i=0}^{p} \sum_{j=0}^{q} b_{ij} z_1^i z_2^j \neq 0, \qquad |z_1| \leqslant 1 \cap |z_2| \leqslant 1. \tag{2}$$

The object of this paper is to present a procedure for checking the stability condition; the procedure is believed to be simpler than those of [2] and [3]. In fact, the procedure of [2] is not finite in the sense that the procedure requires the construction of a theoretically infinite number of mappings. The procedure of [3], though finite, requires application of two bilinear transformations to pose the problem in a form solved by Ansell [4]. In essence, Ansell's main contribution is to couple the use of a Hermite test for checking stability [5] with a series of Sturm tests [6] checking positivity.

Our procedure, like that of [3], is finite. We require no bilinear transformation and we replace the Hermite test component of the main part of Ansell's procedure by a Schur–Cohn matrix test [7]–[9]. Then we allow either a series of Sturm tests, or, what turns out to be equivalent, a series of tests for establishing the root distribution of a polynomial.

The plan of the paper is as follows. In Section II, we recall the initial simplification to the stability problems made by Huang [3] (which is equivalent to a similar simplification made by Ansell [4] for a related problem). Then we indicate the structure of our test. In Sections III–V, we indicate the test in detail. Section VI contains concluding remarks.

II. Outline Solution of the Problem

Huang's simplification of the stability problem rests on the following observation: $B(z_1, z_2) \neq 0$ for $|z_1| \leqslant 1 \cap |z_2| \leqslant 1$ if and only if the following two conditions hold:

$$B(z_1, 0) \neq 0, \qquad |z_1| \leqslant 1 \tag{3}$$

$$B(z_1, z_2) \neq 0, \qquad |z_1| = 1, |z_2| \leqslant 1. \tag{4}$$

Let us now indicate in outline how these conditions may be checked.

First, the checking of (3) is straightforward because $B(z_1, 0)$ is a single variable polynomial and there are a number of tests for determining whether or not its zeros all lie outside the unit circle. One group of tests relies on replacement of $B(z_1, 0)$ by another polynomial through a bilinear transformation which has to be checked for Hurwitz character. (A polynomial is Hurwitz if all its zeros have negative real parts.) There are, of course, many tests for checking the Hurwitz property (see, e.g., [5]). We shall omit further discussion of this approach, this being the philosophy used by Huang.

A second test involves use of the Schur-Cohn matrix [7]–[10]; this matrix is square, Hermitian, of size equal to the degree of $B(z_1, 0)$, and with elements which are simple functions of the coefficients of $B(z_1, 0)$. The matrix is negative definite if and only if $B(z_1, 0)$ has all its zeros in $|z_1| > 1$. Of course, the negative definiteness can be established by examining the signs of the leading principal minors of the matrix.

Other tests may be found in [9]–[12]. These tests include a computationally efficient one involving the recursive construction of a finite set of polynomials, each member of the set having lower degree than the preceding, and with $B(z_1, 0)$ being the first member of the set. Examination of some of the coefficients of these polynomials quickly yields the information as to whether (3) holds.

To avoid burdening the reader with too much detail, we shall restrict consideration in this paper to use of the Schur-Cohn test and the test based on recursive construction of a set of polynomials. We remark that although some simplification in the computations involved in applying the two tests to real polynomials is possible (see [9], [12], and [13]), we avoid stating these simplifications, again to avoid giving too much detail. The tests are given in Section III, together with examples illustrating their application to the checking of condition (3).

Now we comment on how condition (4) is to be checked. We omit further discussion of the approach of Huang, based on a bilinear transformation in both variables. Broadly, the checking falls into two distinct phases. First, we apply a Schur-Cohn test, after a fashion. Second, we check the positivity of a number of polynomials on $|z_1| = 1$.

Application of the Schur-Cohn test is described in Section IV. The idea is to think of z_1 as a parameter, so that $B(z_1, z_2)$ is a polynomial in z_2, whose coefficients are functions of a parameter z_1. For stability, the polynomial cannot be zero inside $|z_2| \leqslant 1$; this means that the associated Schur-Cohn matrix must be negative definite.

We have already noted that the entries of the Schur-Cohn associated with a polynomial are simple functions of the polynomial coefficients. Accordingly, here the entries of the Schur-Cohn matrix become simple functions of z_1. In fact, the entries are polynomial in z_1 and/or z_1^*, as we shall see.

To check negative definiteness of the Schur-Cohn matrix requires determination of the principal minors, which must have certain signs. These minors are again polynomial in z_1 and z_1^* and are real because the Schur-Cohn matrix is Hermitian.

Accordingly, the condition for (4) to hold becomes one requiring the sign definiteness on $|z_1| = 1$ of a set of functions which take real values and are polynomial in z_1 and z_1^*. Bearing in mind that on $|z_1| = 1$, we have $z_1^* = z_1^{-1}$, it follows that the functions are of the form

$$f_i(z_1) = \sum_{j=1}^{N_i} c_j z_1^j + 2c_0 + \sum_{j=1}^{N_i'} d_j z_1^{-j} \tag{5}$$

on $|z_1| = 1$. Because $f_i(z_1)$ is real for all $|z_1| = 1$, one must have c_0 real, $c_j = d_j^*$, and $N_i = N_i'$. In actual fact, because the original $B(z_1, z_2)$ has real coefficients, one finds that c_j is real so that

$$f_i(z_1) = \sum_{j=1}^{N_i} (c_j z_1^j + c_j z_1^{-j}) + 2c_0 = \sum_{j=0}^{N_i} c_j (z_1^j + z_1^{-j}) \tag{6}$$

on $|z_1| = 1$. One still requires a test for this polynomial to be positive or negative as required for all z_1 on $|z_1| = 1$, and this is the second phase in the checking of condition (4).

Before describing this second phase, we deal with one further issue associated with the first phase. Since the zero distribution of a one variable polynomial can be obtained by constructing a sequence of polynomials, one might expect this procedure to be possible for $B(z_1, z_2)$, treating z_1 as a parameter. Indeed, this is so; however, organization of the calculations appears more complex than organization of the calculations based on use of the Schur-Cohn matrix, at least for other than very simple $B(z_1, z_2)$. Therefore, we prefer use only of the Schur-Cohn ideas for the first phase.

Turning to the second phase, we note first that there is more than one approach to checking sign definiteness, covered in detail in Section V. First, one may use a Sturm test; this comes about as follows: Setting $z_1 = \exp(i\theta)$ in (6), where $i = \sqrt{-1}$, one finds, following [16]

$$f_i(\exp i\theta) = f_i(\cos\theta) = \sum_{j=0}^{N_i} 2c_j \cos j\theta. \qquad (7)$$

Then one may set $x = \cos\theta$ and use the Chebyshev polynomials to obtain a real polynomial $g_1(x)$ in x which has constant sign on $(-1, 1)$ if and only if $f_i(z_1)$ in (6) has constant sign on $|z_1| = 1$; the sign of $g_i(x)$ can be checked with a Sturm test.

A second approach is based on examining the zero distributions of $z_1^{N_i} f_i(z_1)$. Observe first that if $f_i(z_1)$ has constant sign on $|z_1| = 1$, it has no zero there, and all zeros are in $|z_1| < 1$ or $|z_1| > 1$. If z_1^0 is a zero, so is $(z_1^0)^{-1}$, as is easily seen, irrespective of sign definiteness. So the number of zeros of $z_1^{N_i} f_i(z_1)$ in $|z_1| < 1$ is N_i and in $|z_1| > 1$ is N_i, if and only if $f_i(z_1)$ has constant sign on $|z_1| = 1$. We shall describe in Section V the variant on the zero distribution tests of Section III required for checking positivity. One such variant has been used in [14].

III. Zero Distribution of Single Variable Polynomials and the Checking of Condition (3)

We shall look first at the Schur-Cohn criterion (see [7]-[10]):

Schur-Cohn Criterion

Suppose that

$$f(z) = \sum_{i=0}^{n} a_i z^i \qquad (a_n \neq 0) \qquad (8)$$

and associate with it the $n \times n$ Hermitian matrix $C = (\gamma_{ij})$ defined by

$$\gamma_{ij} = \sum_{p=1}^{i} (a_{n-i+p} a_{n-j+p}^* - a_{i-p}^* a_{j-p}), \qquad i \leqslant j. \qquad (9)$$

Then the number of zeros z_i of $f(z)$ for which $|z_i| < 1$ and for which z_i^{-1} is not also a zero is the number of positive eigenvalues of C; the number of zeros z_i for which $|z_i| > 1$ and for which z_i^{-1} is not also a zero is the number of negative eigenvalues of C; the number of zeros z_i for which either $|z_i| = 1$ or z_i^{-1} is also a zero (or both) is the nullity of C.

Let us now see how this criterion can be used in checking condition (3).

First, the polynomial $B(z_1, 0)$ will be real and so will the matrix C. For (3) to hold, all eigenvalues of C must be negative definite. This will hold if and only if odd-order leading principal minors are negative and even-order leading principal minors are positive.

Example 1: Consider

$$B(z_1, z_2) = 1 + az_1 + bz_2 + cz_1 z_2. \qquad (10)$$

Evidently,

$$B(z_1, 0) = 1 + az_1$$

and it is trivial to see that we require $|a| < 1$ to satisfy (3).

Example 2: Consider

$$B(z_1, z_2) = 12 + 10z_1 + 6z_2 + 5z_1 z_2 + 2z_1^2 + z_1^2 z_2. \qquad (11)$$

Then

$$B(z_1, 0) = 12 + 10z_1 + 2z_1^2.$$

We could simply observe by direct calculation that the roots of this polynomial lie outside $|z_1| = 1$. However, the Schur-Cohn matrix is

$$C = \begin{bmatrix} a_2^2 - a_0^2 & a_1 a_2 - a_1 a_0 \\ a_1 a_2 - a_1 a_0 & a_2^2 - a_0^2 \end{bmatrix} = \begin{bmatrix} -140 & -100 \\ -100 & -140 \end{bmatrix}$$

which is easily seen to have a negative 1×1 leading principal minor and a positive 2×2 leading principal minor. Thus condition (3) is satisfied for $B(z_1, z_2)$ in (11).

Next, we consider a procedure based on forming a sequence of polynomials. To provide us with a result needed for later sections as well as this, we state a theorem of Jury [12], which represents an extension of the original theorem of Cohn [7].

Theorem: Suppose that

$$f(z) = \sum_{i=0}^{n} a_i z^i \qquad (8)$$

and define a sequence of polynomials

$$F_0(z) = f(z), \quad F_1(z) = \sum_{i=0}^{n-1} a_i^{(1)} z^i,$$

$$F_2(z) = \sum_{i=0}^{n-2} a_i^{(2)} z^i, \cdots \qquad (12)$$

by

$$F_{j+1}(z) = a_0^{*(j)} F_j(z) - a_{n-j}^{(j)} F_j^*(z) \qquad (13)$$

where

$$F_j^*(z) = a_0^{*(j)} z^{n-j} + a_1^{*(j)} z^{n-j-1} + \cdots + a_{n-j}^*. \qquad (14)$$

(Thus $F_j^*(z)$ is obtained from $F_j(z)$ by coefficient reversal and conjugation.) Set $\delta_j = F_j(0)$ and $P_j = \delta_1 \delta_2 \cdots \delta_j$. Then

1) all zeros of $f(z)$ lie inside $|z| < 1$ if and only if $P_j < 0$ for all j;
2) all zeros of $f(z)$ lie outside $|z| \leqslant 1$ if and only if $P_j > 0$ for all j, or, equivalently, $\delta_j > 0$ for all j;
3) if all P_j are nonzero, the number of negative P_j is the number of zeros of $f(z)$ inside $|z| < 1$ and the number of positive P_j is the number of zeros outside $|z| \leqslant 1$.

We comment that statement 2 above covers the requirement for checking the zero distribution of $B(z_1, 0)$. The theorem, however, is incomplete in the sense that it does not discuss what happens when some P_j are zero; note though that this cannot happen if all zeros of $f(z)$ are in $|z| < 1$ or outside $|z| \leqslant 1$.

Example 1 (continued): We have

$$B(z_1, 0) = 1 + az_1 = F_0(z).$$

Then

$$F_1(z) = (1 + az_1) - a(a + z_1)$$
$$= 1 - a^2.$$

Then $P_1 = F_1(0) = 1 - a^2$, and $P_1 > 0$ if $|a| < 1$. This is the required condition.

Example 2 (continued): We have

$$B(z_1, 0) = 12 + 10z_1 + 2z_1^2.$$

Then

$$F_1(z) = 12(12 + 10z_1 + 2z_1^2) - 2(2 + 10z_1 + 12z_1^2)$$
$$= 140 + 100z_1$$
$$F_2(z) = 140(140 + 100z_1) - 100(100 + 140z_1)$$
$$= 9600.$$

We have $\delta_1 = 140$, $\delta_2 = 9600$. The positivity indicates satisfaction of condition (3).

IV. Application of the Schur-Cohn Test to the Checking of Condition (4)

The Schur-Cohn test associates with the polynomial

$$f(z) = \sum_{i=0}^{n} a_i z^i \qquad (8)$$

the Hermitian matrix $C = (\gamma_{ij})$ with

$$\gamma_{ij} = \sum_{p=1}^{i} (a_{n-i+p} a_{n-j+p}^* - a_{i-p}^* a_{j-p}), \qquad i \leqslant j. \quad (9)$$

To check condition (4), we replace $f(z)$ by $B(z_1, z_2)$, written as a polynomial in z_2:

$$B(z_1, z_2) = \sum_{j=0}^{q} \left(\sum_{i=0}^{p} b_{ij} z_1^i \right) z_2^j. \qquad (15)$$

The coefficient a_j is therefore $\Sigma_{i=0}^{p} b_{ij} z_1^i$ and the matrix C is $q \times q$, with entries which from (9) are seen to be polynomial in z_1 and z_1^* with real coefficients.

The condition $B(z_1, z_2) \neq 0$ for $|z_1| = 1$, $|z_2| \leqslant 1$ holds if and only if for all $|z_1| = 1$, C is negative definite, i.e., if and only if the leading principal minors of C have appropriate signs. These principal minors being linear combinations of products of the γ_{ij} are themselves polynomial in z_1 and z_1^* with real coefficients; they are also real, since C is Hermitian. On $|z_1| = 1$, we may set $z_1^* = z_1^{-1}$ and the polynomials have the form $\Sigma_{j=0}^{N} c_j (z_1^j + z_1^{-j})$. Such polynomials are termed self-inversive; if $z_1 = z_{1\alpha}$ is a zero, so is $z_1 = z_{1\alpha}^{-1}$.

Example 1 (continued): We write

$$B(z_1, z_2) = (1 + az_1) + (b + cz_1) z_2.$$

This is a degree one polynomial in z_2 and, accordingly, the Schur-Cohn matrix is of size 1×1; it is simply

$$C = (b + cz_1^*)(b + cz_1) - (1 + az_1^*)(1 + az_1).$$

For it to be negative definite on $|z_1| = 1$, we require

$$(1 + az_1^*)(1 + az_1) - (b + cz_1^*)(b + cz_1) > 0, \qquad |z_1| = 1$$

or, recognizing that $z_1^* = z_1^{-1}$ on $|z_1| = 1$, we have

$$(a - bc) z_1 + (1 + a^2 - b^2 - c^2) + (a - bc) \frac{1}{z_1} > 0.$$

Notice that, as claimed, the coefficients of this expression are real. Conditions guaranteeing the inequality are easy to find in this case because of the simplicity of $B(z_1, z_2)$. General methods for checking the inequality are described in Section V.

Example 2 (continued): In this case, we have

$$B(z_1, z_2) = (12 + 10z_1 + 2z_1^2) + (6 + 5z_1 + z_1^2) z_2.$$

This leads to a 1×1 Schur-Cohn matrix:

$$C = [(6 + 5z_1 + z_1^2)(6 + 5z_1^{-1} + z_1^{-2}) - (12 + 10z_1 + 2z_1^2)(12 + 10z_1^{-1} + 2z_1^{-2})]$$
$$= [-18z_1^2 - 105z_1 - 186 - 105z_1^{-1} - 18z_1^{-2}], \qquad |z_1| = 1$$

and C is negative definite if

$$18z_1^2 + 105z_1 + 106 + 105z_1^{-1} + 10z_1^{-2} > 0, \qquad |z_1| = 1.$$

The checking of this condition is discussed in Section V.

Note that, in general, the Schur-Cohn matrix will be larger than 1×1; in case it has dimension q, there will be q positivity conditions to be checked, one associated with each leading principal minor of C.

V. Checking Positivity of Self-Inversive Polynomials

In this section, we discuss two techniques for checking the positivity on $|z_1| = 1$ of a self inversive polynomial:

$$f(z_1) = \sum_{j=0}^{N} c_j (z_1^j + z_1^{-j}). \quad (16)$$

Here, the c_j are real constants. Of course, one has immediate necessary conditions, obtained by putting $z_1 = 1, -1$:

$$\sum_{j=0}^{N} c_j > 0 \qquad \sum_{j=0}^{N} (-1)^j c_j > 0. \quad (17)$$

(Equally simply conditions can be obtained by setting $z_1 = \sqrt{-1}$ also. See [17].)

The first approach requires us to set $z_1 = \exp(i\theta)$; then (16) becomes

$$f(\cos\theta) = 2 \sum_{j=0}^{N} c_j \cos j\theta. \quad (18)$$

Positivity is to be checked for $0 \leqslant \theta \leqslant 2\pi$ (or $-\pi \leqslant \theta \leqslant \pi$). Since $\cos j\theta = \cos(-j\theta)$, *it is enough to* check positivity for $0 \leqslant \theta \leqslant \pi$. Let us set

$$x = \cos\theta \qquad T_k(x) = \cos k\theta \quad (19)$$

where $T_k(x)$ is the kth Chebyshev polynomial of the *first kind, defined recursively by*

$$T_{k+1}(x) = 2x\, T_k(x) - T_{k-1}(x), \qquad T_1(x) = x, \quad T_0(x) = 1. \quad (20)$$

Then (dropping the unessential factor 2) we require

$$g(x) = \sum_{j=0}^{N} c_j T_j(x) > 0 \quad (21)$$

for $-1 \leqslant x \leqslant 1$. Observe that $T_j(x)$ is a polynomial in x of degree j; therefore, $g(x)$ has the form

$$g(x) = \sum_{j=0}^{N} d_j x^j \quad (22)$$

for some real d_j. Positivity can be checked by forming a Sturm chain, see, e.g., [6]. Set $f_0(x) = g(x)$, $f_1(x) = g'(x)$ and define $f_2(x), f_3(x), \cdots$ by

$$f_0(x) = q_1(x) f_1(x) - f_2(x)$$
$$f_1(x) = q_2(x) f_2(x) - f_3(x)$$

with $f_{i+2}(x)$ the remainder obtained when dividing $f_i(x)$ by $f_{i+1}(x)$, and with $f_{i+2}(x)$ of lower degree than $f_{i+1}(x)$. Assuming $f_0(x)$ and $f_1(x)$ have no common factors, the algorithm terminates with $f_N(x)$ equal to a constant, and $f_0(x)$ is positive on $[-1, 1]$ if and only if $f_0(0) > 0$ and

$$V\{f_0(-1), f_1(-1), \cdots, f_N(-1)\} = V\{f_0(1), f_1(1), \cdots, f_N(1)\}$$

where $V\{\cdots\}$ denotes the number of variations in sign of the numbers inside $\{\cdots\}$. In case $f_0(x)$ and $f_1(x)$ have a common zero (i.e., $g(x)$ has repeated zeros), some modification is necessary (see [6]).

Example 1 (continued): We are required to check that

$$(a - bc) z_1 + (1 + a^2 - b^2 - c^2) + (a - bc)\frac{1}{z_1} > 0$$

on $|z_1| = 1$, *or, equivalently,*

$$(a - bc) T_1(x) + \tfrac{1}{2}(1 + a^2 - b^2 - c^2) > 0, \qquad -1 \leqslant x \leqslant 1$$

or

$$(a - bc) x + \tfrac{1}{2}(1 + a^2 - b^2 - c^2) > 0, \qquad -1 \leqslant x \leqslant 1.$$

The conditions for this are obvious;

$$|a - bc| < \tfrac{1}{2}(1 + a^2 - b^2 - c^2).$$

This condition is easily rewritten in the form stated by Huang [3]:

$$\left|\frac{1-a}{b-c}\right| > 1 \qquad \left|\frac{1+a}{b+c}\right| > 1.$$

Example 2 (continued): We are required to check the inequality

$$18z_1^2 + 105z_1 + 186 + 105z_1^{-1} + 18z_1^{-2} > 0$$

or

$$18\, T_2(x) + 105\, T_1(x) + 93 > 0, \qquad -1 \leqslant x \leqslant 1$$

or

$$36x^2 + 87x + 93 > 0, \qquad -1 \leqslant x \leqslant 1.$$

We have

$$f_0(x) = 36x^2 + 87x + 93$$
$$f_1(x) = 72x + 87$$
$$f_2(x) = \frac{93 \times 144 - 87 \times 87}{144}.$$

Then $V\{f_0(-1), f_1(-1), f_2(-1)\} = 0 = V\{f_0(1), f_1(1), f_2(1)\}$. This establishes that condition (4) is satisfied.

The second approach to checking positivity is based on the determination of the zero distribution of $f(z_1)$ in (16). Because $f(z_1)$ is self-inversive, there are as many zeros of $z_1^N f(z_1)$ inside $|z_1| < 1$ as outside. Therefore, $f(z_1)$ is positive on $|z_1| = 1$ if and only if $f(1) > 0$ [or $f(z_1)$ is positive at any one point of $|z_1| = 1$] and $z_1^N f(z_1)$ has N zeros inside $|z_1| < 1$.

(For then $z_1^N f(z_1)$ has N zeros outside, and thus no zeros on $|z_1| = 1$. Hence $f(z_1)$ has constant sign on $|z_1| = 1$.)

Unfortunately, the Schur–Cohn matrix for a self-inversive polynomial is zero, and the other procedure given in Section III based on setting up a sequence of polynomials leads to a zero polynomial at the first recursion. Hence, neither procedure, as it stands, is of help. However, the following result is of assistance:

Theorem [7], [12]: Let $f(z_1)$ be as in (16). The number of zeros of $g(z_1) = z_1^N f(z_1)$ in $|z_1| < 1$ is the same as the number of zeros of $z_1^{2N-1} g'(1/z_1)$ in $|z_1| < 1$, where $g'(1/z_1)$ is obtained by differentiating $g(z_1)$ with respect to z_1 and then substituting z_1^{-1} for z_1.

Assuming that $z_1^{2N-1} g'(1/z_1)$ has neither zeros reciprocal with respect to $|z_1| = 1$ nor zeros on $|z_1| = 1$, the Schur–Cohn criterion will yield the number of zeros inside $|z_1| < 1$ and the procedure based on generating of a recursive set of polynomials may also. If, on the other hand, $z_1^{2N-1} g'(1/z_1)$ does have reciprocal zeros or zeros on the unit circle, the situation must be analyzed further. In view of the easily established relation

$$z_1 g'(z_1) + z_1^{2N-1} g'\left(\frac{1}{z_1}\right) = f(z_1) \tag{23}$$

it follows that such zeros of $z_1^{2N-1} g'(1/z_1)$ are also zeros of $f(z_1)$. Accordingly, the highest common factor of $z_1^{2N-1} g'(1/z_1)$ and $f(z_1)$ can be found, and then its zero properties can in turn be studied, it too being self-inversive. Proceeding in this way, we can determine ultimately the number of zeros of $z_1^{2N-1} g'(1/z_1)$ and therefore $z_1^N f(z_1)$ inside $|z_1| < 1$, to conclude whether or not $f(z_1)$ is positive.

Example 1 (continued): We have

$$z_1 f(z_1) = (a - bc) z_1^2 + (1 + a^2 - b^2 - c^2) z_1 + (a - bc)$$

and this has the same number of zeros inside $|z_1| = 1$ as does

$$2(a - bc) + (1 + a^2 - b^2 - c^2) z_1.$$

Evidently, this has a zero inside $|z_1| = 1$ if and only if

$$|2(a - bc)| < |1 + a^2 - b^2 - c^2|.$$

Further, for $f(z_1)$ to be positive we require $f(1) > 0$ or

$$2(a - bc) + (1 + a^2 - b^2 - c^2) > 0.$$

Example 2 (continued): We have

$$g(z_1) = z_1^2 f(z_1) = 18z_1^4 + 105z_1^3 + 186z_1^2 + 105z_1 + 18.$$

This has the same number of zeros inside $|z_1| = 1$ as does

$$z_1^3 g'\left(\frac{1}{z_1}\right) = z_1^3 [72z_1^{-3} + 315z_1^{-2} + 372z_1^{-1} + 105]$$

or

$$105z_1^3 + 372z_1^2 + 315z_1 + 72$$

or

$$h(z_1) = 35z_1^3 + 124z_1^2 + 105z_1 + 24.$$

The Schur–Cohn matrix is

$$C = \begin{bmatrix} a_3^2 - a_0^2 & a_2 a_3 - a_0 a_1 & a_1 a_3 - a_0 a_2 \\ a_2 a_3 - a_0 a_1 & a_3^2 + a_2^2 - a_0^2 - a_1^2 & a_2 a_3 - a_0 a_1 \\ a_1 a_3 - a_0 a_2 & a_2 a_3 - a_0 a_1 & a_3^2 - a_0^2 \end{bmatrix} = \begin{bmatrix} 649 & 1820 & 699 \\ 1820 & 5000 & 1820 \\ 699 & 1820 & 649 \end{bmatrix}.$$

The signs of the leading principal minors of C are positive, negative, negative. Jacobi's theorem [15] states that the number of positive eigenvalues of an $n \times n$ Hermitian C is the number of permanences of sign of $\{1, D_1, D_2, \cdots D_n\}$ where D_i is the ith leading principal minor, provided all D_i are nonzero. In our case, C therefore has two positive eigenvalues and one negative eigenvalue. By the Schur–Cohn theorem, the polynomial $h(z_1)$ has two zeros inside $|z_1| = 1$; hence $g(z_1)$ has two zeros inside $|z_1| = 1$. Because $f(1) > 0$, it follows that $f(z_1) > 0$ for all $|z_1| = 1$.

Finally, we comment that conditions (3) and (4) can obviously be replaced by conditions involving interchange of z_1 and z_2 which may be simpler:

$$B(0, z_2) \neq 0, \qquad |z_2| \leqslant 1 \tag{24}$$

$$B(z_1, z_2) \neq 0, \qquad |z_1| \leqslant 1, \quad |z_2| = 1. \tag{25}$$

As a further illustration of the techniques presented, which of course apply to (24) and (25), we shall verify that (24) and (25) hold for the same polynomial in z_1 and z_2 as appeared in Example 2. This time we shall use shortcuts.

Example 3:

$$B(z_1, z_2) = 12 + 10z_1 + 6z_2 + 5z_1 z_2 + 2z_1^2 + z_1^2 z_2. \tag{11}$$

We have

$$B(0, z_2) = 12 + 6z_2.$$

Obviously, condition (24) holds. Next,

$$B(z_1, z_2) = (12 + 6z_2) + (10 + 5z_2) z_1 + (2 + z_2) z_1^2.$$

The Schur–Cohn matrix obtained by regarding this as a polynomial in z_1 and by setting $z_2^{-1} = z_2^*$ is

$$C = \begin{bmatrix} |a_2|^2 - |a_0|^2 & a_2 a_1^* - a_0^* a_1 \\ a_2^* a_1 - a_0 a_1^* & |a_2^2| - |a_0|^2 \end{bmatrix} = \begin{bmatrix} -175 - 70z_2 - 70z_2^{-1} & -125 - 50z_2 - 50z_2^{-1} \\ -125 - 50z_2 - 50z_2^{-1} & -175 - 70z_2 - 70z_2^{-1} \end{bmatrix}.$$

For (25) to hold, the 1,1 entry of C must be negative definite and C must have positive determinant, i.e.,

$$175 + 70z_2 + 70z_2^{-1} > 0, \quad |z_2| = 1$$

$$(175 + 70z_2 + 70z_2^{-1})^2 - (125 + 50z_2 + 50z_2^{-1})^2 > 0, \quad |z_2| = 1.$$

The first inequality is obviously satisfied. To check the second, set $x = \cos\theta$. The inequality which has to be satisfied becomes

$$(175 + 140x)^2 - (125 + 100x)^2 > 0, \qquad -1 \leqslant x \leqslant 1$$

i.e.,

$$(300 + 240x)(50 + 40x) > 0, \qquad -1 \leqslant x \leqslant 1$$

and clearly this inequality holds.

VI. Conclusion

Our method is really to be compared against that of Huang. By and large, the methods involve the same sort of calculations, save that we avoid the bilinear transformation component of Huang's method. Undoubtedly, this represents a substantial computational load for any but the simplest two-variable polynomials. On the other hand, as soon as the two-variable polynomials under test become at all complex, presumably computers will be used to do the checking, and it might well prove the case that programming considerations determine which is the better method. Nevertheless, for those who have some knowledge of the Schur–Cohn and related theory, our method would probably be conceptually preferable. For those whose knowledge encompasses Hurwitz testing but not the Schur–Cohn material, Huang's method might be conceptually preferable.

Of course, in programming either algorithm, one would want to take full advantage of the fact that a number of the component tests involve real polynomials, and some simplification may then be possible.

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A Low-Sensitivity Active *RC* Low-Pass Filter

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Abstract—A new active canonic *RC* low-pass filter is given. The sensitivities, with respect to all passive and active circuit components, are low. The frequency limitations of the filter are derived based on the one-pole rolloff model of the operational amplifier (*OA*) and the results arrived at are compared with other well-known low-pass filters. Experimental results agree with the theoretical ones.

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I. Introduction

Several active *RC* networks are available to realize second-order low-pass transfer characteristics having a low-pole Q [1], [2]. In comparing these networks, the effect of the nonideal operational amplifier must be taken into consideration. Recently, a unified approach, based on the one-pole rolloff model of the operational amplifier (*OA*) to determine the upper bound on the frequency ω_0 that these networks can effectively realize, was developed [3].