On the Connectivity of Wireless Multi-Hop Networks with Arbitrary Wireless Channel Models

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***Abstract*—Considering a wireless multi-hop network where a total of n nodes are randomly, independently and uniformly distributed in a unit square in $\Re^2$ and each node has a uniform transmission power, a fundamental problem is to investigate the connectivity of such networks. In this letter, we prove that the probability of having a connected network and the probability of having no isolated node asymptotically converges to the same value as n goes to infinity for an arbitrary wireless channel model satisfying certain intuitively reasonable conditions.**

***Index Terms*—Connectivity, isolated node, wireless multi-hop network, channel model.**

I. Introduction

ONE of the most fundamental properties of wireless multi-hop networks, e.g. wireless ad hoc/sensor network, is connectivity [1], [2], [3]. A network is said to be connected iff (if and only if) for any pair of two nodes, there is at least one path between them. There has been significant research on network connectivity during the last few years [1], [2], [3], [4]. The results proved by Gupta *et al.* indicate that under the *unit disk communication model*, the probability of having a connected network asymptotically converges to the probability of having no isolated node as the number of nodes goes to infinity in $\Re^2$ [1]. Similar results were proved under the *log-normal shadowing model* in $\Re^2$ by Ta *et al.* in [5]. These two results raise an interesting question: whether the asymptotic property, i.e. the above two probabilities converge as the number of nodes goes to infinity, holds under more *generic* channel models. The results obtained under a specific channel model may not necessarily apply for other channel models. Hence, it is important to investigate whether this asymptotic result is valid for generic channel models satisfying certain common conditions.

In this letter, we investigate analytically the network connectivity in a wireless multi-hop network for wireless channel models which are arbitrary apart from obeying certain intuitively reasonable general conditions on node-pair connectivity. We assume that a total of n nodes are randomly, independently and uniformly distributed in a unit square in $\Re^2$, and each node has a uniform transmission power. Any two nodes can communicate with each other directly with probability $g^{\mathcal{C}}(x)$, where x is the Euclidean distance between the two nodes, $\mathcal{C}$ represents any arbitrary channel model (e.g. log-normal shadowing model), and $g^{\mathcal{C}}(x)$ is the probability that any two nodes with distance x apart from each other are directly connected. The function $g^{\mathcal{C}}(x)$ can be derived using the channel model, the transmission power and the threshold power (or SNR) above which a receiving node can correctly receive the packet. Different channel models may lead to different $g^{\mathcal{C}}(x)$. For example, if the channel model $\mathcal{C}$ is the unit disk communication model with the transmission range r, we have $g^{\mathcal{C}}(x) = 1$ when $x \leq r$; $g^{\mathcal{C}}(x) = 0$ when $x > r$. For the log-normal shadowing model, $g^{\mathcal{C}}(x)$ can be found in [2], [5]. Based on this network model, we shall prove that the probability of having a connected network asymptotically converges to the probability of having no isolated node as $n \to \infty$ for any channel model satisfying the following conditions of rotational and translation invariance, monotonicity and integral boundedness:

$$\begin{cases} g^{(\mathcal{C})}(x) = g^{(\mathcal{C})}(y) \quad whenever \quad x = y; \\ g^{(\mathcal{C})}(x) \leq g^{(\mathcal{C})}(y) \quad whenever \quad x \geq y; \\ \qquad 0 < \int_{\Re^2} g^{(\mathcal{C})}(x)dx < \infty. \end{cases} \tag{1}$$

The first restriction indicates that the propagation path is symmetric; the second restriction indicates that $g^{(\mathcal{C})}(x)$ must be a non-increasing function of the distance x; the third restriction avoids the trivial case that any two nodes are directly connected with probability one [6]. It can be easily shown that both the unit disk communication model and the log-normal shadowing model satisfy the condition Eq. 1. The result in this letter extends the result derived in [1] for the unit disk communication model and the result derived in [5] for the log-normal shadowing model.

Throughout this letter, we represent a wireless multi-hop network by an undirected graph with each vertex of its vertex set uniquely representing a node and each edge of its edge set uniquely representing a wireless link, and vice versa. The graph is then called the *underlying graph* of the network, and is defined as follows.

Definition 1. Let X_1, X_2, ..., X_n be n points which are independently, randomly and uniformly distributed in a unit square in $\Re^2$; let $\mathcal{X}_n = \{X_1, X_2, ..., X_n\}$. The underlying graph $G(\mathcal{X}_n, \mathcal{C})$ is an undirected graph having $\mathcal{X}_n$ as its vertex set, and with an edge connecting each pair of vertices X_i and X_j in $\mathcal{X}_n$ with probability $g^{(\mathcal{C})}(\|X_i - X_j\|)$, where $\mathcal{C}$ represents the channel model, function $g^{(\mathcal{C})}(\cdot)$ satisfies the conditions Eq. 1, and $\| \cdot \|$ means the Euclidean norm.

II. Connectivity

In this section, we present the main result in this letter, which is given in the following theorem.

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Theorem 1. *Let $P_c(\mathcal{X}_n,\mathcal{C})$ denote the probability that the graph $G(\mathcal{X}_n,\mathcal{C})$ of Definition 1 is connected, and $P_0(\mathcal{X}_n,\mathcal{C})$ denote the probability that the graph $G(\mathcal{X}_n,\mathcal{C})$ of Definition 1 has no isolated node. Ignore the boundary effect. If the limit of $P_0(\mathcal{X}_n,\mathcal{C})$ exists when $n\to\infty$, then the limit of $P_c(\mathcal{X}_n,\mathcal{C})$ also exists when $n\to\infty$, vice versa, and*

$$\lim_{n\to\infty} P_c(\mathcal{X}_n,\mathcal{C}) = \lim_{n\to\infty} P_0(\mathcal{X}_n,\mathcal{C}).$$

Theorem 1 implies that for a given graph $G(\mathcal{X}_n,\mathcal{C})$ as defined in Definition 1, if n is large enough, then with high probability, the graph becomes connected at the moment when the last isolated node vanishes (or equivalently, the network achieves a minimum degree of one). The result, derived for an arbitrary channel model satisfying the conditions Eq. 1, is consistent with the result derived for the unit disk communication model in [1] and the result derived for the log-normal shadowing model in [5].

It is well known that in two or higher dimensional networks, it is difficult to derive an closed-form analytical formula for computing the probability $P_c(\mathcal{X}_n,\mathcal{C})$. A widely used method is to study $P_c(\mathcal{X}_n,\mathcal{C})$ by analyzing $P_0(\mathcal{X}_n,\mathcal{C})$ which can be easily derived. Theorem 1 indicates that $P_0(\mathcal{X}_n,\mathcal{C})$ is a close approximation of $P_c(\mathcal{X}_n,\mathcal{C})$ for large n for an arbitrary channel model satisfying Eq. 1, which provides a theoretical basis for estimating $P_c(\mathcal{X}_n,\mathcal{C})$ by $P_0(\mathcal{X}_n,\mathcal{C})$. Existing solutions of $P_0(\mathcal{X}_n,\mathcal{C})$ for the unit disk communication model and the log-normal shadowing model can be found in [2], [3], [5].

III. Proof of Theorem 1

To prove Theorem 1, we apply some results derived for the *Poisson random-connection model*, as defined below, in continuum percolation [6].

Definition 2 ([6])**.** Let $\{X_1, X_2, X_3, ...\}$ be an infinite series of points which are randomly, independently and uniformly distributed in a unit square in $\Re^2$; let N_λ be a Poisson random variable with mean $\lambda > 0$; let $\mathcal{P}_\lambda = \{X_1, X_2, ..., X_{N_\lambda}\}$. The Poisson random-connection model, denoted as $G(\mathcal{P}_\lambda, g)$, is an undirected graph having $\mathcal{P}_\lambda$ as its vertex set, and with an edge connecting each pair of vertices X_i and X_j in $\mathcal{P}_\lambda$ with probability $g(\|X_i - X_j\|)$, where $g(\cdot)$ is a function from the positive reals into $[0,1]$ and satisfies the conditions Eq. 1.

Throughout this letter, let $F(\mathcal{C})$ denote the probability that two randomly selected nodes in the graph $G(\mathcal{X}_n,\mathcal{C})$ are directly connected. In principle at least $F(\mathcal{C})$ can be derived using the function $g^C(x)$ and the node distribution. For example, for nodes distributed randomly and uniformly in a unit square, ignoring the boundary effect, $F(\mathcal{C}) = \pi r^2$ when $\mathcal{C}$ is the unit disk communication model with transmission range r [1], [3]. In the rest of this section, we first derive three lemmas, viz., Lemmas 1, 2 and 3, and then prove Theorem 1 using these lemmas. We shall use standard mathematical notations: $y(n) = o(g(n))$ if $\lim_{n\to\infty}\frac{y(n)}{g(n)} = 0$; $g(n) \gg y(n)$ if $y(n) = o(g(n))$; $y(n) \sim g(n)$ if $\lim_{n\to\infty}\frac{y(n)}{g(n)} = 1$.

Lemma 1. *Let $P_d(\mathcal{X}_n,\mathcal{C})$ be the probability that the graph $G(\mathcal{X}_n,\mathcal{C})$ is disconnected. Let $F(\mathcal{C})$ be the probability that two randomly selected nodes in $G(\mathcal{X}_n,\mathcal{C})$ are directly connected. Ignore the boundary effect. Given any fixed $c\in\Re$, if $F(\mathcal{C}) = \frac{\log n + c}{n}$, then*

$$\liminf_{n\to\infty} P_d(\mathcal{X}_n,\mathcal{C}) \geq 1 - \exp\left(-e^{-c}\right).$$

Proof: Let $P_{iso}(\mathcal{X}_n,\mathcal{C})$ be the probability that an arbitrary node in $G(\mathcal{X}_n,\mathcal{C})$ is isolated. Ignoring boundary effects, then

$$P_{iso}(\mathcal{X}_n,\mathcal{C}) \sim (1 - F(\mathcal{C}))^{n-1}, \quad as \quad n\to\infty.$$

Because $F(\mathcal{C}) = \frac{\log n + c}{n} \to 0$ as $n\to\infty$, we have

$$P_{iso}(\mathcal{X}_n,\mathcal{C}) \sim \frac{e^{-c}}{n}, \quad as \quad n\to\infty.$$

Since $n \gg 1$ and $F(\mathcal{C}) = \frac{\log n + c}{n} \to 0$ as $n\to\infty$, the event that a randomly selected node has i neighbors is almost independent of the event that another randomly selected node has j neighbors [3], [4], [6]. Ignoring boundary effects, then

$$P_0(\mathcal{X}_n,\mathcal{C}) = (1 - P_{iso}(\mathcal{X}_n,\mathcal{C}))^n \sim \left(1 - \frac{e^{-c}}{n}\right)^n \sim \exp\left(-e^{-c}\right), \quad as \quad n\to\infty. \quad (2)$$

Using Eq. 2 and the fact that $P_d(\mathcal{X}_n,\mathcal{C}) \geq 1 - P_0(\mathcal{X}_n,\mathcal{C})$, the result follows. ■

Lemma 2. *Let $P_d(\mathcal{P}_{m(n)},\mathcal{C})$ be the probability that the graph $G(\mathcal{P}_{m(n)},\mathcal{C})$ is disconnected, where $m(n) = \lfloor n - n^{\frac{3}{4}}\rfloor$. Let $F(\mathcal{C})$ be the same probability as defined in Lemma 1. Ignore boundary effects. Given any fixed $c\in\Re$, if $F(\mathcal{C}) = \frac{\log n + c}{n}$,*

$$\lim_{n\to\infty} P_d(\mathcal{P}_{m(n)},\mathcal{C})) = 1 - \exp\left(-e^{-c}\right).$$

Proof: A *component* of a graph is a maximally connected subgraph of the graph. The *order* of a component is the number of vertices in the component. It has been proved that a.a.s. $G(\mathcal{P}_\lambda, g)$ has at most one infinite-order component for each $\lambda \geq 0$ (Theorem 6.3 of [6]). In addition, as $\lambda\to\infty$, a.a.s. any given point of $\mathcal{P}_\lambda$ either lies in an infinite-order component or is isolated (Theorem 6.4 of [6]). Since $m(n)\to\infty$ as $n\to\infty$, the graph $G(\mathcal{P}_{m(n)},\mathcal{C})$ only consists of isolated vertices and an infinite-order component a.a.s..

Let $Y(\mathcal{P}_{m(n)},\mathcal{C})$ be the probability that $G(\mathcal{P}_{m(n)},\mathcal{C})$ has at least one isolated node. Using the above analysis, we have

$$P_d(\mathcal{P}_{m(n)},\mathcal{C}) = (1 + o(1))Y(\mathcal{P}_{m(n)},\mathcal{C}), \quad as \quad n\to\infty. \quad (3)$$

For a Poisson random variable M_n with mean $m(n) = \lfloor n - n^{\frac{3}{4}}\rfloor$, using the Chebyshev's inequality, we have [5]

$$\lim_{n\to\infty} Pr\{\lfloor n - 2n^{\frac{3}{4}}\rfloor \leq M_n \leq n\} = 1. \quad (4)$$

For any integer $j \geq 0$, define $Y(\mathcal{X}_j,\mathcal{C})$ as the probability that the graph $G(\mathcal{X}_j,\mathcal{C})$ has at least one isolated node. Define $J_n = \{j : j\in\mathbb{N},\ \lfloor n - 2n^{\frac{3}{4}}\rfloor \leq j \leq n\}$. Then, using Eq. 4, it can be easily shown that

$$Y(\mathcal{P}_{m(n)},\mathcal{C}) = \sum_{j=0}^{\infty}\frac{(m(n))^j}{j!}e^{-m(n)}Y(\mathcal{X}_j,\mathcal{C}) = \sum_{j\in J_n}\frac{(m(n))^j}{j!}e^{-m(n)}Y(\mathcal{X}_j,\mathcal{C}) + o(1), \quad as \quad n\to\infty. \quad (5)$$

Ignoring the boundary effect, for any $j \in J_n$, we have

$$Y(\mathcal{X}_j, \mathcal{C}) = 1 - (1 - P_{iso}(\mathcal{X}_j, \mathcal{C}))^j .$$

Since $j \in J_n$, we have $\frac{j}{n} \to 1$ as $n \to \infty$. In the same way as in the proof of Lemma 1, we have

$$Y(\mathcal{X}_j, \mathcal{C}) \sim 1 - \exp\left(-e^{-c}\right), \quad as \quad n \to \infty. \tag{6}$$

Substituting Eq. 6 into Eq. 5, and using Eq. 4, we have

$$Y(\mathcal{P}_{m(n)}, \mathcal{C}) = 1 - \exp\left(-e^{-c}\right) + o(1), \quad as \quad n \to \infty. \tag{7}$$

By Eq. 3 and Eq. 7, the result follows immediately. ■

Lemma 3. *Adopt the same hypothesis as Lemma 1, then*

$$\limsup_{n\to\infty} P_d(\mathcal{X}_n, \mathcal{C}) \le 1 - \exp\left(-e^{-c}\right).$$

Proof: For any integer $j \ge 0$, define $P_d(\mathcal{X}_j, \mathcal{C})$ as the probability that the graph $G(\mathcal{X}_j, \mathcal{C})$ is disconnected. In the same way as in the proof of Lemma 2, we have

$$\begin{aligned} P_d(\mathcal{P}_{m(n)}, \mathcal{C}) &= \sum_{j=0}^{\infty} \frac{(m(n))^j}{j!} e^{-m(n)} P_d(\mathcal{X}_j, \mathcal{C}) \\ &= \sum_{j\in J_n} \frac{(m(n))^j}{j!} e^{-m(n)} P_d(\mathcal{X}_j, \mathcal{C}) + o(1), \ as \ n \to \infty. \end{aligned} \tag{8}$$

For fixed transmission power, channel $\mathcal{C}$, and any $l > 0$, we have the following inequality (similar to the one in [1]),

$$\begin{aligned} P_d(\mathcal{X}_l, \mathcal{C}) \quad &\le \quad Pr\{node\ l\ is\ isolated\ in\ G(\mathcal{X}_l, \mathcal{C})\} \\ &\quad + P_d(\mathcal{X}_{l-1}, \mathcal{C}). \end{aligned} \tag{9}$$

By taking recursion on Eq. 9, we have that for any $j \in \left[\lfloor n - 2n^{\frac{3}{4}} \rfloor - 1, n-1\right]$,

$$\begin{aligned} & P_d(\mathcal{X}_n, \mathcal{C}) \\ & \le \sum_{l=j+1}^{n} Pr\{node\ l\ is\ isolated\ in\ G(\mathcal{X}_l, \mathcal{C})\} + P_d(\mathcal{X}_j, \mathcal{C}) \\ & \sim \sum_{l=j+1}^{n} (1 - F(\mathcal{C}))^{l-1} + P_d(\mathcal{X}_j, \mathcal{C}) \\ & \le \frac{(1 - F(\mathcal{C}))^j}{F(\mathcal{C})} + P_d(\mathcal{X}_j, \mathcal{C}). \end{aligned} \tag{10}$$

Since $j \in J_n$, $\frac{j}{n} \to 1$ as $n \to \infty$. Using the condition $F(\mathcal{C}) = \frac{\log n + c}{n}$, we have

$$\frac{(1 - F(\mathcal{C}))^j}{F(\mathcal{C})} \sim \frac{e^{-c}}{\log n + c} = o(1) \quad as \quad n \to \infty. \tag{11}$$

Inserting Eq. 10, Eq. 11 into Eq. 8, it can be obtained that

$$\begin{aligned} P_d(\mathcal{P}_{m(n)}, \mathcal{C}) &\ge P_d(\mathcal{X}_n, \mathcal{C}) \sum_{j\in J_n} \frac{(m(n))^j}{j!} e^{-m(n)} + o(1) \\ &= P_d(\mathcal{X}_n, \mathcal{C}) + o(1), \quad as \quad n \to \infty. \end{aligned} \tag{12}$$

By Lemma 2 and Eq. 12, we have

$$P_d(\mathcal{X}_n, \mathcal{C}) \le 1 - \exp\left(-e^{-c}\right) + o(1), \quad as \quad n \to \infty,$$

from which the result follows. ■

Proof of Theorem 1: Note that both probabilities $P_c(\mathcal{X}_n, \mathcal{C})$ and $P_0(\mathcal{X}_n, \mathcal{C})$ are monotonically increasing functions of $F(\mathcal{C})$. In the following, we shall use this monotonic property and Lemmas 1, 3 and Eq. 2 to prove Theorem 1 in three steps.

Suppose the limit of $P_0(\mathcal{X}_n, \mathcal{C})$ exists when $n \to \infty$. Then, there are only three cases that $\lim_{n\to\infty} P_0(\mathcal{X}_n, \mathcal{C})$ can be, i.e. 0, 1 and any value $\beta \in (0, 1)$.

1) If $\lim_{n\to\infty} P_0(\mathcal{X}_n, \mathcal{C}) = 0$: it is trivial to obtain that $\lim_{n\to\infty} P_c(\mathcal{X}_n, \mathcal{C}) = 0$ since $P_c(\mathcal{X}_n, \mathcal{C}) \le P_0(\mathcal{X}_n, \mathcal{C})$ for all n.
2) If $\lim_{n\to\infty} P_0(\mathcal{X}_n, \mathcal{C}) = 1$: because $\exp\left(-e^{-c}\right) \to 1$ as $c \to \infty$, by the monotonic property of $P_0(\mathcal{X}_n, \mathcal{C})$ with respect to $F(\mathcal{C})$ and Eq. 2, it must be true that for any arbitrary $c_2 \in \Re$, there exists an integer N_{c_2} such that

$$F(\mathcal{C}) \ge \frac{\log n + c_2}{n}, \quad \forall \quad n \ge N_{c_2}.$$

Then, using the above requirement on $F(\mathcal{C})$, the fact that $\exp\left(-e^{-c_2}\right) \to 1$ as $c_2 \to \infty$ *and* c_2 can be arbitrarily large, and the monotonic property of $P_c(\mathcal{X}_n, \mathcal{C})$ with respect to $F(\mathcal{C})$ and Lemma 3, it can be readily obtained that $\lim_{n\to\infty} P_c(\mathcal{X}_n, \mathcal{C}) = 1$.
3) If $\lim_{n\to\infty} P_0(\mathcal{X}_n, \mathcal{C}) = \beta \in (0, 1)$: by the strict monotonicity of $\exp\left(-e^{-c}\right)$ with respect to c and Eq. 2, it must be true that

$$F(\mathcal{C}) = \frac{\log n + c_\beta}{n},$$

where $c_\beta \in \Re$ satisfies $\exp\left(-e^{-c_\beta}\right) = \beta$. Using this requirement on $F(\mathcal{C})$, and Lemmas 1 and 3, it can be obtained that $\lim_{n\to\infty} P_c(\mathcal{X}_n, \mathcal{C}) = \beta$.

Combining the above three cases, it is clear that if the limit of $P_0(\mathcal{X}_n, \mathcal{C})$ exists when $n \to \infty$, then the limit of $P_c(\mathcal{X}_n, \mathcal{C})$ also exists when $n \to \infty$, and $\lim_{n\to\infty} P_0(\mathcal{X}_n, \mathcal{C}) = \lim_{n\to\infty} P_c(\mathcal{X}_n, \mathcal{C})$. In the same way, we can prove that if the limit of $P_c(\mathcal{X}_n, \mathcal{C})$ exists when $n \to \infty$, then the limit of $P_0(\mathcal{X}_n, \mathcal{C})$ also exists when $n \to \infty$, and $\lim_{n\to\infty} P_0(\mathcal{X}_n, \mathcal{C}) = \lim_{n\to\infty} P_c(\mathcal{X}_n, \mathcal{C})$. Hence, the result follows. ■

Remark. Since Theorems 6.3-6.4 of [6] are also valid in 3-dimensional space, in the same way as shown in the proofs of Lemmas 1, 2, 3 and Theorem 1, we can prove that Theorem 1 is also valid in 3-dimensional space.

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