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Quantitative effects of weight adjustments in $\mathscr{H}_\infty$ control[‡]

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SUMMARY

In an $\mathscr{H}_\infty$ design, when weights are adjusted, corresponding modifications occur in the synthesized $\mathscr{H}_\infty$ controller and the resulting closed-loop transfer function matrices of interest. In this paper, we seek to understand and provide quantitative results on how weight adjustments directly affect an $\mathscr{H}_\infty$ controller and, more importantly, the corresponding closed-loop transfer function matrices. Here, we explore issues such as whether one can find the new controller as a perturbation of the original controller.

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1. INTRODUCTION

In this paper, we present quantitative results on how weight adjustments directly affect an $\mathscr{H}_\infty$ controller and the corresponding closed-loop transfer function matrices and extend the applicability

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of earlier results [1] to multi-input multi-output systems and to general $\mathscr{H}_\infty$ problems. This includes the innovation of using a quadratically convergent algorithm and proposing an iterative method for handling extended-sized weight changes (e.g. larger weight changes as opposed to only small weight changes). Of course, given a weight adjustment, one can redesign an $\mathscr{H}_\infty$ controller from scratch. However, it is of interest and importance to explore whether one can find the new controller as a perturbation of the first controller via a computationally simple mechanism. Our proposed algorithm thus constitutes an important tool, which can be used in iterative adaptive control algorithms that are based on small weight adjustments at each iteration [2]. Even when the weight change is significant, it is still relevant to consider if the initial design can be used as a basis for the calculation of the new design. Furthermore, the results of this work could potentially be useful in studying the related problem of finding the reverse mapping, i.e. how to find the immediate corresponding changes in the weights given a desired change in the closed loop. This is, however, beyond the scope of the present paper.

It has been argued [3, 4] that safe iterative adaptive control (i.e. guaranteeing that no instability is introduced when one makes step changes in the controller) can be secured through limitation of the size of the controller change. This can be achieved by limiting the size of the weight change. In these iterative adaptive control schemes, one typically wishes to update the synthesized $\mathscr{H}_\infty$ controller online, possibly at each sampling instant. Thus, whenever high-order systems are involved or for numerically ill-conditioned systems, directly solving two new Riccati equations (involved in the analysis of a standard $\mathscr{H}_\infty$ problem [5, Chapter 15]) online at each sampling instant may be computationally infeasible. The formulae in this paper (which are based on linear analysis) offer designers the possibility of implementing such iterative adaptive control schemes online using approximate updates due to small weight changes.

It is well known that if the state-space matrices of the generalized plant change by a small amount, then the stabilizing solutions to the Riccati equations of the relevant $\mathscr{H}_\infty$ control problem also change by a small amount, and there are various methods for deriving the resulting small change using linearization of these equation solutions. Thus, an approximation of the change in controller may be easily derivable in those cases. However, if there is a small change in the frequency response of a weight that is absorbed in the generalized plant and if this change in frequency response causes a change in McMillan degree of the generalized plant, it is not clear how to demonstrate that the resulting $\mathscr{H}_\infty$ controller changes by a small amount. Certainly, the Riccati equation approach would seem unpromising. Another contribution of this paper is to show how to find the resulting controller change. In order to address this problem, we use the chain-scattering approach to $\mathscr{H}_\infty$ control of [6]. This approach is very similar (and in fact equivalent in some sense) to the J-spectral factorization approach to $\mathscr{H}_\infty$ control of [7–12]. Adjustments of the weighting functions that may result in a change in McMillan degree are easily handled in the frequency domain operator-theoretic framework of chain-scattering, but are more cumbersome and not easily cast in state-space descriptions.

Section 2 sets up the considered problem, states our objectives and analyzes the effects of weight adjustments on the central controller. Then, the related effects on the corresponding closed-loop transfer functions can be readily calculated. In Section 3, we propose an iterative algorithm that guarantees quadratic convergence to the exact solution (i.e. controller and closed-loop transfer functions) provided the initial error lies in a region of attraction. Section 5 includes concluding remarks and presents potential future research direction.

2. CONSIDERED PROBLEM

Consider the following weighted general $\mathscr{H}_\infty$ control design problem consisting of finding an internally stabilizing controller K such that

$$\left\| \begin{bmatrix} W_2 & 0 \\ 0 & W_4 \end{bmatrix} \mathscr{F}_l(G,K) \begin{bmatrix} W_3 & 0 \\ 0 & W_1 \end{bmatrix} \right\|_\infty < \gamma \tag{1}$$

where γ is some positive number, $W_i \in \mathscr{R}\mathscr{H}_\infty$ are square weighting functions (not restricted to bi-proper weights only) and G is a generalized plant defined by

$$G := \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \in \mathscr{R}$$

The weights W_i specified in Equation (1) can be all independent or can be related (e.g. if one chooses $W_3 = W_2^{-1}$ and $W_4 = W_1^{-1}$, then this will give the $\mathscr{H}_\infty$ loop-shaping paradigm).

Some algebraic manipulations should convince the reader that the above $\mathscr{H}_\infty$-norm objective can be restated as: Synthesize an internally stabilizing controller K such that

$$\|\mathscr{F}_l(\Sigma, K)\|_\infty < \gamma \tag{2}$$

where the generalized plant $\Sigma \in \mathscr{R}\mathscr{L}_\infty$ is defined by

$$\begin{aligned}\Sigma := \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \underbrace{\Sigma_{21}}_{r} & \underbrace{\Sigma_{22}}_{p} \end{bmatrix} \begin{matrix} \}m \\ \}q \end{matrix} &= \begin{bmatrix} W_2 & 0 & 0 \\ 0 & W_4 & 0 \\ 0 & 0 & I \end{bmatrix} G \begin{bmatrix} W_3 & 0 & 0 \\ 0 & W_1 & 0 \\ 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} \begin{pmatrix} W_2 & 0 \\ 0 & W_4 \end{pmatrix} G_{11} \begin{pmatrix} W_3 & 0 \\ 0 & W_1 \end{pmatrix} & \begin{pmatrix} W_2 & 0 \\ 0 & W_4 \end{pmatrix} G_{12} \\ G_{21} \begin{pmatrix} W_3 & 0 \\ 0 & W_1 \end{pmatrix} & G_{22} \end{bmatrix}\end{aligned} \tag{3}$$

From this representation, it is clear that the results presented here can all be readily specialized to one-block, namely both Σ_{21} and Σ_{12} are square, and two-block problems, namely either Σ_{21} or Σ_{12} is square. The following lemma characterizes the set of all admissible controllers in terms of a solution of a J-cospectral factorization problem when the normalized $\mathscr{H}_\infty$ control problem is solvable. Here, J_{pq} denotes the signature matrix defined by $J_{pq} := \mathrm{diag}(I_p, -I_q)$ and $\mathrm{HM}(\Xi, S)$ denotes a homographic transformation defined by $\mathrm{HM}(\Xi, S) := (\Xi_{11}S + \Xi_{12})(\Xi_{21}S + \Xi_{22})^{-1}$. The notation $(\cdot)^\sim$ should be taken to mean the Hermitian para-conjugate of $(\cdot)$. The reader is referred to [6, 8] for further information on homographic transformation ideas and the underlying ideas specialized for use in the following lemma.

Lemma 1

Suppose that the normalized $\mathscr{H}_\infty$ control problem in Equation (2) is solvable for a generalized plant Σ, which satisfies $\mathrm{rank}[\Sigma_{21}(j\omega)] = q \leqslant r$ and $\mathrm{rank}[\Sigma_{12}(j\omega)] = p \leqslant m$ for all $\omega \in \mathbb{R} \cup \{\infty\}$. Then

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there exists a unimodular Ξ in $\mathcal{RH}_\infty$ (with a $(q\times q)$ bi-proper Ξ_{22}) satisfying

$$\Xi J_{pq}\Xi^{\sim}=\begin{bmatrix} I_p & 0\\ \Sigma_{22} & \Sigma_{21}\end{bmatrix}\begin{bmatrix}\Sigma_{12}^{\sim}\Sigma_{12} & \Sigma_{12}^{\sim}\Sigma_{11}\\ \Sigma_{11}^{\sim}\Sigma_{12} & \Sigma_{11}^{\sim}\Sigma_{11}-I_r\end{bmatrix}^{-1}\begin{bmatrix} I_p & \Sigma_{22}^{\sim}\\ 0 & \Sigma_{21}^{\sim}\end{bmatrix} \tag{4}$$

In this case, all admissible controllers are given by

$$K=\mathrm{HM}(\Xi,S)$$

for some $S\in\mathcal{RH}_\infty$ satisfying $\|S\|_\infty<1$.

The unimodular matrix Ξ in $\mathcal{RH}_\infty$ satisfying Equation (4) is unique up to right multiplication by a constant non-singular real matrix $\hat\Psi$, which satisfies $\hat\Psi J_{pq}\hat\Psi^{\mathrm{T}}=J_{pq}$ (i.e. there is an infinite family of unimodular matrices Ξ solving Equation (4)).

Reference [13] describes in detail how to appropriately select a particular unimodular matrix Ξ by fixing the choice of $\Xi(j\infty)$. It is cardinal for the analysis of Section 2.1 to pin down one particular unimodular matrix Ξ that solves (4) because first-order approximations make sense only when considering the effect of small changes on the *same* unimodular matrix Ξ.

Let us now begin to quantitatively analyze the effect of adjustments in the weighting functions of an $\mathcal{H}_\infty$ design on the central controller K_{c} (which is uniquely defined by taking $S=0$ after Ξ has been pinned down at infinite frequency). Note that, strictly speaking, we could study the effect of weight changes on any controller, not necessarily the central one, as one could argue some benefits associated with non-central controllers. This could be done through an appropriate re-parameterization. The cardinal point here is that we must pick one (and just one) controller from the admissible controller set and then study the resulting changes in this controller. In this paper, we shall choose the central controller, which is simply defined by

$$K_{\mathrm{c}}:=\mathrm{HM}(\Xi,0) \tag{5}$$

for a particular choice of unimodular matrix Ξ (see [13]) that satisfies Equation (4).

Using Lemma 1 and rewriting Equation (4) for the $\mathcal{H}_\infty$ control problem posed, we obtain

$$\Xi J_{pq}\Xi^{\sim}=\Gamma(W_{i=\{1,\ldots,4\}}) \tag{6}$$

where

$$\begin{aligned}\Gamma(W_i):=&\left[\begin{pmatrix} I & 0\\ G_{22} & G_{21}\end{pmatrix}\begin{pmatrix} I & 0\\ 0 & \begin{pmatrix} W_3 & 0\\ 0 & W_1\end{pmatrix}\end{pmatrix}\right]\\ &\times\begin{bmatrix} G_{12}^{\sim}\begin{pmatrix} W_2^{\sim}W_2 & 0\\ 0 & W_4^{\sim}W_4\end{pmatrix}G_{12} & G_{12}^{\sim}\begin{pmatrix} W_2^{\sim}W_2 & 0\\ 0 & W_4^{\sim}W_4\end{pmatrix}G_{11}\begin{pmatrix} W_3 & 0\\ 0 & W_1\end{pmatrix}\\ \begin{pmatrix} W_3^{\sim} & 0\\ 0 & W_1^{\sim}\end{pmatrix}G_{11}^{\sim}\begin{pmatrix} W_2^{\sim}W_2 & 0\\ 0 & W_4^{\sim}W_4\end{pmatrix}G_{12} & \begin{pmatrix} W_3^{\sim} & 0\\ 0 & W_1^{\sim}\end{pmatrix}G_{11}^{\sim}\begin{pmatrix} W_2^{\sim}W_2 & 0\\ 0 & W_4^{\sim}W_4\end{pmatrix}G_{11}\begin{pmatrix} W_3 & 0\\ 0 & W_1\end{pmatrix}-I\end{bmatrix}^{-1}\\ &\times\left[\begin{pmatrix} I & 0\\ G_{22} & G_{21}\end{pmatrix}\begin{pmatrix} I & 0\\ 0 & \begin{pmatrix} W_3 & 0\\ 0 & W_1\end{pmatrix}\end{pmatrix}\right]^{\sim}\end{aligned} \tag{7}$$

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Suppose that the posed weighted $\mathscr{H}_\infty$ control problem is solvable and that we have a uniquely specified unimodular matrix Ξ of interest that satisfies Equation (6), the corresponding central controller K_c and the resulting closed-loop transfer function matrix $T_{zw} = \mathscr{F}_l(G, K_c)$. Then we adjust weighting function W_i by an amount ΔW_i to give $W_i^{new} := W_i + \Delta W_i$. After solving the new $\mathscr{H}_\infty$ control problem that results from this change in weight, the unimodular matrix Ξ changes to $\Xi_{new} := \Xi + \Delta\Xi$, the corresponding central controller K_c changes to $K_c^{new} := K_c + \Delta K_c$ and hence the resulting closed-loop transfer function matrix T_{zw} changes to $T_{zw}^{new} := T_{zw} + \Delta T_{zw}$.

We will now effectively construct an approximation of the mapping $\Delta W_i \mapsto \Delta\Xi \mapsto \Delta K_c \mapsto \Delta T_{zw}$ based on first-order approximations. It will be shown that this mapping is in general not memoryless in the sense that $\Delta K_c(j\omega_1)$ depends not only on $\Delta W_i(j\omega_1)$ but also on $\Delta W_i(j\omega)$ for, in principle, all $\omega \in [0, \infty)$.

Consequently, the problem considered in the sequel can be formulated as follows: 'Suppose that an $\mathscr{H}_\infty$ control problem for a generalized plant G (as described in the beginning of this section) has been solved for weights W_i $(i = 1, \ldots, 4)$ resulting in a central controller K_c. Furthermore, suppose that the same $\mathscr{H}_\infty$ control problem is now considered but with new weights $W_i^{new} := W_i + \Delta W_i$. Find an approximation of the new controller K_c^{new} as a function of ΔW_i and the variables of the first $\mathscr{H}_\infty$ control problem via some calculations rather than solving an $\mathscr{H}_\infty$ control problem from scratch'.

2.1. *The effects of weight adjustments*

Analysis of the effects of weight adjustments ΔW_i on the object Ξ is discussed below. Given that Ξ defines the central controller K_c through Equation (5), the effects of ΔW_i on K_c and T_{zw} can then be easily deduced. In this sense, we will derive linearizations of Equation (6) since it relates W_i to Ξ. An approximation for $\Delta\Xi$ is given in Theorem 1.

Theorem 1

Suppose a number $\gamma > 0$, a nominal generalized plant $G \in \mathscr{RL}_\infty$ and some square weights $W_i \in \mathscr{RH}_\infty \forall i \in \{1, \ldots, 4\}$ are given for which the normalized $\mathscr{H}_\infty$ control problem stated in Equation (2) is solvable. Let Ξ (unimodular in $\mathscr{RH}_\infty$) denote the solution of Equation (6) and force uniqueness on Ξ by pinning down $\Xi(j\infty)$ as described in [13].

Then consider the adjustment of each weight W_i by some amount ΔW_i to give corresponding new weights $W_i^{new} := W_i + \Delta W_i$ for which the new $\mathscr{H}_\infty$ control problem remains solvable. As a result of these weight changes, the selected Ξ changes to Ξ_{new} and a first-order approximation of the change $\Delta\Xi := \Xi_{new} - \Xi$ is given by

$$\Delta\Xi \approx \Xi\Phi J_{pq} \tag{8}$$

where $\Phi \in \mathscr{RH}_\infty$ satisfies $\Phi(j\infty) = \frac{1}{2}[\Xi^{-1}\Gamma(W_i^{new})\Xi^{-\sim} - J_{pq}](j\infty)$ and

$$\Phi + \Phi^\sim = \Xi^{-1}(\Gamma(W_i^{new}) - \Gamma(W_i))\Xi^{-\sim} \tag{9}$$

and $\Gamma(\cdot)$ is defined in Equation (7).

Proof

The unimodular matrix Ξ corresponding to weights W_i satisfies Equation (6), whereas the unimodular matrix Ξ_{new} corresponding to weights W_i^{new} satisfies

$$\Xi_{new} J_{pq} \Xi_{new}^\sim = \Gamma(W_i^{new}) \tag{10}$$

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Using the above equation and Equation (9), we obtain[§]

$$\begin{aligned}\Phi+\Phi^{\sim} &= \Xi^{-1}(\Gamma(W_i^{\mathrm{new}})-\Gamma(W_i))\Xi^{-\sim} = \Xi^{-1}(\Xi_{\mathrm{new}}J_{pq}\Xi_{\mathrm{new}}^{\sim}-\Xi J_{pq}\Xi^{\sim})\Xi^{-\sim}\\ &= \Xi^{-1}(\Xi+\Delta\Xi)J_{pq}(\Xi+\Delta\Xi)^{\sim}\Xi^{-\sim}-J_{pq}\\ &= \Xi^{-1}\Delta\Xi J_{pq}+J_{pq}\Delta\Xi^{\sim}\Xi^{-\sim}+\Xi^{-1}\Delta\Xi J_{pq}\Delta\Xi^{\sim}\Xi^{-\sim} \end{aligned}\tag{11}$$

Then, letting[‡] $f\in\mathscr{RH}_\infty$ with $f(j\infty)=\frac{1}{2}\Xi^{-1}\Delta\Xi J_{pq}\Delta\Xi^{\sim}\Xi^{-\sim}(j\infty)$ satisfy

$$f+f^{\sim}=\Xi^{-1}\Delta\Xi J_{pq}\Delta\Xi^{\sim}\Xi^{-\sim}$$

Equation (11) can be rearranged as

$$(\Xi^{-1}\Delta\Xi J_{pq}+f-\Phi)=-(\Xi^{-1}\Delta\Xi J_{pq}+f-\Phi)^{\sim}$$

and since the term on the left-hand side belongs to $\mathscr{RH}_\infty$ while the term on the right-hand side belongs to $\mathscr{RH}_\infty^-$, it follows that

$$\Xi^{-1}\Delta\Xi J_{pq}+f-\Phi=L \tag{12}$$

where L is a constant skew-symmetric matrix. It is easy to determine the value of L by simply reading the value of the left-hand side of Equation (12) at one frequency. That is,

$$\begin{aligned} L &= \Xi(j\infty)^{-1}\Delta\Xi(j\infty)J_{pq}+f(j\infty)-\Phi(j\infty)\\ &= [\Xi^{-1}\Delta\Xi J_{pq}+\tfrac{1}{2}\Xi^{-1}\Delta\Xi J_{pq}\Delta\Xi^{\sim}\Xi^{-\sim}-\tfrac{1}{2}(\Xi^{-1}\Gamma(W_i^{\mathrm{new}})\Xi^{-\sim}-J_{pq})](j\infty)\\ &= [\Xi^{-1}\Delta\Xi J_{pq}+\tfrac{1}{2}\Xi^{-1}\Delta\Xi J_{pq}\Delta\Xi^{\sim}\Xi^{-\sim}\\ &\quad -\tfrac{1}{2}(\Xi^{-1}\Delta\Xi J_{pq}+J_{pq}\Delta\Xi^{\sim}\Xi^{-\sim}+\Xi^{-1}\Delta\Xi J_{pq}\Delta\Xi^{\sim}\Xi^{-\sim})](j\infty)\ \text{via (11)}\\ &= \tfrac{1}{2}[\Xi^{-1}\Delta\Xi J_{pq}-J_{pq}\Delta\Xi^{\sim}\Xi^{-\sim}](j\infty) \end{aligned}\tag{13}$$

Note that if ΔW_i were all strictly proper, then $\Gamma(W_i^{\mathrm{new}}(j\infty))=\Gamma(W_i(j\infty))$ and consequently $\Xi_{\mathrm{new}}(j\infty)=\Xi(j\infty)$, in turn implying that $\Delta\Xi(j\infty)=0$ and $L=0$.

We shall now show that f is negligible compared with the other objects in Equation (12) when sufficiently small weight adjustments are considered, and also that removing the constant skew-symmetric matrix L from Equation (12) effectively amounts to picking a different Ξ_{new} that still satisfies Equation (10) from the one picked according to reference [13]. As a consequence of these arguments, Equation (12) will reduce to $\Xi^{-1}\Delta\Xi J_{pq}-\Phi\approx 0$, which yields the required result given in (8) after a simple rearrangement.

[§]Note that any self-adjoint function in $\mathscr{RL}_\infty$ can be split additively into a function in $\mathscr{RH}_\infty$ plus its adjoint. This is essentially nothing more than decomposing into partial fractions. A Lyapunov equation, which always has a unique solution, can also be used; see e.g. [14].

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To show negligibility of f compared with the other objects in Equation (12) consider that

$$
\begin{aligned}
\|f\|_\infty &\leqslant \overline{\sigma}[f(j\infty)]+2\sum_{i=1}^{N}\sigma_i(f)\\
&\leqslant \overline{\sigma}[f(j\infty)]+2N\sigma_1(f)=\frac{1}{2}\overline{\sigma}[f(j\infty)+f(j\infty)^*]+2N\inf_{\eta\in\mathscr{RH}_\infty^-}\|f+\eta\|_\infty\\
&\leqslant \left(2N+\frac{1}{2}\right)\|f+f^\sim\|_\infty=\left(2N+\frac{1}{2}\right)\|\Xi^{-1}\Delta\Xi J_{pq}\Delta\Xi^\sim\Xi^{-\sim}\|_\infty\\
&\leqslant \left(2N+\frac{1}{2}\right)\|\Xi^{-1}\|_\infty^2\|\Delta\Xi\|_\infty^2
\end{aligned}\tag{14}
$$

where $\sigma_i(\cdot)$, for $i\in\{1,2,\dots,N\}$, denote distinct Hankel singular values of $(\cdot)$ ordered as $\sigma_i(\cdot)>\sigma_{i+1}(\cdot)$ for all $i\in\{1,2,\dots,N-1\}$ ignoring multiplicities. Consequently, since $\|f\|_\infty\leqslant k\|\Delta\Xi\|_\infty^2$ for some constant k, it is clear that f is negligible compared with the other objects in Equation (12) when the weight adjustment is sufficiently small (i.e. f is a second-order term and all the other terms are at most first order).

Second, we show the effects of deleting L from Equation (12), i.e. replacing L by zero. To do this, first note that since L is small when small weight adjustments are considered (see Equation (13)), it follows that

$$(I-LJ_{pq})J_{pq}(I-LJ_{pq})^*=J_{pq}-L-L^*+LJ_{pq}L^*=J_{pq}+LJ_{pq}L^*\approx J_{pq}$$

Hence, $(I-LJ_{pq})$ is approximately J-unitary. Furthermore, consider that

$$\Xi_{\text{new}}=\Xi+\Xi(\Phi-f+L)J_{pq}\quad\text{and}\quad\hat{\Xi}_{\text{new}}=\Xi+\Xi(\Phi-f)J_{pq}$$

where the formula for Ξ_{new} is obtained from Equation (12) and $\Xi_{\text{new}}=\Xi+\Delta\Xi$ and the formula for $\hat{\Xi}_{\text{new}}$ is obtained in the same manner but replacing L with zero. We are seeking to understand how Ξ_{new} and $\hat{\Xi}_{\text{new}}$ are related. Towards this end, note that

$$
\begin{aligned}
\hat{\Xi}_{\text{new}}&=\Xi+\Xi(\Phi-f)J_{pq}=\Xi+\Xi(\Phi-f+L)J_{pq}-\Xi LJ_{pq}=\Xi_{\text{new}}-\Xi LJ_{pq}\\
&=\Xi_{\text{new}}-(\Xi_{\text{new}}-\Delta\Xi)LJ_{pq}\approx\Xi_{\text{new}}-\Xi_{\text{new}}LJ_{pq}\quad\text{(removing second-order terms)}\\
&=\Xi_{\text{new}}(I-LJ_{pq})
\end{aligned}
$$

Consequently, we have shown that $\hat{\Xi}_{\text{new}}$ is approximately equal to Ξ_{new} post-multiplied by a constant matrix $(I-LJ_{pq})$ that is approximately J-unitary and close to the identity matrix (since L is small). It is acceptable that the above argument is also based on a first-order approximation as the result of this theorem is a first-order approximation.

Given the freedom in picking any matrix Ξ_{new} that satisfies Equation (10), and not restricted to the one picked according to reference [13], the constant skew-symmetric matrix L can be removed from Equation (12) (i.e. replaced by zero). Note that Ξ was pinned down at infinite frequency as described in [13]. It is only Ξ_{new} that will be allowed to be picked differently from [13] in order

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to simplify the formulae and remove the extra constant L. It was also pointed out in [13] that right multiplication by a skew-symmetric matrix yields a reparametrization of the set of all admissible $\mathscr{H}_\infty$ controllers. Thus, the controller $\mathrm{HM}(\Xi_{\mathrm{new}}, 0)$ that we obtain when L is removed from the formulation would still be a central controller in the chain-scattering framework as it is obtained by the formula $\mathrm{HM}(\Xi_{\mathrm{new}}, 0)$, but it may not always correspond to the central minimum-entropy controller widely studied in the literature (see [13] for details). □

The following two subsections give first-order approximation changes in the central controller K_{c} and the considered closed-loop transfer functions T_{zw}. These formulae and theorems are given for completeness and insight, but are not necessary in an eventual calculation. This is because once a change in Ξ is obtained via Theorem 1, it is equally easy to compute $K_{\mathrm{c}}^{\mathrm{new}}$ and T_{zw}^{new} directly using Ξ_{new} rather than finding first-order approximations.

2.2. *The effect of weight adjustments on central controller K_{c}*

Theorem 1 provides an approximation of the difference between Ξ_{new} and Ξ. This approximation is a function of ΔW_i and the variables involved in the original (i.e. the one with weights W_i) problem. Using this result, we shall now derive an approximation for $\Delta K_{\mathrm{c}} := K_{\mathrm{c}}^{\mathrm{new}} - K_{\mathrm{c}}$ using Equation (5), which relates Ξ and K_{c}.

Theorem 2

Let the suppositions of Theorem 1 hold with Ξ partitioned as follows

$$\Xi = \begin{array}{c} \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{bmatrix} \begin{matrix} \}p \\ \}q \end{matrix} \\ \begin{matrix} \underbrace{\phantom{\Xi_{11}}}_{p} & \underbrace{\phantom{\Xi_{12}}}_{q} \end{matrix} \end{array}$$

and define the central controller as in Equation (5). Then consider the adjustment of each weight W_i by ΔW_i to give the corresponding new weights $W_i^{\mathrm{new}} := W_i + \Delta W_i$. As a result of these weight changes, K_{c} changes to $K_{\mathrm{c}}^{\mathrm{new}}$ and a first-order approximation of $\Delta K_{\mathrm{c}} := K_{\mathrm{c}}^{\mathrm{new}} - K_{\mathrm{c}}$ is given by

$$\Delta K_{\mathrm{c}} \approx -(\Xi_{11} - \Xi_{12}\Xi_{22}^{-1}\Xi_{21})\Phi_{12}\Xi_{22}^{-1} \tag{15}$$

where Φ_{12} and Φ_{21} are stable transfer function sub-matrices of Φ in (9) that solve

$$\Phi_{12} + \Phi_{21}^{\sim} = [I_p \ \ 0]\Xi^{-1}(\Gamma(W_i^{\mathrm{new}}) - \Gamma(W_i))\Xi^{-\sim} \begin{bmatrix} 0 \\ I_q \end{bmatrix} \tag{16}$$

with

$$\Phi_{12}(j\infty) = \frac{1}{2}[I_p \ \ 0]\Xi^{-1}\Gamma(W_i^{\mathrm{new}})\Xi^{-\sim} \begin{bmatrix} 0 \\ I_q \end{bmatrix} (j\infty)$$

Proof

Note that

$$K_{\mathrm{c}} = \mathrm{HM}(\Xi, 0) = \Xi_{12}\Xi_{22}^{-1} \quad \text{and} \quad K_{\mathrm{c}}^{\mathrm{new}} = \mathrm{HM}(\Xi_{\mathrm{new}}, 0) = \Xi_{12,\mathrm{new}}\Xi_{22,\mathrm{new}}^{-1}$$

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Thus, using the fact that $(A+\Delta A)^{-1} \approx A^{-1} - A^{-1}\Delta A A^{-1}$ when $\overline{\sigma}(\Delta A) \ll \underline{\sigma}(A)$ for any matrix A, we obtain

$$\begin{aligned}
K_{\mathrm{c}}^{\mathrm{new}} &= (\Xi_{12}+\Delta\Xi_{12})(\Xi_{22}+\Delta\Xi_{22})^{-1} \approx (\Xi_{12}+\Delta\Xi_{12})(\Xi_{22}^{-1} - \Xi_{22}^{-1}\Delta\Xi_{22}\Xi_{22}^{-1}) \\
&\approx \Xi_{12}\Xi_{22}^{-1} + \Delta\Xi_{12}\Xi_{22}^{-1} - \Xi_{12}\Xi_{22}^{-1}\Delta\Xi_{22}\Xi_{22}^{-1} \\
&= K_{\mathrm{c}} + \Delta\Xi_{12}\Xi_{22}^{-1} - \Xi_{12}\Xi_{22}^{-1}\Delta\Xi_{22}\Xi_{22}^{-1}
\end{aligned}$$

However, from Theorem 1,

$$\Delta\Xi_{12} \approx -[\Xi_{11} \;\; \Xi_{12}]\begin{bmatrix}\Phi_{12}\\ \Phi_{22}\end{bmatrix} \quad \text{and} \quad \Delta\Xi_{22} \approx -[\Xi_{21} \;\; \Xi_{22}]\begin{bmatrix}\Phi_{12}\\ \Phi_{22}\end{bmatrix}$$

and hence

$$\begin{aligned}
\Delta K_{\mathrm{c}} = K_{\mathrm{c,new}} - K_{\mathrm{c}} &\approx ([\Xi_{12}\Xi_{22}^{-1}\Xi_{21} \;\; \Xi_{12}] - [\Xi_{11} \;\; \Xi_{12}])\begin{bmatrix}\Phi_{12}\\ \Phi_{22}\end{bmatrix}\Xi_{22}^{-1} \\
&= -(\Xi_{11} - \Xi_{12}\Xi_{22}^{-1}\Xi_{21})\Phi_{12}\Xi_{22}^{-1}
\end{aligned}$$

The proof is concluded by observing that Equation (16) is obtained by simply selecting the $(1,2)$-block entries of Equation (9), similarly for $\Phi_{12}(j\infty)$ from $\Phi(j\infty)$. □

If the weight adjustments ΔW_i happen to be all strictly proper, then $\Phi_{12}(j\infty) = 0$ and hence the right-hand side of approximation (15) is equal to zero at infinite frequency (i.e. no change in controller at infinite frequency). It should also be pointed out that Theorem 2 is interesting only from a pedagogical (as opposed to numerical) point of view since $\Delta K_{\mathrm{c}} \approx [\mathrm{HM}(\Xi + \Xi\Phi J_{pq}, 0) - K_{\mathrm{c}}]$ is equally easy to compute and does not introduce additional approximations.

2.3. The effect of weight adjustments on the closed-loop transfer functions T_{zw}

Theorem 2 introduces an approximation of the difference between $K_{\mathrm{c}}^{\mathrm{new}}$ and K_{c} and allows us to derive an approximation for the difference between the new closed-loop transfer function matrices T_{zw}^{new} and T_{zw}. Let us choose the generalized plant G in Equation (1) such that the closed-loop transfer functions of interest are (for example) given by

$$T_{zw} := \begin{bmatrix}P\\ I\end{bmatrix}(I - K_{\mathrm{c}}P)^{-1}[-K_{\mathrm{c}} \;\; I] \tag{17}$$

Theorem 3

Let the suppositions of Theorem 2 hold and define the closed-loop transfer function matrices of interest T_{zw} as in Equation (17).

Then consider the adjustment of each weight W_i by ΔW_i to give corresponding new weights $W_i^{\mathrm{new}} := W_i + \Delta W_i$. As a result of these weight changes, T_{zw} changes to T_{zw}^{new} and a first-order approximation of the change $\Delta T_{zw} := T_{zw}^{\mathrm{new}} - T_{zw}$ is given by

$$\Delta T_{zw} \approx \begin{bmatrix}P\\ I\end{bmatrix}(I - K_{\mathrm{c}}P)^{-1}\Delta K_{\mathrm{c}}(I - PK_{\mathrm{c}})^{-1}[-I \;\; P] \tag{18}$$

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Proof
The proof trivially follows after the following algebraic manipulations:

$$\Delta T_{zw} = \begin{bmatrix} P \\ I \end{bmatrix} (I-(K_c+\Delta K_c)P)^{-1}[-(K_c+\Delta K_c) \;\; I] - \begin{bmatrix} P \\ I \end{bmatrix} (I-K_c P)^{-1}[-K_c \;\; I]$$

$$= \begin{bmatrix} P \\ I \end{bmatrix} (I-(K_c+\Delta K_c)P)^{-1}\Delta K_c (I-PK_c)^{-1}[-I \;\; P]$$

$$\approx \begin{bmatrix} P \\ I \end{bmatrix} (I-K_c P)^{-1}\Delta K_c (I-PK_c)^{-1}[-I \;\; P]$$

The last expression follows by considering just a first-order approximation. □

Clearly, Theorem 3 is relevant only if one seeks a first-order approximation of ΔT_{zw} given a change ΔK_c, since an exact calculation of T_{zw}^{new} using $K_c+\Delta K_c$ is equally easy to compute.

In the sequel, we utilize the approximation derived in Theorem 1 to propose an iterative algorithm, which guarantees quadratic convergence to the exact solution of the problem with changed weights if an initial quantity lies in a certain region of attraction. We will also explicitly derive an under-bound on the size of this region of attraction, which provides a handle on precisely what acceptable 'weight changes' are.

3. A QUADRATICALLY CONVERGENT ALGORITHM FOR WEIGHT ADJUSTMENTS

The proposed iterative algorithm with quadratic convergence is outlined below.

Algorithm 1

i. Set the counter $k=0$ and $\Xi_0=\Xi$.
ii. Set $\Gamma_k=\Xi_k J_{pq}\Xi_k^{\sim}$.
iii. Solve[¶] for $\Phi_k \in \mathscr{RH}_\infty$ with $\Phi_k(j\infty)=\frac{1}{2}[\Xi_k^{-1}\Gamma(W_i^{\text{new}})\Xi_k^{-\sim}-J_{pq}](j\infty)$ the following equation:

$$\Phi_k+\Phi_k^{\sim}=\Xi_k^{-1}(\Gamma(W_i^{\text{new}})-\Gamma_k)\Xi_k^{-\sim} \tag{19}$$

iv. Let $\Xi_{k+1}=\Xi_k(I+\Phi_k J_{pq})$.
v. If $\|\Phi_k\|_\infty \ll 1$, then EXIT. Otherwise, increment the counter k by 1 and go to Step ii.

It should be clear that the steps in this algorithm come from the equations and approximations in Theorem 1. Note that, for example, Φ_0 in Equation (19) is equivalent to Φ in Theorem 1. The

[¶]Although Step iii involves a spectrum factorization, this can be easily accomplished by using just the solution of a (linear) Lyapunov equation. This is in contrast to solving the modified problem from scratch which would involve solving two (quadratic) Riccati equations.

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stopping criterion for this algorithm is chosen as $\|\Phi_k\|_\infty \ll 1$ since this guarantees that $\Xi_{k+1} \approx \Xi_k$ (i.e. practically no improvement in the solution) through the equation $\Xi_{k+1} = \Xi_k(I+\Phi_k J_{pq})$.

On preliminary inspection of the algorithm, one may be concerned that the McMillan degrees of Φ_k and Ξ_k increase at every iteration. This is *not* the case as will be shown in the sequel.

3.1. Boundedness of the McMillan degrees of Φ_{k+1} and Φ_k

Let us first study the relationship between the McMillan degree of Φ_{k+1} and Φ_k. The technique through which we show that the proposed algorithm does not have explosion of degree is also of independent interest because it suggests a computationally simpler way of performing the algorithm. Towards this end, from Equation (19) (used with indices $k+1$ and k) and other quantities in the above algorithm, observe that

$$\begin{aligned}\Phi_{k+1}+\Phi_{k+1}^{\sim}+J_{pq} &= \Xi_{k+1}^{-1}\Gamma(W_i^{\text{new}})\Xi_{k+1}^{-\sim} = (I+\Phi_k J_{pq})^{-1}\Xi_k^{-1}\Gamma(W_i^{\text{new}})\Xi_k^{-\sim}(I+\Phi_k J_{pq})^{-\sim}\\ &= (I+\Phi_k J_{pq})^{-1}(\Phi_k+\Phi_k^{\sim}+J_{pq})(I+\Phi_k J_{pq})^{-\sim}\\ &= J_{pq}-[I-(I+\Phi_k J_{pq})^{-1}]J_{pq}[I-(I+\Phi_k J_{pq})^{-\sim}]\end{aligned}$$

and hence

$$\Phi_{k+1}+\Phi_{k+1}^{\sim} = -[I-(I+\Phi_k J_{pq})^{-1}]J_{pq}[I-(I+\Phi_k J_{pq})^{-\sim}] \tag{20}$$

Observe that Equation (20) can be used as a replacement of Equation (19) in the computation of $\Phi_k \forall k\in\mathbb{Z}_+$ (although Φ_0 still needs to be computed via Equation (19)). Given Φ_k, one could solve Equation (20) for Φ_{k+1} directly and easily using state-space data, for example, as follows: Letting

$$\Phi_k = \left[\begin{array}{c|c}\hat{A} & \hat{B}\\ \hline \hat{C} & \hat{D}\end{array}\right] \tag{21}$$

be a minimal state-space realization, it is easy to see that a state-space realization for the right-hand side of Equation (20) is given by

$$\left[\begin{array}{cc|c}\hat{A}-\hat{B}J_{pq}R^{-1}\hat{C} & (\hat{B}J_{pq}R^{-1})J_{pq}(\hat{B}J_{pq}R^{-1})^{\mathrm{T}} & -(\hat{B}J_{pq}R^{-1})J_{pq}(I-R^{-1})^{\mathrm{T}}\\ 0 & -(\hat{A}-\hat{B}J_{pq}R^{-1}\hat{C})^{\mathrm{T}} & (R^{-1}\hat{C})^{\mathrm{T}}\\ \hline R^{-1}\hat{C} & (I-R^{-1})J_{pq}(\hat{B}J_{pq}R^{-1})^{\mathrm{T}} & -(I-R^{-1})J_{pq}(I-R^{-1})^{\mathrm{T}}\end{array}\right]$$

where $R=(I+\hat{D}J_{pq})$ and $(\hat{A}-\hat{B}J_{pq}R^{-1}\hat{C})$ is Hurwitz since $(I+\Phi_k J_{pq})^{-1}\in\mathscr{RH}_\infty$.[‖] Then, there clearly always exists an X that solves

$$X(\hat{A}-\hat{B}J_{pq}R^{-1}\hat{C})^{\mathrm{T}}+(\hat{A}-\hat{B}J_{pq}R^{-1}\hat{C})X+(\hat{B}J_{pq}R^{-1})J_{pq}(\hat{B}J_{pq}R^{-1})^{\mathrm{T}}=0 \tag{22}$$

[‖]It will be shown in Section 3.2 via inequalities (39) and (42) that $\|\Phi_k\|_\infty<1\,\forall k\in\{0\}\cup\mathbb{Z}_+$ (provided the initial quantities lie in a region of attraction), which automatically guarantees that $(I+\Phi_k J_{pq})^{-1}\in\mathscr{RH}_\infty\forall k\in\{0\}\cup\mathbb{Z}_+$.

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so that a similarity transform $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$ on the above state-space realization yields

$$\Phi_{k+1} = \left[\begin{array}{c|c} \hat{A} - \hat{B}J_{pq}R^{-1}\hat{C} & -X(R^{-1}\hat{C})^{\mathrm{T}} - (\hat{B}J_{pq}R^{-1})J_{pq}(I-R^{-1})^{\mathrm{T}} \\ \hline R^{-1}\hat{C} & -\frac{1}{2}(I-R^{-1})J_{pq}(I-R^{-1})^{\mathrm{T}} \end{array}\right] \tag{23}$$

via Equation (20). It is unclear whether this last state-space realization is minimal or not. However, letting $\deg(\cdot)$ denote the McMillan degree of $(\cdot)$, we have shown that

$$\deg(\Phi_{k+1}) \leqslant \deg(\Phi_k) \quad \forall k \in \{0\} \cup \mathbb{Z}_+ \tag{24}$$

It is worth pointing out at this stage that one could use Equations (21)–(23) to calculate Φ_k at each $k \in \mathbb{Z}_+$ (except $k=0$) instead of Step iii of Algorithm 1.** This clearly is computationally more efficient and ensures that the McMillan degree of Φ_k does not increase at each iteration. In addition, if one wishes, it is also possible to calculate Φ_k offline (i.e. independently from the rest of the algorithm) for all $k \in \mathbb{Z}_+$ via Equation (20) (or equivalently via Equations (21)–(23)).

Second, let us also check that the McMillan degree of Ξ_k does not increase at each iteration. Towards this end, note that

$$\begin{aligned} \Phi_k + \Phi_k^{\sim} &= \Xi_k^{-1}\Gamma(W_i^{\text{new}})\Xi_k^{-\sim} - J_{pq} \\ &\Leftrightarrow \Xi_k \Phi_k J_{pq} + \Xi_k \Phi_k^{\sim} J_{pq} = \Gamma(W_i^{\text{new}})\Xi_k^{-\sim} J_{pq} - \Xi_k \\ &\Leftrightarrow \Xi_{k+1} = \Gamma(W_i^{\text{new}})\Xi_k^{-\sim} J_{pq} - \Xi_k \Phi_k^{\sim} J_{pq} \\ &\Leftrightarrow \Xi_{k+1} = [\Xi_k \;\; \breve{\Gamma}^s_{\text{new}}] \begin{bmatrix} -\Phi_k^{\sim} J_{pq} \\ \breve{\Gamma}^a_{\text{new}} \Xi_k^{-\sim} J_{pq} \end{bmatrix} \end{aligned} \tag{25}$$

where $\breve{\Gamma}^s_{\text{new}} \in \mathscr{RH}_\infty$ and $\breve{\Gamma}^a_{\text{new}} \in \mathscr{RH}^-_\infty$ satisfy $\breve{\Gamma}^s_{\text{new}}\breve{\Gamma}^a_{\text{new}} = \Gamma(W_i^{\text{new}})$. Note that such a decomposition is easily computed, in state-space data for example, as we do not require $\breve{\Gamma}^s_{\text{new}}$ and $\breve{\Gamma}^a_{\text{new}}$ to be square or units! It also follows easily that

$$[\Xi_{k+1} \;\; \breve{\Gamma}^s_{\text{new}}] = [\Xi_k \;\; \breve{\Gamma}^s_{\text{new}}] \begin{bmatrix} -\Phi_k^{\sim} J_{pq} & 0 \\ \breve{\Gamma}^a_{\text{new}} \Xi_k^{-\sim} J_{pq} & I \end{bmatrix} \tag{26}$$

The following lemma facilitates our discussion.

Lemma 2

Given $F, G \in \mathscr{RH}_\infty$ and $H \in \mathscr{RH}^-_\infty$ satisfying $F = GH$, then

$$\deg(F) \leqslant \deg(G)$$

Furthermore, let

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad \text{and} \quad H = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$$

**Note that the calculations giving the state-space formula for Φ_{k+1} in Equation (23) also work when the given realization for Φ_k in Equation (21) is not necessarily minimal but $\hat{A}$ is Hurwitz.

be state-space realizations with A and $(-\bar{A})$ Hurwitz. Then

$$F=\left[\begin{array}{c|c} A & B\bar{D}-X\bar{B} \\ \hline C & D\bar{D} \end{array}\right] \tag{27}$$

where matrix X satisfies $AX-X\bar{A}+B\bar{C}=0$.

Proof
Given the state-space realizations of G and H with A and $(-\bar{A})$ Hurwitz, we have

$$F=GH=\left[\begin{array}{cc|c} A & B\bar{C} & B\bar{D} \\ 0 & \bar{A} & \bar{B} \\ \hline 0 & \bar{A} & D\bar{D} \end{array}\right]$$

and it is always possible to find an X such that $AX-X\bar{A}+B\bar{C}=0$. Then, using the similarity transform $\left[\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right]$ on the state-space description of F, we obtain

$$F=\left[\begin{array}{cc|c} A & AX-X\bar{A}+B\bar{C} & B\bar{D}-X\bar{B} \\ 0 & \bar{A} & \bar{B} \\ \hline C & CX+D\bar{C} & D\bar{D} \end{array}\right]=\left[\begin{array}{c|c} A & B\bar{D}-X\bar{B} \\ \hline C & D\bar{D} \end{array}\right]+\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline CX+D\bar{C} & 0 \end{array}\right]$$

$$\Leftrightarrow F-\left[\begin{array}{c|c} A & B\bar{D}-X\bar{B} \\ \hline C & D\bar{D} \end{array}\right]=\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline CX+D\bar{C} & 0 \end{array}\right] \tag{28}$$

Since the function on the left-hand side of Equation (28) belongs to $\mathscr{RH}_\infty$ (i.e. is analytic in $\overline{\mathbb{C}}_+$) and the function on the right-hand side of Equation (28) belongs to $\mathscr{RH}_\infty^-$ (i.e. is analytic in $\overline{\mathbb{C}}_-$), it follows that the function must be analytic everywhere (i.e. all of $\mathbb{C}$) and hence needs to be a constant. By inspection, it is obvious that this constant is zero. Therefore, Equation (27) holds.

This realization gives the desired construction. Furthermore, even when the realization of G is chosen to be minimal, it is unknown whether this realization for F is minimal or not. Thus, on choosing the realization of G to be minimal, this realization for F also immediately yields the desired result $\deg(F)\leqslant\deg(G)$. □

Using Lemma 2 in Equation (25), we obtain

$$\deg(\Xi_{k+1})\leqslant\deg([\Xi_k \;\; \breve{\Gamma}^s_{\mathrm{new}}]) \tag{29}$$

and doing the same in Equation (26), we obtain

$$\deg([\Xi_{k+1} \;\; \breve{\Gamma}^s_{\mathrm{new}}])\leqslant\deg([\Xi_k \;\; \breve{\Gamma}^s_{\mathrm{new}}]) \tag{30}$$

Hence, letting $\alpha:=\deg([\Xi_0 \;\; \breve{\Gamma}^s_{\mathrm{new}}])$, it follows from inequality (30) that

$$\deg([\Xi_k \;\; \breve{\Gamma}^s_{\mathrm{new}}])\leqslant\alpha \quad \forall k\in\{0\}\cup\mathbb{Z}_+$$

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and thus through inequality (29) it follows that

$$\deg(\Xi_k)\leqslant\alpha \quad \forall k\in\mathbb{Z}_+ \tag{31}$$

Consequently, the McMillan degree of Ξ_k does not increase beyond that of $[\Xi_0\ \ \breve{\Gamma}^s_{\text{new}}]$.

It is also worth noting at this stage that the state-space construction in Equation (27) given in Lemma 2 can be used to construct a state-space realization for Ξ_k at each $k\in\mathbb{Z}_+$ that is of non-increasing order. This is clearly more advantageous than the direct computation in Step iv of Algorithm 1. In order to do this, let

$$[\Xi_k\ \ \breve{\Gamma}^s_{\text{new}}]=\left[\begin{array}{c|c}\grave{A} & \grave{B}\\ \hline \grave{C} & \grave{D}\end{array}\right] \tag{32}$$

and

$$\begin{bmatrix}-\Phi_k^{\sim}J_{pq}\\ \breve{\Gamma}^a_{\text{new}}\Xi_k^{-\sim}J_{pq}\end{bmatrix}=\left[\begin{array}{c|c}\bar{A} & \bar{B}\\ \hline \bar{C} & \bar{D}\end{array}\right] \tag{33}$$

be realizations with $\grave{A}$, $(-\bar{A})$ Hurwitz. Then, applying the state-space construction of Equation (27) in Lemma 2 on Equations (25) and (26), there exists a Y satisfying

$$\grave{A}Y-Y\bar{A}+\grave{B}\bar{C}=0 \tag{34}$$

and consequently state-space realizations for Ξ_{k+1} and $[\Xi_{k+1}\ \ \breve{\Gamma}^s_{\text{new}}]$ are given by

$$\Xi_{k+1}=\left[\begin{array}{c|c}\grave{A} & \grave{B}\bar{D}-Y\bar{B}\\ \hline \grave{C} & \grave{D}\bar{D}\end{array}\right] \tag{35}$$

and

$$[\Xi_{k+1}\ \ \breve{\Gamma}^s_{\text{new}}]=\left[\begin{array}{c|cc}\grave{A} & \grave{B}\bar{D}-Y\bar{B} & \grave{B}\begin{bmatrix}0\\ I\end{bmatrix}\\ \hline \grave{C} & \grave{D}\bar{D} & \grave{D}\begin{bmatrix}0\\ I\end{bmatrix}\end{array}\right] \tag{36}$$

These formulae, from Equations (32) to (35), can be used repeatedly to generate Ξ_k at each $k\in\mathbb{Z}_+$ as their individual components Φ_k, Ξ_k and $\breve{\Gamma}^a_{\text{new}}$ are all available.

3.2. Quadratic convergence of the algorithm

Let us study how $\Gamma_k=\Xi_kJ_{pq}\Xi_k^{\sim}$ changes at each iteration and whether it approaches $\Gamma(W_i^{\text{new}})$ as k increases. If this is the case, then it would also implicitly imply that Ξ_k approaches Ξ_{new} as k increases, which is the desired outcome from this algorithm. To this end, note that from Step iii of Algorithm 1 we have

$$\|\Xi_k^{-1}\Gamma(W_i^{\text{new}})\Xi_k^{-\sim}-J_{pq}\|_\infty\leqslant 2\|\Phi_k\|_\infty \tag{37}$$

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Therefore, $\Xi_k^{-1}\Gamma(W_i^{\mathrm{new}})\Xi_k^{-\sim} \to J_{pq}$, or equivalently $\Gamma_k = \Xi_k J_{pq} \Xi_k^{\sim} \to \Gamma(W_i^{\mathrm{new}})$, as $\Phi_k \to 0$. Consequently, we need to show only that $\Phi_k \to 0$ as $k \to \infty$, as this will in turn guarantee an algorithm solution Ξ_k, which converges to Ξ_{new} as $k \to \infty$.

Thus, to prove that the algorithm of Section 3 has the property that $\Phi_k \to 0$ as $k \to \infty$, consider the line of argument in and around Equation (14) and let $\beta = [\deg(\Xi_0^{-1}\Gamma(W_i^{\mathrm{new}})\Xi_0^{-\sim}) + \frac{1}{2}]$ to obtain

$$\begin{aligned}
\|\Phi_{k+1}\|_\infty &\leqslant \overline{\sigma}[\Phi_{k+1}(j\infty)] + 2\sum_{j=1}^{N_{k+1}} \sigma_j(\Phi_{k+1}) \leqslant \overline{\sigma}[\Phi_{k+1}(j\infty)] + 2N_{k+1}\sigma_1(\Phi_{k+1}) \\
&= \frac{1}{2}\overline{\sigma}[\Phi_{k+1}(j\infty) + \Phi_{k+1}(j\infty)^*] + 2N_{k+1} \inf_{\eta \in \mathscr{H}_\infty^-} \|\Phi_{k+1} + \eta\|_\infty \\
&\leqslant \left(2N_{k+1} + \frac{1}{2}\right) \|\Phi_{k+1} + \Phi_{k+1}^{\sim}\|_\infty \leqslant \left(2N_{k+1} + \frac{1}{2}\right) \|I - (I + \Phi_k J_{pq})^{-1}\|_\infty^2 \text{ via (20)} \\
&= \left(2N_{k+1} + \frac{1}{2}\right) \|(I + \Phi_k J_{pq})^{-1}\Phi_k\|_\infty^2 \leqslant \beta \frac{\|\Phi_k\|_\infty^2}{(1 - \|\Phi_k\|_\infty)^2}
\end{aligned} \tag{38}$$

where N_{k+1} is the number of distinct Hankel singular values of Φ_{k+1}. Note that $\beta \geqslant (2N_k + \frac{1}{2})\ \forall k \in \mathbb{Z}_+$ since the McMillan degree of Φ_k is non-increasing at each k; see (24).

We shall now use inequality (38) to show that provided the initial Φ_0 lies in a certain region of attraction, then $\|\Phi_k\|_\infty$ decreases at each k down to zero and in fact converges to zero quadratically. Thus, let

$$\|\Phi_0\|_\infty < \frac{1}{\beta\varepsilon} \quad (\leqslant 1) \tag{39}$$

where

$$\varepsilon := \frac{1}{\beta} + \frac{1}{2} + \sqrt{\frac{1}{\beta} + \frac{1}{4}} \tag{40}$$

If Φ_0 lies in the region specified by inequality (39), then it follows that

$$(1 - \|\Phi_0\|_\infty)^{-2} < \varepsilon \tag{41}$$

as $(\varepsilon/(\varepsilon - 1/\beta))^2 = \varepsilon$ from Equation (40). Now using (38), (41) and (39) and noting that the function $1/(1-x)^2$ is monotonically increasing as x increases in the interval $x \in [0, 1)$, we obtain

$$\|\Phi_{k+1}\|_\infty < \|\Phi_k\|_\infty \quad \forall k \in \{0\} \cup \mathbb{Z}_+ \tag{42}$$

$$(1 - \|\Phi_k\|_\infty)^{-2} < \varepsilon \quad \forall k \in \{0\} \cup \mathbb{Z}_+ \tag{43}$$

Hence, $\|\Phi_k\|_\infty$ decreases at each k and $\|\Phi_k\|_\infty < 1$ for all $k \in \{0\} \cup \mathbb{Z}_+$ (implying that $(I + \Phi_k J_{pq})^{-1} \in \mathscr{RH}_\infty$ is automatically guaranteed by $\Phi_k \in \mathscr{RH}_\infty$). In addition, using inequalities (38) and (43), it is easy to see that

$$\|\Phi_{k+1}\|_\infty < \beta\varepsilon \|\Phi_k\|_\infty^2 \tag{44}$$

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which in turn yields

$$\|\Phi_k\|_\infty < \frac{1}{\beta\varepsilon}(\beta\varepsilon\|\Phi_0\|_\infty)^{2^k} \quad \forall k \in \mathbb{Z}_+ \tag{45}$$

This shows that $\|\Phi_k\|_\infty \to 0$ quadratically as $k \to \infty$, as claimed!

Note that the bound on the region of attraction given in inequality (39) is very conservative due to the series of inequalities used to obtain it. One might expect that the region of attraction is much larger. Thus, the algorithm is likely to converge for initial errors outside the region identified, as the region identified establishes only a sufficient condition for convergence. It does not preclude convergence occurring for larger initial errors.

If one insists on using the proposed algorithm for an initial Φ_0 outside the guaranteed region of attraction (39) and notes that the algorithm still converges to a fixed solution, then this solution must be the correct solution Ξ_{new} even though the algorithm started outside the guaranteed region of attraction. This is because from Steps iv and iii of Algorithm 1 we can see that as $\Xi_{k+1} \to \Xi_k$ (i.e. the algorithm is converging) then $\Phi_k \to 0$ and consequently $\Gamma_k \to \Gamma(W_i^{\text{new}})$ yielding the required conclusion. Note that this observation is independent of whether the weight adjustment satisfied inequality (39) or not!

In the sequel we shall develop an expression to help us indicate directly acceptable weight changes ΔW_i that fall within the identified region of attraction given by inequality (39). By considering the line of argument in and around Equation (14) we have

$$\|\Phi_0\|_\infty \leqslant \beta\|\Phi_0 + \Phi_0^\sim\|_\infty = \beta\|\Xi_0^{-1}\Gamma(W_i^{\text{new}})\Xi_0^{-\sim} - J_{pq}\|_\infty$$

and if we require the weight adjustments ΔW_i to satisfy

$$\|\Xi_0^{-1}\Gamma(W_i^{\text{new}})\Xi_0^{-\sim} - J_{pq}\|_\infty < \frac{1}{\beta^2\varepsilon} \tag{46}$$

then inequality (39) will automatically be satisfied. This last condition given by inequality (46) helps us express directly acceptable weight changes ΔW_i that fall within the identified region of attraction given by inequality (39).

It is important to note that the first-order approximation of $\Delta\Xi$ works well in approximating the required objects when the weight changes ΔW_i satisfy $\|\Phi\|_\infty = \|\Phi_0\|_\infty \ll (1/\beta\varepsilon)$ $(\leqslant 1)$, in which case the quadratically convergent algorithm of Section 3 terminates after one pass. The aforementioned discussion stated the acceptable weight changes ΔW_i, see inequality (46), in the proposed iterative algorithm for which a sequence of these linear approximations converges quadratically to the exact solution. Furthermore, one can also envisage larger weight adjustments by dividing the large weight change into a number of weight changes, which satisfy inequality (46), for which the quadratically convergent algorithm of Section 3 can be readily utilized. Recall that if the algorithm of Section 3 converges, then it must converge to the correct solution Ξ_{new}, irrespective of whether inequality (39) is satisfied or not. Consequently, for convenience when dividing a large weight adjustment, it may sometimes be easier to simply postulate a particular weight change and check if the algorithm converges or not, without testing whether inequality (39) (or inequality (46)) is satisfied.

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4. NUMERICAL EXAMPLE

The first-order approximations in Section 2.1 are intended for online implementation of iterative adaptive control schemes that involve small weight adjustment at each sampling interval. However, for the purpose of illustration, this example will show that our approximations do indeed converge to the accurate solution when iterated or when sufficiently small changes are envisaged. For example, the windsurfing approach [15–17] entails several controller designs to gradually and safely widen the closed-loop bandwidth through repetition of a two-step procedure using identification and controller re-design steps. For each identified plant model, our proposed method can be used to make consecutive weight adjustments such that the maximum achievable closed-loop bandwidth is achieved for that model before re-identification becomes necessary.

We shall now illustrate the effectiveness of our results presented in this paper via the following example. Consider the following system

$$P(s)=\frac{10}{(s-1)(0.2s+1)} \tag{47}$$

and design an $\mathscr{H}_\infty$ controller for the system P using the following weights:

$$W_1(s)=\frac{0.1s+1}{0.003(100s+1)},\quad W_2=\frac{1}{20},\quad W_3=\frac{1}{30},\quad W_4=1 \tag{48}$$

Using a standard Riccati-based $\mathscr{H}_\infty$ control design method, we obtain the following central controller

$$K_\mathrm{c}=\frac{164.1246(s+4.993)(s+0.8347)}{(s+0.01)(s^2+46.53s+1053)} \tag{49}$$

which results in a closed-loop transfer function with $T_{21}=K_\mathrm{c}/(1+K_\mathrm{c}P)$ having a high peak as shown in Figure 1. In order to constrain this resonance peak, we shall choose the following new weight:

$$W_3^\mathrm{new}=\frac{\frac{1}{6}s+1}{6(\frac{1}{30}s+1)} \tag{50}$$

The $\mathscr{H}_\infty$ problem is solved with this new weight keeping the other three weights unmodified. This results in the following new central controller

$$K_\mathrm{c}^\mathrm{new}=\frac{27.8746(s+4.996)(s+30)(s+0.7208)}{(s+25.53)(s+0.01)(s^2+25.15s+307.7)} \tag{51}$$

with its frequency response shown in Figure 2, and the resulting new closed-loop transfer function $T_{21,\mathrm{new}}$ shown in Figure 1.

These figures provide a comparison between the modified transfer functions with the corresponding transfer function of the initial $\mathscr{H}_\infty$ problem (i.e. the one with the weights in (48)).

Now, we will utilize the results presented in this paper to accurately compute the controller $K_\mathrm{c}^\mathrm{new}$. For that, we require, as starting point, the knowledge of the matrix $\Xi(s)$ corresponding to the initial $\mathscr{H}_\infty$ control design (i.e. the one with the weights in (48)) to compute $K_\mathrm{c}^\mathrm{new}$ (see

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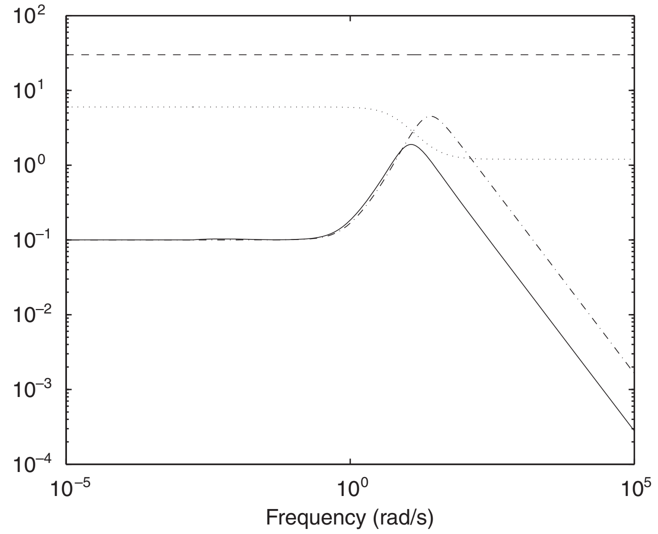

Figure 1. $|T_{21}(j\omega)|$ (dashed-dotted) and $|T_{21}^{\text{new}}(j\omega)|$ (solid), $|W_3(j\omega)^{-1}|$ (dashed), $|W_3^{\text{new}}(j\omega)^{-1}|$ (dotted).

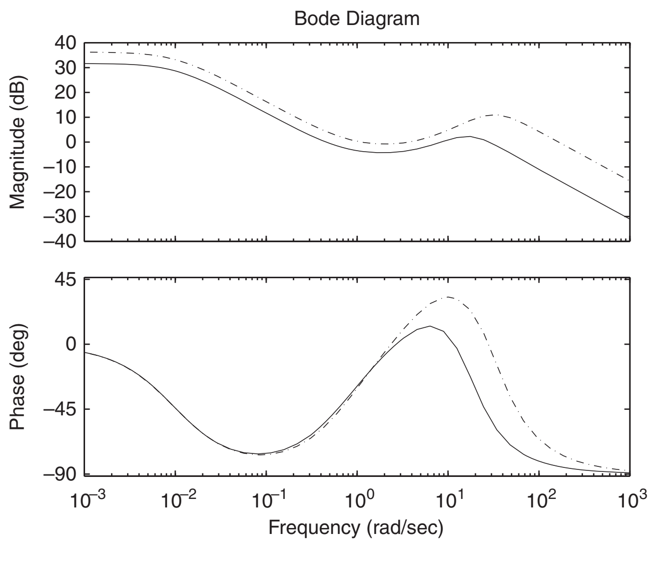

Figure 2. $|K_c(j\omega)|$ (dashed-dotted) and $|K_c^{\text{new}}(j\omega)|$ (solid).

Theorem 2). To compute the new controller, consider Figure 1 and note that the change of weight $W_3^{\text{new}} - W_3$ is quite significant. The significance of the weight change can also be evidenced from the large $\mathscr{H}_\infty$ norm of the quantity $\Phi(s)$ computed via (9) $\|\Phi(s)\|_\infty = 3003.8 \ggg 1$. In this case, to be able to compute K_c^{new}, we shall divide the weight adjustment $W_3^{\text{new}} - W_3$ into a number of

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smaller weight changes. As a first guess, we choose to adjust $W_3 = \frac{1}{30} - \frac{1}{6}$. We then verify whether the iterative algorithm of Section 3 can be used to compute the change $\Delta\Xi$ corresponding to this weight change. The algorithm does not converge, and we consequently have to reduce the weight change. This forces a weight change from $W_3 = \frac{1}{30}$ to $\frac{1}{10}$. Algorithm 1 converges in three iterations and results in an accurate matrix Ξ corresponding to a weight $W_3 = \frac{1}{10}$. We then proceed further by modifying $W_3 = \frac{1}{10} - \frac{1}{6}$, which after three iterations leads to a new matrix Ξ corresponding to $W_3 = \frac{1}{6}$. By continuing this procedure, we finally observe that the requested large weight change W_3 to W_3^{new} can be divided into four weight changes in the sense that the algorithm of Section 3 converges for each of the following intermediate weight adjustments:

$$W_{3,0} = W_3 \rightarrow W_{3,1} = \frac{1}{10} \rightarrow W_{3,2} = \frac{1}{6} \rightarrow W_{3,3} = \frac{\frac{1}{10}s+1}{6(\frac{1}{30}s+1)} \rightarrow W_{3,4} = W_3^{\text{new}} \tag{52}$$

For each of these four steps, only three iterations were necessary to obtain an accurate expression for the corresponding matrix Ξ. We obtain Ξ_{new} corresponding to the design with weight W_3^{new}, and consequently via (5) the central controller:

$$K_c^{\text{new}} = \frac{27.8673(s+0.7208)(s+4.996)(s+30)}{(s+25.53)(s+0.01)(s^2+25.15s+307.6)} \tag{53}$$

It can be readily verified that by using our results in the paper we could obtain the same controller as previously obtained via a standard Riccati-based $\mathscr{H}_\infty$ control design method.

5. CONCLUSIONS

We derived first-order approximations in Section 2.1 to shed lights on how to establish the approximate quantitative effects of a weight change on an $\mathscr{H}_\infty$ controller and the associated closed-loop transfer functions. These formulae can, for example, be very useful for online implementation of iterative adaptive control schemes that involve small weight adjustments at each sampling instance. The domain of applicability of the derived formulae was further stretched out by proposing a quadratically convergent iterative algorithm in Section 3, which enabled us to handle extended-sized weight changes. We have also shown that in this algorithm a sequence of successive iterations of the above-mentioned linear approximations leads to quadratic convergence to the exact solution. Our results here are also independent of any structure imposed on the weighting functions. Thus, adjustments of diagonal and non-diagonal weights are treated in the same framework with similar ease.

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REFERENCES

1. Bombois X, Anderson BDO. On the influence of weight modification in $\mathscr{H}_\infty$ control design. *Proceedings of the 41st IEEE Conference on Decision and Control*, Las Vegas, NV, U.S.A., December 2002; 1958–1963.
2. Graham MR, de Callafon RA. Performance weight adjustment for iterative cautious control design. *Proceedings of the European Control Conference 2007*, Kos, Greece, July 2007; 217–222.
3. Lecchini A, Lanzon A, Anderson BDO. Safe adaptive controller changes based on reference model adjustments. *Proceedings of the 43rd IEEE Conference on Decision and Control*, vol. 4, Maui, HI, U.S.A., December 2003; 4297–4302.
4. Lecchini A, Lanzon A, Anderson BDO. Safe controller changes with additional guaranteed model reference performance improvement for the unknown plant. *Proceedings of the 43rd IEEE Conference on Decision and Control*, Paradise Island, Bahamas, December 2004; 2692–2693.
5. Zhou K, Doyle JC, Glover K. *Robust and Optimal Control*. Prentice-Hall: Englewood Cliffs, NJ, 1996.
6. Kimura H. *Chain-scattering Approach to $\mathscr{H}_\infty$-control*. Birkhäuser: Basel, 1997.
7. Ball JA, Cohen N. Sensitivity minimisation in an $\mathscr{H}_\infty$ norm: parametrization of all sub-optimal solutions. *International Journal of Control* 1987; **46**:785–816.
8. Green M, Glover K, Limebeer D, Doyle J. A J-spectral factorization approach to $\mathscr{H}_\infty$ control. *SIAM Journal on Control and Optimization* 1990; **28**(6):1350–1371.
9. Kimura H, Lu Y, Kawatani R. On the structure of $\mathscr{H}_\infty$ control systems and related extensions. *IEEE Transactions on Automatic Control* 1991; **36**(6):653–667.
10. Kwakernaak H. Robust control and $\mathscr{H}_\infty$-optimization: tutorial paper. *Automatica* 1993; **29**(2):255–273.
11. Kwakernaak H. The polynomial approach to $\mathscr{H}_\infty$-optimal regulation. In *$\mathscr{H}_\infty$ Control Theory*, Mosca E, Pandolfi L (eds). Lecture Notes in Mathematics, vol. 1496. Springer: Berlin, 1990.
12. Green M. $\mathscr{H}_\infty$ controller synthesis by J-lossless coprime factorisation. *SIAM Journal on Control and Optimization* 1992; **30**:522–547.
13. Lanzon A, Bombois X, Anderson BDO. Selection of a single uniquely specifiable $\mathscr{H}_\infty$ controller in the chain-scattering framework. *Automatica* 2004; **40**(6):985–994.
14. Francis BA. *A Course in $\mathscr{H}_\infty$ Control Theory* (1st edn). Lecture Notes in Control and Information Sciences, vol. 88. Springer: Berlin, 1987.
15. Anderson BDO, Kosut RL. Adaptive robust control: on-line learning. *Proceedings of the 30th IEEE Conference on Decision and Control*, Brighton, U.K., 1991; 297–298.
16. Dehghani A, Lanzon A, Anderson BDO. An $\mathscr{H}_\infty$ algorithm for the windsurfer approach to adaptive robust control. *International Journal of Adaptive Control and Signal Processing* 2004; **18**(8):607–628.
17. Anderson BDO. Windsurfing approach to iterative control design. In *Iterative Identification and Control*, Albertos P, Sala A (eds). Springer: London, 2002; 143–166.

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