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A game theoretic algorithm to compute local stabilizing solutions to HJBI equations in nonlinear H_∞ control ☆

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ABSTRACT

In this paper, an iterative algorithm to solve Hamilton–Jacobi–Bellman–Isaacs (HJBI) equations for a broad class of nonlinear control systems is proposed. By constructing two series of nonnegative functions, we replace the problem of solving an HJBI equation by the problem of solving a sequence of Hamilton–Jacobi–Bellman (HJB) equations whose solutions can be approximated recursively by existing methods. The local convergence of the algorithm and local quadratic rate of convergence of the algorithm are guaranteed and a proof is given. Numerical examples are also provided to demonstrate the effectiveness of the proposed algorithm. A game theoretical interpretation of the algorithm is given.

1. Introduction

Traditionally, in linear H_2 control, one needs to solve LQ-type Algebraic Riccati Equations (AREs) with a negative semidefinite quadratic term; in linear H_∞ control, for a given disturbance attenuation level $\gamma > 0$, one needs to solve AREs with an indefinite quadratic term. Some iterative procedures to solve such AREs were proposed in Kleinman (1968), Lanzon, Feng, and Anderson (2007) and Lanzon, Feng, Anderson, and Rotkowitz (2008). In Kleinman (1968), a sequence of monotonically non-increasing matrices is constructed to obtain the unique stabilizing solution of an ARE with a negative semidefinite quadratic term. In Lanzon et al. (2007, 2008), an ARE with a sign-indefinite quadratic term is replaced by a sequence of AREs with a negative semidefinite quadratic term and each of them can be solved by the Kleinman algorithm in Kleinman (1968); then the solution of the original ARE can be approximated by the sum of the solutions of these AREs. In some sense, the iteration scheme in Lanzon et al. (2007, 2008) is an extension to the one in Kleinman (1968), since both algorithms are Newton algorithms and enjoy similar characteristics such as high numerical reliability, local quadratic rate of convergence (see Kleinman (1968), Lanzon et al. (2007, 2008)) and, as noted, the algorithm in Lanzon et al. (2007, 2008) can be applied in more general cases (i.e. solving AREs with an indefinite quadratic term). Interestingly, an iterative procedure to solve differential Riccati equations with a negative semidefinite quadratic term and time-varying coefficient matrices can be found in Reid (1972). After further observation, it can be seen that the algorithm in Reid (1972) has similar properties to the Kleinman algorithm in Kleinman (1968), for example, both of them obtain the solutions by constructing a series of monotonically non-increasing matrices. A natural conjecture is that the algorithm in Reid (1972) is an extension of the Kleinman algorithm in Kleinman (1968) for the linear time-varying case, but we will not explore this point more in this paper.

Although linear optimal control theory, as well as linear H_∞ control theory, has been well developed in the past decades, matters become more complicated when a nonlinear control system is considered. For example, in nonlinear optimal control, HJB equations may need to be solved to obtain an optimal control law. However, HJB equations are first order, nonlinear partial differential equations that have been proven to be impossible to solve in general and are often very difficult to solve for specific nonlinear systems. Since these equations are difficult to solve analytically, there has been much research directed toward

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approximating their solutions. For example, the technique of successive approximation in policy space (Bellman, 1957, 1971; Bellman & Kalaba, 1965) can be used to approximate the solutions of HJB equations iteratively. In fact, it can be shown (see Leake and Liu (1967)) that the technique of policy space iteration can be used to replace a nonlinear HJB partial differential equation by a sequence of linear partial differential equations. Also, in some sense, the iterative procedure to solve HJB equations in Leake and Liu (1967) is a generalization of the Kleinman algorithm in Kleinman (1968), since both of them obtain solutions by constructing a sequence of monotonic functions or matrices while the algorithm in Leake and Liu (1967) can be used in more general cases than just the LQ problem.

In nonlinear H_∞ control, given a disturbance attenuation level $\gamma > 0$, in order to solve the H_∞ suboptimal control problem, one needs to solve Hamilton–Jacobi–Bellman–Isaacs (HJBI) equations. It is clear that HJBI equations are generally more difficult to solve than HJB equations, since the disturbance inputs are additionally reflected in HJBI equations. Recall the iterative algorithm in Lanzon et al. (2007, 2008): an ARE with an indefinite quadratic term is reduced to a sequence of AREs with a negative semidefinite quadratic term, which are more easily solved by an existing algorithm (e.g. the Kleinman algorithm). If we regard HJB equations as the general version of AREs with a negative semidefinite quadratic term and HJBI equations as the general version of AREs with an indefinite quadratic term, then the question arising here is: "can we approximate the solution of an HJBI equation by obtaining the solutions of a sequence of HJB equations and thereby extend the algorithm in Lanzon et al. (2007, 2008) to nonlinear control systems?" In this paper, we will answer this question to some degree, that is, we extend the algorithm in Lanzon et al. (2007, 2008) for a broad class of nonlinear control systems and develop an iterative procedure to solve a broad class of HJBI equation associated with the nonlinear H_∞ control problem.

In fact, our motivation for this paper comes both from the "gap" between linear optimal control and nonlinear optimal control, as well as the "gap" between linear H_∞ control and nonlinear H_∞ control. Roughly speaking, the conclusions obtained in linear systems cannot always be naturally extended to nonlinear systems in general and some well-established conclusions in linear systems have not been well extended to their nonlinear counterpart. For example, in the LQ problem, to obtain optimal control laws, one needs the stabilizing solutions of LQ-type AREs and necessary and sufficient conditions for the existence of the stabilizing solutions, indeed unique stabilizing solutions, of such LQ-type Riccati equations have been well set up (see Zhou, Doyle, and Glover (1996)); in nonlinear optimal control problems, to obtain optimal control laws, one needs solutions of HJB equations, so the questions arise as to the existence of stabilizing solutions to HJB equations and the uniqueness of such solutions. Furthermore, in linear H_∞ control, one needs to solve AREs with a sign-indefinite quadratic term; in nonlinear H_∞ control, one needs to solve HJBI equations. If we compare AREs with a sign-indefinite quadratic term with HJBI equations, similar questions will arise. Although a sufficient condition for the existence of smooth H_∞ state feedback control has been given in van der Schaft (1991), there was no explicit discussion on stabilizing solutions of HJBI equations in van der Schaft (1991), except for a discussion relevant to existence of an equilibrium of an HJBI equation, and based on linearization. In this paper, to develop our proposed algorithm, we will firstly set up such some results about the stabilizing solution for both a broad class of HJB equation and a broad class of HJBI equation. Although our results are only restricted to a class of HJB equation and HJBI equation, they at least go some way in extending well-known results about the stabilizing solutions of AREs to their nonlinear counterpart (i.e. the stabilizing solutions of (HJB) HJBI equations).

One method of solving HJB and HJBI equations is the higher order expansion method (see Albrecht (1961), Hu, Yang, and Chang (1999), Huang and Lin (1995), Kang, De, and Isidori (1992) and Navasca and Krener (2000)). In this method, system equations are assumed to be analytic in the state and a Taylor series expansion of the value function is formed to approximate solutions of HJB and HJBI equations. However, there are some difficulties if a higher order expansion method is used to solve HJB and HJBI equations. For example, it is generally hard to solve for high order terms in approximations. Furthermore, as noted in Navasca and Krener (2000), the power series constructed in this method does not converge quickly on a large neighborhood of 0. In our proposed algorithm, we can show that the function sequence (which is obtained by solving a sequence of HJB equations) converges locally quadratically to the stabilizing solution of the given HJBI equation (see Section 6). Meanwhile, compared with the higher order expansion method, our proposed algorithm has a natural game theoretic interpretation which is also an advantage. In our proposed algorithm, we replace the problem of solving an HJBI equation by the problem of solving a sequence of HJB equations. As mentioned above, HJBI equations are generally more difficult to solve than HJB equations; so by using our proposed algorithm, we reduce a difficult problem to a sequence of less difficult problems. Even so, it is well known that HJB equations are in general difficult to solve and there appears to be no standard packages except the work in Mitchell (2007).

Besides the higher order expansion method, there are some other existing methods to solve HJBI equations, such as the method of characteristics (see Wise and Sedwick (1994)) and finite element method (see Bardi and Capuzzo-Dolcetta (1997) and Kushner and Dupuis (1992)). However, there are problems with each of the methods listed above (see Beard and McLain (1998)). In view of this, Beard and McLain Beard and McLain (1998) developed a new approximation method to solve HJBI equations. The basic idea of this method consists of two steps: first, Bellman's idea of approximation in policy space is used to reduce an HJBI equation to a sequence of linear partial differential equations; second, Galerkin's approximation method is used, with basis functions defined globally on some compact set, to approximate these PDEs. Although the approximation method in Beard and McLain (1998) has some advantages compared with traditional methods (method of characteristics, series approximation and finite element method) in solving HJBI equations, there are still some problems in implementing such a method in practice. For example, when the approximation method in Abu-Khalaf, Huang, and Lewis (2006); Abu-Khalaf, Lewis, and Huang (2006) and Beard and McLain (1998)is used to solve HJBI equations, it must be initialized with a stabilizing control law, but an initial stabilizing control law is not always straightforward to obtain in many situations. In contrast, our proposed algorithm in this paper can be started with a simple choice $V_0 = 0$ (see Section 5). Note however that if we use an iterative algorithm (such as that of the algorithm in Leake and Liu (1967)) to solve an HJB equation, we will still need an initial stabilizing feedback law and this is not always straightforward to obtain. In such a situation, we can regard our algorithm as a dual case of the algorithm in Abu-Khalaf, Huang, et al. (2006), Abu-Khalaf, Lewis, et al. (2006) and Beard and McLain (1998) since both our algorithm and the algorithm in Abu-Khalaf, Huang, et al. (2006), Abu-Khalaf, Lewis, et al. (2006) and Beard and McLain (1998) include an inner iteration and an outer iteration. However, the initial conditions for the algorithm in Abu-Khalaf, Huang, et al. (2006), Abu-Khalaf, Lewis, et al. (2006) and Beard and McLain (1998) are typically more difficult to obtain than those of our algorithm (assuming we use an iterative algorithm to solve HJB equations) since the execution of the algorithm in Abu-Khalaf, Huang, et al. (2006), Abu-Khalaf, Lewis, et al. (2006) and Beard and

McLain (1998) requires an additional gain boundedness condition (apart from the stabilizing condition) for initialization. Besides the advantage of a simple initialization, as already noted, our proposed algorithm has a local quadratic rate of convergence. It appears that the rate of convergence of the algorithms in Abu-Khalaf, Huang, et al. (2006), Abu-Khalaf, Lewis, et al. (2006) and Beard and McLain (1998) has not been investigated. It is possible that the algorithms in Abu-Khalaf, Huang, et al. (2006), Abu-Khalaf, Lewis, et al. (2006) and Beard and McLain (1998) also have a local quadratic rate of convergence since the structure of both of these two algorithms is similar to the structure of our algorithm. However, this paper does not contain theoretical discussion of these two algorithms.

Besides the advantages of our algorithm mentioned above, our algorithm can be supposed to have a higher accuracy and numerical stability than existing algorithms to solve HJBI equations since our algorithm in the linear time-invariant case (i.e. solving H_∞-type algebraic Riccati equations) has shown higher accuracy and numerical reliability, see Example 3 in Lanzon et al. (2008) for a demonstration of this. Meanwhile, the accuracy of our algorithm can also be illustrated by Example 2 in Section 7.

The structure of the paper is as follows. Section 2 introduces the HJBI equation we want to solve in this paper. Section 3 recalls some existing definitions and results, and then establishes some preliminary results which will be used in the main theorem. Section 4 presents the main result. Section 5 states the algorithm. Section 6 provides a result for the local quadratic rate of convergence. Section 7 presents some numerical examples. Section 8 contains concluding remarks.

2. Hamilton–Jacobi–Isaacs–Bellman equations

In this section, we introduce the HJBI equation we want to solve in this paper.

We omit standard notation in this paper for the sake of brevity and we only introduce following notation which are not standard but will be extensively used in the paper.

Define $\mathbb{X}_0$ as a neighborhood of the origin in $\mathbb{R}^n$ and define the function space $\mathcal{X}_0$ as:

$$\mathcal{X}_0 = \left\{ x : \mathbb{R}^+ \to \mathbb{X}_0 \,\middle|\, \int_{t_0}^{t_1} \|x(t)\|^2 \mathrm{d}t < \infty \quad \forall t_0, t_1 \in \mathbb{R}^+ \right\}.$$

The function spaces $\mathcal{U}_0$, $\mathcal{W}_0$ and $\mathcal{Y}_0$ are defined similarly as $\mathcal{X}_0$. The function spaces $\mathcal{X}_0$, $\mathcal{W}_0$, $\mathcal{U}_0$ and $\mathcal{Y}_0$ are indeed extended spaces defined so as to include signals with unbounded energy, as long as they have finite energy over any finite interval. We define the partial derivative of a function $V : \mathbb{R}^n \to \mathbb{R}^+$ as

$$\nabla V := \frac{\partial V(x)}{\partial x}.$$

We work with the nonlinear control system $\Gamma : \mathcal{U}_0 \times \mathcal{W}_0 \to \mathcal{Y}_0$ given by the following equations:

$$x(0) = x_0 \quad (1)$$

$$\dot{x}(t) = f(x(t)) + g_1(x(t))w(t) + g_2(x(t))u(t) \quad (2)$$

$$y(t) = h(x(t)) \quad (3)$$

where $x \in \mathcal{X}_0$ is the state; $x_0 \in \mathbb{X}_0$ is the initial state; $u \in \mathcal{U}_0$ is the input; $w \in \mathcal{W}_0$ is the disturbance; $y \in \mathcal{Y}_0$ is the output. $f : \mathbb{X}_0 \to \mathbb{R}^n$, $g_1 : \mathbb{X}_0 \to \mathbb{R}^{n\times q}$, $g_2 : \mathbb{X}_0 \to \mathbb{R}^{n\times m}$ and $h : \mathbb{X}_0 \to \mathbb{R}^p$ are smooth functions with $f(0) = 0$ and $h(0) = 0$. It is assumed further that f, g_1, g_2 are such that (2) has a unique solution for any $u \in \mathcal{U}_0$, $w \in \mathcal{W}_0$, and $x_0 \in \mathbb{X}_0$. Throughout this paper, it is further assumed that the functions f, g_1, g_2, h defined in the system Γ can be represented in the following form:

$$f(\tilde{x}) = F\tilde{x} + f_r(\tilde{x}), \quad (4)$$

$$g_1(\tilde{x}) = G_1 + g_{1r}(\tilde{x}), \quad (5)$$

$$g_2(\tilde{x}) = G_2 + g_{2r}(\tilde{x}), \quad (6)$$

$$h(\tilde{x}) = H\tilde{x} + h_r(\tilde{x}), \quad (7)$$

where $\tilde{x} := x(t)$, F, G_1, G_2, H are real constant matrices with suitable dimensions and $f_r(\tilde{x})$, $g_{1r}(\tilde{x})$, $g_{2r}(\tilde{x})$, $h_r(\tilde{x})$ are higher order remainders in power expansions.

The steady-state HJBI equation associated with the system Γ we treat in this paper is

$$\begin{aligned} 0 \;=\; & 2\,(\nabla\Pi)^{\mathrm{T}} f(\tilde{x}) + (\nabla\Pi)^{\mathrm{T}} \left(g_1(\tilde{x})g_1^{\mathrm{T}}(\tilde{x}) - g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\right)(\nabla\Pi) \\ & + h^{\mathrm{T}}(\tilde{x})h(\tilde{x}), \quad (8) \\ \Pi(0) = 0 & \end{aligned}$$

where f, g_1, g_2, h are real functions in the system Γ, $\tilde{x} \in \mathbb{X}_0$ is the state vector of the system Γ and $\Pi : \mathbb{X}_0 \to \mathbb{R}^+$ is the unique local nonnegative stabilizing solution we seek. Here, a solution of (8) is called a local ***stabilizing solution*** if this solution is such that the closed-loop of the system Γ is locally exponentially stable under the feedback inputs $u^*(t) = -g_2^{\mathrm{T}}(x(t))\,\nabla\Pi|_{\tilde{x}=x(t)}$ and $w^*(t) = g_1^{\mathrm{T}}(x(t))\,\nabla\Pi|_{\tilde{x}=x(t)}$. In such a situation, the vector field

$$\tilde{f}(\tilde{x}) = f(\tilde{x}) + g_1(\tilde{x})g_1^{\mathrm{T}}(\tilde{x})\nabla\Pi - g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla\Pi \quad (9)$$

is locally exponentially stable at the equilibrium point $x^* = 0$.

3. Definitions and preliminary results

In this section, we firstly recall some existing definitions and results in nonlinear control theory and then set up some lemmas.

The system $\Delta : \mathcal{U}_0 \to \mathcal{Y}_0$ is used in this paper, and it is defined by letting $w(t) = 0$ for all $t \geq 0$ in the system Γ.

We now define stabilizability of a matrix function pair by the stabilizability of the linear parts of this matrix function pair.

Definition 1 (*Lukes, 1969*). Let f, g_2 be the real functions defined in the system Δ and suppose (4) and (6) hold. The pair (f, g_2) is called linearly ***stabilizable*** if there exists a matrix $\hat{D}$ such that $(F + G_2\hat{D})$ is a Hurwitz matrix.

We now define three functions Θ, $\hat{f}_V$ and $\bar{f}_V$, which will be useful throughout the paper.

Definition 2. Let f, g_1, g_2, h be the real vector functions defined in the system Γ, and $\tilde{x} \in \mathbb{X}_0$ be the state value of Γ. Let $\mathbb{T}$ be the set which includes all smooth mappings from $\mathbb{X}_0$ to $\mathbb{R}$, and define $\Theta : \mathbb{T} \to \mathbb{T}$ as

$$\begin{aligned} (\Theta(V))(\tilde{x}) \;=\; & 2\,(\nabla V)^{\mathrm{T}} f(\tilde{x}) + (\nabla V)^{\mathrm{T}} \left(g_1(\tilde{x})g_1^{\mathrm{T}}(\tilde{x})\right. \\ & \left. - g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\right)(\nabla V) + h^{\mathrm{T}}(\tilde{x})h(\tilde{x}) \quad (10) \end{aligned}$$

for all $V \in \mathbb{T}$, $\tilde{x} \in \mathbb{X}_0$. Suppose there exists a local nonnegative stabilizing solution $\Pi \in \mathbb{T}$ to (8). Let $\hat{f}_V : \mathbb{X}_0 \to \mathbb{R}$ be defined as

$$\hat{f}_V(\tilde{x}) = f(\tilde{x}) + g_1(\tilde{x})g_1^{\mathrm{T}}(\tilde{x})\nabla V - g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla V$$

for all $\tilde{x} \in \mathbb{X}_0$, and let $\bar{f}_V : \mathbb{X}_0 \to \mathbb{R}$ be defined as

$$\bar{f}_V(\tilde{x}) = f(\tilde{x}) + g_1(\tilde{x})g_1^{\mathrm{T}}(\tilde{x})\nabla V - g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla\Pi$$

for all $\tilde{x} \in \mathbb{X}_0$.

We define a function U which will be used to simplify the expressions of formulas in Lemma 8 which will be extensively used in our proposed algorithm.

Definition 3. Let F, G_1, G_2, H be the real matrices appearing in (4)–(7). Define $U : \mathbb{S}^{n\times n} \to \mathbb{S}^{n\times n}$ as

$$U(\hat{Q}) = \hat{Q}F + F^{\mathrm{T}}\hat{Q} - \hat{Q}(G_2G_2^{\mathrm{T}} - G_1G_1^{\mathrm{T}})\hat{Q} + H^{\mathrm{T}}H. \quad (11)$$

In the remainder of this section, we will set up some preliminary results which will be used in our main results.

The following lemma sets up the results regarding the existence and uniqueness of the local nonnegative stabilizing solutions of a broad class of HJB equations.

Lemma 4. *Consider the system Δ, suppose (4)–(7) hold, and let $\tilde{x} \in \mathbb{X}_0$ be the state value of Δ. Let $x_0 \in \mathbb{X}_0$ be the initial state of the system Δ. Let f, g_2, h be the real vector functions defined in Δ and let F, G_2, H be the real matrices appearing in (4)–(7). If (F, G_2) is stabilizable and (H, F) is detectable, then*

(i) *there exists a unique stabilizing solution P, which is also nonnegative, to the following ARE:*

$$0 = PF + F^{\mathrm{T}}P - PG_2G_2^{\mathrm{T}}P + H^{\mathrm{T}}H. \tag{12}$$

Here, a solution P of (12) is called a ***stabilizing solution*** *of (12) if it is such that the matrix $F - G_2G_2^{\mathrm{T}}P$ is Hurwitz;*

(ii) *there exists at least locally[1] a stabilizing solution $Z(\tilde{x}) \geq 0$ to the following equation:*

$$0 = f^{\mathrm{T}}(\tilde{x})\nabla Z - \frac{1}{2}(\nabla Z)^{\mathrm{T}} g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla Z + \frac{1}{2}h^{\mathrm{T}}(\tilde{x})h(\tilde{x}) \tag{13}$$

with $0 = Z(0)$, $0 = \nabla Z|_{\tilde{x}=0}$, $P = \nabla^2 Z|_{\tilde{x}=0}$, where $P \geq 0$ is the unique stabilizing solution to (12);

(iii) *the solution $Z(\tilde{x})$ appearing in (ii) is unique and also nonnegative. Here, a solution of (13) is called the local* ***stabilizing solution*** *of (13) if it is such that the system Δ is locally exponentially stable under the input $u^*(t) = -g_2^{\mathrm{T}}(x(t))\nabla Z|_{\tilde{x}=x(t)}$.*

Proof. See Zhou et al. (1996) and Willems (1971) for (i). See van der Schaft (1999) for (ii). (iii) can be shown by using a similar contradiction argument to that in William Helton and James (1999). For brevity, the proof is omitted here. □

The following lemma gives an existence and uniqueness result regarding the HJBI equation (8).

Lemma 5. *Consider the system Γ, and let F, G_1, G_2, H be the real matrices appearing in (4)–(7). If (H, F) is detectable, and there exists a symmetric stabilizing solution K, which is also unique and nonnegative, to the following ARE:*

$$0 = KF + F^{\mathrm{T}}K - K(G_2G_2^{\mathrm{T}} - G_1G_1^{\mathrm{T}})K + H^{\mathrm{T}}H, \tag{14}$$

then there exists a local[2] solution $\Pi(\tilde{x})$, which is also unique and nonnegative, to the steady-state HJBI equation (8) for $\tilde{x} \in \mathbb{X}_0$ with $\Pi(0) = 0$, $\nabla\Pi|_{\tilde{x}=0} = 0$ and $K = \nabla^2\Pi|_{\tilde{x}=0}$. Furthermore, if there does not exist a stabilizing solution to (14), then there does not exist a smooth local stabilizing solution to (8).

Proof. See van der Schaft (1999) for the existence of $\Pi(\tilde{x})$. By using a similar argument to that for Lemma 4, we conclude that $\Pi(\tilde{x})$ is the unique smooth local stabilizing solution of (8). If there does not exist a smooth local stabilizing solution to (14), then by (4)–(6) and $\Pi(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}K\tilde{x} + \Pi_r(\tilde{x})$ (see van der Schaft (1992)), we conclude that (9) is not locally exponentially stable, hence $\Pi(\tilde{x})$ is not a smooth local stabilizing solution of (8). □

The next lemma establishes some relations that will be very useful in the proof of the main theorem.

Lemma 6. *Let f, g_1, g_2, h be the real vector functions defined in the system Γ, and $\tilde{x} \in \mathbb{X}_0$ be the state value of Γ. Let $\mathbb{T}$ and Θ be as defined in Definition 2, and let addition in $\mathbb{T}$ be defined in the obvious way. Given $V, Z \in \mathbb{T}$, then*

$$\begin{aligned}(\Theta(V+Z))(\tilde{x}) = {} & (\Theta(V))(\tilde{x}) + 2(\nabla Z)^{\mathrm{T}}\hat{f}_V(\tilde{x}) \\ & - (\nabla Z)^{\mathrm{T}}\left(g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x}) - g_1(\tilde{x})g_1^{\mathrm{T}}(\tilde{x})\right)\nabla Z.\end{aligned}$$

Furthermore, if $V, Z \in \mathbb{T}$ satisfy

$$0 = 2(\nabla Z)^{\mathrm{T}}\hat{f}_V(\tilde{x}) - (\nabla Z)^{\mathrm{T}} g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla Z + (\Theta(V))(\tilde{x}) \tag{15}$$

for all $\tilde{x} \in \mathbb{X}_0$, then

$$(\Theta(V+Z))(\tilde{x}) = (\nabla Z)^{\mathrm{T}} g_1(\tilde{x})g_1^{\mathrm{T}}(\tilde{x})\nabla Z$$

for all $\tilde{x} \in \mathbb{X}_0$.

Proof. The first result can be obtained by direct computations; the second claim is then trivial. □

The next lemma sets up some basic relationships between the local nonnegative stabilizing solution of Eq. (8) and the functions $V, Z \in \mathbb{T}$ satisfying Eq. (15).

Lemma 7. *Let f, g_1, g_2, h be the real vector functions defined in the system Γ. Let $\Theta, \mathbb{T}$ be defined in Definition 2. Let $V \in \mathbb{T}$ be of the form $V(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}A\tilde{x} + V_r(\tilde{x})$ for all $\tilde{x} \in \mathbb{X}_0$, where $V_r(\tilde{x})$ are terms of higher order than quadratic and $A \geq 0$ is an $n \times n$ symmetric matrix. Suppose there exists a local nonnegative stabilizing solution $Z(\tilde{x})$ to (13) and there exists a local nonnegative stabilizing solution $\Pi(\tilde{x})$ to Eq. (8). Suppose $V, Z \in \mathbb{T}$ satisfy Eq. (15). Let $\hat{f}_V$ be the function defined in Definition 2. Let $\Sigma : \mathbb{X}_0 \to \mathbb{R}$ and $\Sigma(\tilde{x}) = \Pi(\tilde{x}) - V(\tilde{x}) - Z(\tilde{x})$ for all $\tilde{x} \in \mathbb{X}_0$. Then*

(i) *$\Pi(\tilde{x}) \geq V(\tilde{x}) + Z(\tilde{x})$ for all $\tilde{x} \in \mathbb{X}_0$ if $x^* = 0$ is a locally exponentially stable equilibrium point of the vector field $\bar{f}_V(\tilde{x})$,*

(ii) *$x^* = 0$ is a locally exponentially stable equilibrium point of vector field $\bar{f}_{V+Z}(\tilde{x})$ if $\Pi(\tilde{x}) \geq V(\tilde{x}) + Z(\tilde{x})$ for all $\tilde{x} \in \mathbb{X}_0$.*

Proof. The proof of this lemma is an evolution of the proof of Lemma 2 in Lanzon et al. (2007, 2008) and it can be completed by using Lemma 6 and some well-known results with some algebraic manipulations. For brevity, the proof is omitted. □

The following lemma constructs two matrix sequences which will be used in our main results.

Lemma 8 (*Lanzon et al., 2007, 2008*). *Let F, G_1, G_2, H be the real matrices appearing in (4)–(7). Let U be the function defined by (11). Suppose (H, F) is detectable, (F, G_2) is stabilizable and there exists a stabilizing solution $K \geq 0$ to Eq. (14). Then*

(I) *three square matrix sequences J_k, F_k, and D_k can be defined for all $k \in \mathbb{Z}_{\geq 0}$ which satisfy*

$$J_0 = 0, \tag{16}$$

$$F_k = F + G_1G_1^{\mathrm{T}}J_k - G_2G_2^{\mathrm{T}}J_k, \tag{17}$$

D_k is the unique positive semidefinite and stabilizing solution of

$$0 = D_kF_k + F_k^{\mathrm{T}}D_k - D_kG_2G_2^{\mathrm{T}}D_k + U(J_k), \tag{18}$$

and then

$$J_{k+1} = J_k + D_k; \tag{19}$$

(II) *the series defined in part (I) have the following additional properties:*
(1) *$(F + G_1G_1^{\mathrm{T}}J_k, G_2)$ is stabilizable, $\forall k \in \mathbb{Z}_{\geq 0}$,*
(2) *$D_k \geq 0$ $\forall k \in \mathbb{Z}_{\geq 0}$,*
(3) *$U(J_{k+1}) = D_kG_1G_1^{\mathrm{T}}D_k$, $\forall k \in \mathbb{Z}_{\geq 0}$,*
(4) *$F + G_1G_1^{\mathrm{T}}J_k - G_2G_2^{\mathrm{T}}J_{k+1}$ is Hurwitz $\forall k \in \mathbb{Z}_{\geq 0}$,*
(5) *$\Pi \geq J_{k+1} \geq J_k \geq 0$, $\forall k \in \mathbb{Z}_{\geq 0}$,*
(6) *$(G_1^{\mathrm{T}}D_k, F_{k+1})$ is detectable, $\forall k \in \mathbb{Z}_{\geq 0}$;*

(III) *the limit*

$$J_\infty := \lim_{k\to\infty} J_k,$$

exists and $J_\infty = K$.

Proof. See Lanzon et al. (2007, 2008). □

[1] In this case, "locally" means around the origin.

[2] In this case, "local" means around the origin.

4. Main results

In this section, we set up the main theorem by constructing two nonnegative function series $Z_k(\tilde{x})$ and $V_k(\tilde{x})$, and we also prove that $V_k(\tilde{x})$ is monotonically increasing and converges to the unique local nonnegative stabilizing solution $\Pi(\tilde{x})$ of (8) if such a solution exists.

Theorem 9. *Consider the system Γ, and let F, G_1, G_2, H be the real matrices appearing in (4)–(7). Let $\tilde{x} \in \mathbb{X}_0$ be the state of the system Γ. Define $\Theta : \mathbb{T} \to \mathbb{T}$ as in (10). Suppose (H, F) is detectable, (F, G_2) is stabilizable and there exists a stabilizing solution $K \geq 0$ to Eq. (14). Let F_k, D_k and J_k be the matrix sequences appearing in Lemma 8. Then*

(I) *there exists a unique local nonnegative stabilizing solution $\Pi(\tilde{x})$ to Eq. (8);*

(II) *two unique real function sequences $Z_k(\tilde{x})$ and $V_k(\tilde{x})$ for all $k \in \mathbb{Z}_{\geq 0}$ can be defined recursively as follows:*

$$V_0(\tilde{x}) = 0 \quad \forall \tilde{x} \in \mathbb{X}_0, \tag{20}$$

$Z_k(\tilde{x})$ is the unique local nonnegative stabilizing solution of

$$0 = 2\hat{f}^{\mathrm{T}}_{V_k}(\tilde{x})\nabla Z - (\nabla Z)^{\mathrm{T}} g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla Z + (\Theta(V_k))(\tilde{x}), \quad \forall \tilde{x} \in \mathbb{X}_0 \tag{21}$$

with $0 = Z_k(0)$, $0 = \nabla Z|_{\tilde{x}=0}$, and then

$$V_{k+1} = V_k + Z_k; \tag{22}$$

(III) *the two series $V_k(\tilde{x})$ and $Z_k(\tilde{x})$ in part (II) have the following properties:*
  (1) *$(f(\tilde{x}) + g_1(\tilde{x})g_1^{\mathrm{T}}(\tilde{x})\nabla V_k, g_2(\tilde{x}))$ is linearly stabilizable $\forall k \in \mathbb{Z}_{\geq 0}$ $\forall \tilde{x} \in \mathbb{X}_0$,*
  (2) *$(\Theta(V_{k+1}))(\tilde{x}) = (\nabla Z_k)^{\mathrm{T}} g_1(\tilde{x})g_1^{\mathrm{T}}(\tilde{x})\nabla Z_k$ $\forall k \in \mathbb{Z}_{\geq 0}$ $\forall \tilde{x} \in \mathbb{X}_0$,*
  (3) *$f(\tilde{x}) + g_1(\tilde{x})g_1^{\mathrm{T}}(\tilde{x})\nabla V_k - g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla V_{k+1}$ is locally exponentially stable at the origin $\forall k \in \mathbb{Z}_{\geq 0}$ $\forall \tilde{x} \in \mathbb{X}_0$,*
  (4) *$\Pi(\tilde{x}) \geq V_{k+1}(\tilde{x}) \geq V_k(\tilde{x}) \geq 0$ $\forall k \in \mathbb{Z}_{\geq 0}$ $\forall \tilde{x} \in \mathbb{X}_0$,*
  (5) *$Z_k(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}D_k\tilde{x} + O_0(\tilde{x})$ $\forall k \in \mathbb{Z}_{\geq 0}$ $\forall \tilde{x} \in \mathbb{X}_0$, $V_k(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}J_k\tilde{x} + O_1(\tilde{x})$ $\forall k \in \mathbb{Z}_{\geq 0}$ $\forall \tilde{x} \in \mathbb{X}_0$, where D_k and J_k are the matrix sequences appearing in Lemma 8, and $O_0(\tilde{x})$ and $O_1(\tilde{x})$ are terms of higher order than quadratic.*

(IV) *For all $\tilde{x} \in \mathbb{X}_0$, the limit*

$$V_\infty(\tilde{x}) := \lim_{k\to\infty} V_k(\tilde{x})$$

exists with $V_\infty(\tilde{x}) \geq 0$. Furthermore, $V_\infty = \Pi$ is the unique local nonnegative stabilizing solution to (8).

Proof. See Lemma 5 for (I). We construct the series for $Z_k(\tilde{x})$ and $V_k(\tilde{x})$ to show results (II) and (III) together by an inductive argument. Firstly we show that (II) and (III) are true when $k = 0$. Then, given q such that for all $k \leq q \in \mathbb{Z}_{\geq 1}$ (II) and (III) are satisfied, we will show that (II) and (III) are also satisfied for $q + 1$.

- **Case** $k = 0$

Since $V_0(\tilde{x}) = 0$ via (16), $f(\tilde{x}) = F\tilde{x} + f_r(\tilde{x})$ by (4), $g_2(\tilde{x}) = G_2 + g_{2r}(\tilde{x})$ by (6), and (F, G_2) is stabilizable, then we obtain (III1) by Definition 1. Since (21) reduces to

$$0 = 2f^{\mathrm{T}}(\tilde{x})\nabla Z_0 - (\nabla Z_0)^{\mathrm{T}} g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla Z_0 + h^{\mathrm{T}}(\tilde{x})h(\tilde{x}), \tag{23}$$
$$0 = Z_0(0), \qquad 0 = \nabla Z_0|_{\tilde{x}=0},$$

then by Lemma 4 there exists a unique nonnegative stabilizing solution $Z_0(\tilde{x})$ to Eq. (23). Since $V_1(\tilde{x}) = V_0(\tilde{x}) + Z_0(\tilde{x})$ via (19), then we have $(\Theta(V_1))(\tilde{x}) = (\nabla Z_0)^{\mathrm{T}} g_1(\tilde{x})g_1^{\mathrm{T}}(\tilde{x})\nabla Z_0$ by Lemma 6 and (III2) is satisfied with $k = 0$. From the proof of Lemma 4, we conclude that $\left(f(\tilde{x}) - g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla Z_0\right)$ is locally exponentially stable at the origin, hence (III3) is satisfied with $k = 0$ on noting that $V_0(\tilde{x}) = 0$ and $V_1(\tilde{x}) = Z_0(\tilde{x})$ for all $\tilde{x} \in \mathbb{X}_0$. We can show that (III4) is satisfied for $k = 0$ by the following steps:

1. Since $Z_0(\tilde{x}) \geq 0$ and $V_1(\tilde{x}) = V_0(\tilde{x}) + Z_0(\tilde{x})$, then $V_1(\tilde{x}) \geq V_0(\tilde{x})$;
2. Since $K \geq 0$ is the stabilizing solution of (14), $(F - G_2G_2^{\mathrm{T}}K)$ is Hurwitz (see Zhou et al. (1996));
3. Since $V_0(\tilde{x}) = 0$, $\bar{f}_{V_0}(\tilde{x}) = f(\tilde{x}) + g_1(\tilde{x})g_1^{\mathrm{T}}(\tilde{x})\nabla V_0 - g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla\Pi = f(\tilde{x}) - g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla\Pi$;
4. Since $\Pi(\tilde{x})$ is the unique stabilizing solution of (8), $f(\tilde{x}) = F\tilde{x} + f_r(\tilde{x})$ by (4), $g_2(\tilde{x}) = G_2 + g_{2r}(\tilde{x})$ by (6), $\Pi(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}K\tilde{x} + \Pi_r(\tilde{x})$ (see van der Schaft (1992)), and $(F - G_2G_2^{\mathrm{T}}K)$ is Hurwitz, then $f(\tilde{x}) - g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla\Pi$ is locally exponentially stable at the origin on noting that $f(\tilde{x}) - g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla\Pi = (F - G_2G_2^{\mathrm{T}}K)\tilde{x} + f_{r1}(\tilde{x})$, where $f_{r1}(\tilde{x})$ are higher order terms than quadratic;
5. Since $\bar{f}_{V_0}(\tilde{x})$ is locally exponentially stable at the origin, then $\Pi(\tilde{x}) \geq (V_0(\tilde{x}) + Z_0(\tilde{x})) = V_1(\tilde{x})$ by Lemma 7 Part (i).

We can show that (III5) is satisfied for $k = 0$ by the following two points:

1. Since (F, G_2) is stabilizable, (H, F) is detectable, then $Z_0(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}D_0\tilde{x} + O_0(\tilde{x})$ by Lemma 4, where $O_0(\tilde{x})$ are higher order terms than quadratic and $D_0 \geq 0$ is the unique stabilizing solution of
$$0 = D_0F + F^{\mathrm{T}}D_0 - D_0G_2G_2^{\mathrm{T}}D_0 + H^{\mathrm{T}}H. \tag{24}$$
2. Since $V_0(\tilde{x}) = 0$ and $V_1(\tilde{x}) = V_0(\tilde{x}) + Z_0(\tilde{x})$, we have $Z_0(\tilde{x}) = V_1(\tilde{x})$. Since $Z_0(\tilde{x}) = V_1(\tilde{x})$, $J_0 = 0$ and $J_1 = D_0 + J_0$ by Lemma 8, we have $V_1(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}J_1\tilde{x} + O_1(\tilde{x})$.

- **Inductive step for** $k \in \mathbb{Z}_{\geq 1}$

Suppose that for all $k \leq q \in \mathbb{Z}_{\geq 1}$, (II) and (III) are satisfied; we now show that (II) and (III) are also satisfied for $k = q + 1$. Under inductive assumptions, firstly we show that (II) holds for $k = q + 1$ by constructing a unique local stabilizing solution $Z_{q+1}(\tilde{x}) \geq 0$ to (21) as argued in the following points:

1. Since (II) and (III) are satisfied for $k \leq q \in \mathbb{Z}_{\geq 1}$ by inductive assumptions, we have
$$Z_q(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}D_q\tilde{x} + O_2(\tilde{x}), \tag{25}$$
$$V_q(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}J_{q+1}\tilde{x} + O_3(\tilde{x}), \tag{26}$$
where $O_2(\tilde{x})$ and $O_3(\tilde{x})$ are higher order terms than quadratic.
2. Since $J_{q+1} = D_q + J_q$ by Lemma 8, $V_{q+1}(\tilde{x}) = V_q(\tilde{x}) + Z_q(\tilde{x})$ by inductive assumption, and (26) holds, we have
$$V_{q+1}(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}J_{q+1}\tilde{x} + O_4(\tilde{x}), \tag{27}$$
where $O_4(\tilde{x})$ are higher order terms than quadratic. Since (27) holds,
$\hat{f}_{V_{q+1}}(\tilde{x}) = f(\tilde{x}) + g_1(\tilde{x})g_1^{\mathrm{T}}(\tilde{x})\nabla V_{q+1} - g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla V_{q+1}$,
$f(\tilde{x}) = F\tilde{x} + f_r(\tilde{x})$ by (4), $g_1(\tilde{x}) = G_1 + g_{1r}(\tilde{x})$ by (5) and $g_2(\tilde{x}) = G_2 + g_{2r}(\tilde{x})$ by (6), we have $\hat{f}_{V_{q+1}}(\tilde{x}) = F_{q+1}\tilde{x} + O_5(\tilde{x})$, where $O_5(\tilde{x})$ are higher order terms than quadratic, and F_{q+1} is as defined in Lemma 8.
3. Since $F + G_1G_1^{\mathrm{T}}J_{q+1} - G_2G_2^{\mathrm{T}}J_{q+2}$ is Hurwitz and $J_{q+2} = J_{q+1} + D_{q+1}$ by Lemma 8, we conclude that (F_{q+1}, G_2) is stabilizable. Also, we note that $(G_1^{\mathrm{T}}D_q, F_{q+1})$ is detectable by Lemma 8.
4. Since $f(\tilde{x}) = F\tilde{x} + f_r(\tilde{x})$ by (4), $g_1(\tilde{x}) = G_1 + g_{1r}(\tilde{x})$ by (5), $g_2(\tilde{x}) = G_2 + g_{2r}(\tilde{x})$ by (6), $h(\tilde{x}) = H\tilde{x} + h_r(\tilde{x})$ by (7), $Z_q(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}D_q\tilde{x} + O_2(\tilde{x})$ by (25), (F_{q+1}, G_2) is stabilizable and $(G_1^{\mathrm{T}}D_q, F_{q+1})$ is detectable, then by Lemma 4 part (II) and (III) we conclude that there exists a unique local stabilizing solution $Z_{q+1}(\tilde{x}) \geq 0$ to the following equations:
$$0 = 2\hat{f}^{\mathrm{T}}_{V_{q+1}}(\tilde{x})\nabla Z_{q+1} - \left(\nabla Z_{q+1}\right)^{\mathrm{T}} g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla Z_{q+1} + \left(\nabla Z_q\right)^{\mathrm{T}} g_1(\tilde{x})g_1^{\mathrm{T}}(\tilde{x})\nabla Z_q, \tag{28}$$
$$0 = Z_{q+1}(0), \qquad 0 = \nabla Z_{q+1}\big|_{\tilde{x}=0}.$$
5. Since (III) are satisfied for all $k \leq q \in \mathbb{Z}_{\geq 1}$, then we have $(\Theta(V_{q+1}))(\tilde{x}) = \left(\nabla Z_q\right)^{\mathrm{T}} g_1(\tilde{x})g_1^{\mathrm{T}}(\tilde{x})\nabla Z_q$. Thus (28) recovers (III2) for $k = q + 1$.

Based on the points we mentioned above, we have shown the existence of the unique stabilizing solution $Z_{q+1}(\tilde{x}) \geq 0$ to (28) under the inductive assumptions, hence (II) is proved for $k = q + 1$. Now we show conclusions in (III) for $k = q + 1$ under the inductive assumptions. Since results (II) and (III) hold for $k \leq q \in \mathbb{Z}_{\geq 1}$ by the inductive assumptions, we have $\Pi(\tilde{x}) \geq V_{q+1}(\tilde{x})$ and $V_{q+1}(\tilde{x}) = V_q(\tilde{x}) + Z_q(\tilde{x})$ for all $\tilde{x} \in \mathbb{X}_0$, thus $\bar{f}_{V_{q+1}}(\tilde{x})$ is locally exponentially stable at the origin via Lemma 7 Part (ii). Since $\bar{f}_{V_{q+1}}(\tilde{x})$ is locally exponentially stable at the origin, $f(\tilde{x}) = F\tilde{x} + f_r(\tilde{x})$ by (4), $g_2(\tilde{x}) = G_2 + g_{2r}(\tilde{x})$ by (6), $\Pi(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}K\tilde{x} + \Pi_r(\tilde{x})$ (see van der Schaft (1992)), and $V_{q+1}(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}J_{q+1}\tilde{x} + O_3(\tilde{x})$ by (27), then we have $\bar{f}_{V_{q+1}}(\tilde{x}) = (F + G_1G_1^{\mathrm{T}}J_{q+1} - G_2G_2^{\mathrm{T}}K)\tilde{x} + O_5(\tilde{x})$, where $O_5(\tilde{x})$ are higher order terms than quadratic. Then by Lemma 2 in Lanzon et al. (2007, 2008), we conclude that $(F + G_1G_1^{\mathrm{T}}J_{q+1} - G_2G_2^{\mathrm{T}}K)$ is Hurwitz, hence $(F + G_1G_1^{\mathrm{T}}J_{q+1}, G_2)$ is stabilizable. Since $(F + G_1G_1^{\mathrm{T}}J_{q+1}, G_2)$ is stabilizable, $f(\tilde{x}) = F\tilde{x} + f_r(\tilde{x})$ by (4), $g_1(\tilde{x}) = G_1 + g_{1r}(\tilde{x})$ by (5), and $V_{q+1}(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}J_{q+1}\tilde{x} + O_3(\tilde{x})$, we conclude that $(f(\tilde{x}) + g_1(\tilde{x})g_1^{\mathrm{T}}(\tilde{x})\nabla V_{q+1}, g_2(\tilde{x}))$ is linearly stabilizable and (III1) holds for $k = q + 1$. Since $V_{q+2}(\tilde{x}) = V_{q+1}(\tilde{x}) + Z_{q+1}(\tilde{x})$, (III2) is trivially satisfied for $k = q+1$ via Lemma 6. Since (28) holds, we have $Z_{q+1}(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}D_{q+1}\tilde{x} + O_6(\tilde{x})$ (see van der Schaft (1992)), where $O_6(\tilde{x})$ are higher order terms than quadratic; then we have

$$\left(\hat{f}_{V_{q+1}}(\tilde{x}) - g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla Z_{q+1}\right) = (F_{q+1} - G_2G_2^{\mathrm{T}}D_{q+1})\tilde{x} + O_7(\tilde{x}), \quad (29)$$

where $O_7(\tilde{x})$ are higher order terms than quadratic. Since $(F_{q+1} - G_2G_2^{\mathrm{T}}D_{q+1}) = F + G_1G_1^{\mathrm{T}}J_{q+1} - G_2G_2^{\mathrm{T}}J_{q+1}$ is Hurwitz by Lemma 8, we conclude $\left(\hat{f}_{V_{q+1}}(\tilde{x}) - g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla Z_{q+1}\right)$ is locally exponentially stable at the origin by (29). Hence (III3) is satisfied for $k = q + 1$. Since $V_{q+2}(\tilde{x}) = V_{q+1}(\tilde{x}) + Z_{q+1}(\tilde{x})$ and $Z_{q+1}(\tilde{x}) \geq 0$, $V_{q+2}(\tilde{x}) \geq V_{q+1}(\tilde{x})$. Also, since $\Pi(\tilde{x}) \geq V_{q+1}(\tilde{x}) = V_q(\tilde{x}) + Z_q(\tilde{x})$ holds under inductive assumptions, it follows that $\bar{f}_{V_{q+1}}(\tilde{x})$ is locally exponentially stable at the origin via Lemma 7 Part (ii) and this in turn gives $\Pi(\tilde{x}) \geq V_{q+2}(\tilde{x})$ via Lemma 7 Part (i). Hence (III4) is satisfied for $k = q + 1$.

- **Inductive Conclusion**

Therefore, together with the base case for $k = 0$ and the inductive step, (II) and (III) are true $\forall k \in \mathbb{Z}_{\geq 0}$, hence the proof for (II) and (III) is completed.

(IV) Since the sequence $V_k(\tilde{x})$ is monotone increasing and bounded above by $\Pi(\tilde{x})$, the sequence converges to a limit $V_\infty(\tilde{x})$, and convergence of the sequence $V_k(\tilde{x})$ to $V_\infty(\tilde{x})$ implies convergence of $Z_k(\tilde{x})$ to 0 since

$$Z_\infty(\tilde{x}) := \lim_{k\to\infty} Z_k(\tilde{x}) = \lim_{k\to\infty} (V_{k+1}(\tilde{x}) - V_k(\tilde{x})) = 0.$$

Then it is clear from (III5) that $V_\infty(\tilde{x}) \geq 0$, from (III3) that $V_\infty(\tilde{x})$ is such that $(\Theta(V_\infty))(\tilde{x}) = 0$, and from (III4) that $V_\infty(\tilde{x})$ must be a function such that $\hat{f}_{V_\infty}(\tilde{x})$ is locally exponentially stable at the origin. Thus $V_\infty(\tilde{x}) \geq 0$ is a local stabilizing solution to $(\Theta(V_\infty))(\tilde{x}) = 0$. Since $\Pi(\tilde{x}) \geq 0$ is the unique local stabilizing solution of $(\Theta(\Pi))(\tilde{x}) = 0$, it is clear that $V_\infty(\tilde{x}) = \Pi(\tilde{x})$ via Lemma 5. □

From Theorem 9 (III1), we know that we can check the existence of the local nonnegative stabilizing solution of (8) by checking the stabilizability of a matrix function pair. By Definition 1, we can check the stabilizability of a matrix function pair by checking its linear part. Hence we have the following corollary which gives a condition under which there does *not* exist a local stabilizing solution $\Pi(\tilde{x}) \geq 0$ to $\Theta(\Pi) = 0$. This is useful for terminating the recursion in finite iterations.

Corollary 10. *Let F, G_1, G_2, H be the real matrices appearing in (4)–(7). Let J_k be the matrix sequence appearing in Lemma 8. Suppose that (H, F) is detectable and (F, G_2) is stabilizable. Let $\tilde{x} \in \mathbb{X}_0$ be the state of the system Γ. Define $\Theta : \mathbb{T} \to \mathbb{T}$ as in Definition 2. If $\exists k \in \mathbb{Z}_{\geq 0}$ such that $(F + G_1G_1^{\mathrm{T}}J_k, G_2)$ is not stabilizable, then there does not exist a local nonnegative stabilizing solution to $\Theta(\Pi) = 0$.*

Proof. If $\exists k \in \mathbb{Z}_{\geq 0}$ such that $(F + G_1G_1^{\mathrm{T}}J_k, G_2)$ is not stabilizable, then by Lemma 8 Part (II1), there does not exist a stabilizing solution to Eq. (14). Then by Lemma 5, there does not exist a local nonnegative stabilizing solution to $\Theta(\Pi) = 0$. □

It is possible to give a game theoretic interpretation of the algorithm, along the same lines as for the linear version of the problem studied in Lanzon et al. (2007, 2008).

5. Algorithm

Let f, g_1, g_2, h be the real functions defined in the system Γ and suppose (4)–(7) hold. Let J_k be the matrix sequence appearing in Lemma 8. Let $\hat{f}_V$ be defined in Definition 2. Suppose (F, G_2) is stabilizable and (H, F) is detectable; an iterative algorithm for finding the local nonnegative stabilizing solution of Eq. (8) is given as follows:

(1) Let $V_0 = 0$ and $k = 0$.
(2) Construct (for example using the algorithm in Albrecht (1961) and Navasca and Krener (2000), though this is not necessary) the unique local nonnegative stabilizing solution $Z_k(\tilde{x})$ which satisfies

$$0 = 2\hat{f}_{V_k}^{\mathrm{T}}(\tilde{x})\nabla Z_k - (\nabla Z_k)^{\mathrm{T}} g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla Z_k + (\Theta(V_k))(x), \quad (30)$$

with $0 = Z_k(0)$, $0 = \nabla Z_k|_{\tilde{x}=0}$, where Θ is defined in Definition 2.
(3) Set $V_{k+1} = V_k + Z_k$.
(4) Rewrite $Z_k(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}D_k\tilde{x} + O_0(\tilde{x})$ (note that this is always possible from Theorem 9 if $Z_k(\tilde{x})$ exists), where $O_0(\tilde{x})$ are terms of higher order than quadratic and $D_k \geq 0$ is the matrix sequence appearing in Lemma 8.
(5) If $\overline{\sigma}(D_k) < \mu$ where μ is a specified accuracy, then set $\Pi = V_{k+1}$ and exit. Otherwise, go to step 6.
(6) If $(F + G_1G_1^{\mathrm{T}}J_k, G_2)$ is stabilizable, then increment k by 1 and go back to step 2. Otherwise, exit as there does not exist a local nonnegative stabilizing solution Π satisfying $\Theta(\Pi) = 0$.

From Corollary 10 we see that if the stabilizability condition in step 6 fails for some $k \in \mathbb{Z}_{\geq 0}$, then there does not exist a local nonnegative stabilizing solution Π to $\Theta(\Pi) = 0$ and the algorithm should terminate (as required by step 5). But when this stabilizability condition is satisfied $\forall\, k \in \mathbb{Z}_{\geq 0}$, construction of the series $V_k(\tilde{x})$ and $Z_k(\tilde{x})$ is always possible and either $V_k(\tilde{x})$ converges to $\Pi(\tilde{x})$ (which is captured by step 5) or $V_k(\tilde{x})$ just diverges to infinity, which again means that there does not exist a stabilizing solution $\Pi(\tilde{x}) \geq 0$ to (8).

6. Rate of convergence

The following theorem states that the local rate of convergence of the algorithm given in Section 5 is quadratic.

Theorem 11. *Given the suppositions of Theorem 9, and two real function series $V_k(\tilde{x})$, $Z_k(\tilde{x})$ as defined in Theorem 9 Part (II), then there exists a $\eta > 0$ such that the rate of convergence of the series $V_k(\tilde{x})$ is quadratic in the region $\sup_{\tilde{x}\in\mathbb{X}_0} \|V_k(\tilde{x}) - \Pi(\tilde{x})\| < \eta$.*

Proof. The proof is in parallel with the counterpart in Lanzon et al. (2008) and it is omitted here due to space limitations. □

7. Numerical examples

In this section, two examples will be given. Example 1 shows that our algorithm works well for nonlinear systems.

Example 2 provides a numerical comparison between the method of characteristics (Wise & Sedwick, 1994) and our algorithm to solve an HJBI equation arising in nonlinear systems, and it shows that our proposed algorithm converges faster than the method of characteristics for this particular example.

Example 1. We consider the following system

$$\begin{pmatrix}\dot{\tilde{x}}_1\\ \dot{\tilde{x}}_2\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\end{pmatrix}+\begin{pmatrix}\frac{\sqrt{2}}{2}\tilde{x}_1\\ 0\end{pmatrix}w+\begin{pmatrix}\frac{\sqrt{2}}{2}\\ 0\end{pmatrix}u$$

$$y=\sqrt{2}\begin{pmatrix}\tilde{x}_1 & \tilde{x}_2\end{pmatrix}^{\mathrm{T}}.$$

With obvious definition of $F,\ G_1,\ G_2,\ H,\ f,\ g_1,\ g_2,\ h$, the HJBI equation is

$$0=2(\nabla\Pi)^{\mathrm{T}}F\tilde{x}+\frac{1}{2}(\nabla\Pi)^{\mathrm{T}}\left(g_1(\tilde{x})g_1^{\mathrm{T}}(\tilde{x})-G_2G_2^{\mathrm{T}}\right)(\nabla\Pi)$$
$$+\tilde{x}^{\mathrm{T}}H^{\mathrm{T}}H\tilde{x},\quad \Pi(0)=0. \tag{31}$$

We use our proposed algorithm to solve (31) recursively by solving a sequence of HJB equations. For this particular example, in the first iteration, we have $Z_0(\tilde{x})=\frac{1}{2}\tilde{x}^{\mathrm{T}}D_0\tilde{x}$, where $D_0=\mathrm{diag}\{0.8284,\ 0.5000\}$. Since

$$\hat{f}_{Z_0}(\tilde{x})=\begin{pmatrix}0.4142\tilde{x}_1^3-1.4142\tilde{x}_1\\ -2\tilde{x}_2\end{pmatrix}$$

and $(\Theta(V_1))(\tilde{x})=0.3431\tilde{x}_1^4$, the second HJB equation we want to solve is

$$0=2\hat{f}_{Z_0}^{\mathrm{T}}(\tilde{x})\nabla Z_1-(\nabla Z_1)^{\mathrm{T}}g_2(\tilde{x})g_2^{\mathrm{T}}(\tilde{x})\nabla Z_1+0.3431\tilde{x}_1^4. \tag{32}$$

By using the method in Albrecht (1961) and Navasca and Krener (2000), we obtain the approximation solution of (32) is $Z_1(\tilde{x})=0.0303\tilde{x}_1^4$ and $D_1=0$. Then by our algorithm, we can obtain an approximate solution of (31) by

$$V_2(\tilde{x})=V_1(\tilde{x})+Z_1(\tilde{x})=0.4142\tilde{x}_1^2+0.25\tilde{x}_2^2+0.0303\tilde{x}_1^4. \tag{33}$$

In fact, if we take (33) into the right-hand side of (31) by letting $\Pi(\tilde{x})=V_2(\tilde{x})$, after some straightforward computation, the right-hand side of (31) becomes

$$0.003\tilde{x}_1^4+0.0931\tilde{x}_1^6+0.0073\tilde{x}_1^8. \tag{34}$$

It is clear that the value of (34) is small when $\tilde{x}_1$ is around the origin, which means we can use $V_2(\tilde{x})$ to approximate $\Pi(\tilde{x})$ locally. In fact, if we set the tolerance $\mu=0.1$, after two iterations, we obtain $\overline{\sigma}(D_1)=0<\mu$ and the iteration is stopped.

Example 2. In Wise and Sedwick (1994), the method of characteristics is used to solve HJBI equations recursively. The following example comes from van der Schaft (1992), and it illustrates the proposed algorithm outperforming the method of characteristics in Wise and Sedwick (1994) when solving HJBI equations. The comparison is possible because in this particular case, we are able to obtain the exact solution of the HJBI equation. The scalar system is given by

$$\dot{x}(t)=u(t)+x(t)w(t) \tag{35}$$

with output $y(t)=x(t)$. For this example, we have $f(\tilde{x})=0$, $g_1(\tilde{x})=\tilde{x},\ g_2(\tilde{x})=1,\ h(\tilde{x})=1,\ F=0,\ G_1=0,\ G_2=1,\ H=1$ and it is clear that (F,G_2) is stabilizable and (H,F) is detectable. Now the steady-state HJBI equation becomes

$$\tilde{x}^2-(\nabla\Pi)^2(1-\tilde{x}^2)=0 \tag{36}$$

with $\Pi(0)=0$. We have (without any approximation)

$$\nabla\Pi=\pm\frac{\tilde{x}}{\sqrt{1-\tilde{x}^2}}\quad\forall\tilde{x}\in(-1,1),\quad\Pi(0)=0. \tag{37}$$

a. Exact solution

Since $\Pi(0)=0$ and we seek the solution for which $\Pi(\tilde{x})\geq 0$ in a neighborhood of the origin, we have

$$\nabla\Pi=\frac{\tilde{x}}{\sqrt{1-\tilde{x}^2}} \tag{38}$$

for $-1<\tilde{x}<1$. Now the closed-loop saddle point solution for the system (35) is $u^*(\tilde{x})=-\frac{\tilde{x}}{\sqrt{1-\tilde{x}^2}}$, $w^*(\tilde{x})=\frac{\tilde{x}^2}{\sqrt{1-\tilde{x}^2}}$ and the closed-loop of the system (35) under the saddle point inputs u^* and w^* is

$$\dot{\tilde{x}}=-\tilde{x}\sqrt{1-\tilde{x}^2} \tag{39}$$

for $-1<\tilde{x}<1$. Then it is clear that $x^*=0$ is a local stable equilibrium point for the system (39). We approximate the value of $\Pi(\tilde{x})$ by approximating the value of $\nabla\Pi$. From (38), we know that the value of $\Pi(\tilde{x})$ is symmetric about the origin. In view of this, we only approximate the value of $\nabla\Pi$ for $0\leq\tilde{x}<1$ in the following. The exact solution of $\nabla\Pi$ in (36) can be approximated by both our algorithm and the method of characteristics in Wise and Sedwick (1994).

b. Algorithm of this paper

To approximate $\nabla\Pi$ in (36), we carry out our proposed algorithm from Section 5. For convenience, we denote $(\cdot)_{k,\tilde{x}}=\nabla(\cdot)_k$ in the following for $k=0,1,2,3$. After a straightforward computation, we obtain the first three approximations $V_{1,\tilde{x}},V_{2,\tilde{x}},V_{3,\tilde{x}}$ of $\nabla\Pi$ in (36) as follows:

$$V_{1,\tilde{x}}=Z_{0,\tilde{x}}=\tilde{x},$$
$$Z_{1,\tilde{x}}=\tilde{x}^3-\tilde{x}+\tilde{x}\sqrt{\tilde{x}^4-\tilde{x}^2+1},$$
$$V_{2,\tilde{x}}=\tilde{x}^3+\tilde{x}\sqrt{\tilde{x}^4-\tilde{x}^2+1},$$
$$Z_{2,\tilde{x}}=f_2+\sqrt{f_2^2+\tilde{x}^2Z_{1,\tilde{x}}^2},$$
$$V_{3,\tilde{x}}=\tilde{x}^5+\tilde{x}^3\sqrt{\tilde{x}^4-\tilde{x}^2+1}+\sqrt{f_2^2+\tilde{x}^2Z_{1,\tilde{x}}^2},$$

where $f_2=\tilde{x}^5-\tilde{x}^3+(\tilde{x}^3-\tilde{x})\sqrt{\tilde{x}^4-\tilde{x}^2+1}$.

c. Algorithm of characteristics

If we use the method in Wise and Sedwick (1994) to approximate the local nonnegative stabilizing solution $\Pi(\tilde{x})$ to the HJBI equation (16), the first three approximations $\overline{V}_{1,\tilde{x}},\overline{V}_{2,\tilde{x}},\overline{V}_{3,\tilde{x}}$ of $\nabla\Pi$ in (36) are

$$\overline{V}_{1,\tilde{x}}=\tilde{x},$$
$$\overline{V}_{2,\tilde{x}}=\tilde{x}+\frac{1}{2}\tilde{x}^3,$$
$$\overline{V}_{3,\tilde{x}}=\tilde{x}+\frac{1}{2}\tilde{x}^3+\frac{7}{16}\tilde{x}^5+\frac{9}{80}\tilde{x}^7+\frac{437}{53760}\tilde{x}^9.$$

We plot these approximations together in Fig. 1 (we ignore the first approximations for both algorithms since they are identical) to compare their convergence to the "Exact Solution", which is given by (38).

From Fig. 1, we can see that our algorithm has better accuracy than the method of characteristics in Wise and Sedwick (1994), noting in particular the following points:

1. For both the 2nd approximation and the 3rd approximation, our algorithm is more accurate than the method in Wise and Sedwick (1994).
2. The 2nd approximation (dotted line) of our algorithm is very close to the 3rd approximation (dashed line) of the method in Wise and Sedwick (1994).
3. The 3rd approximation of our algorithm (thin solid line) is very close to the exact solution (thick solid line).

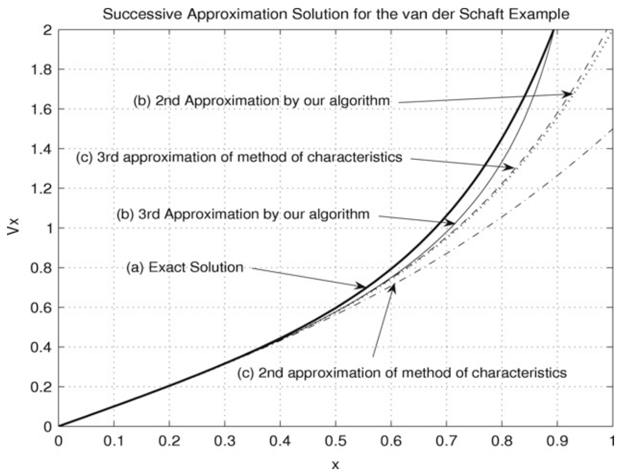

Fig. 1. Demonstration and comparison of algorithm.

8. Concluding remarks and future work

In this paper, we have developed an iterative procedure to solve HJBI equations associated with broad classes of nonlinear system and broad classes of performance indices. Under some suitable assumptions, we can compute local nonnegative stabilizing solutions of HJBI equations recursively. We focused on a particular class of problem arising in H_∞ control; however, our algorithm may be effective for broader classes of systems and performance indices. It is expected that the algorithm developed here could be extended to more general classes of systems and performance indices. Furthermore, it might also be possible that our algorithm in this paper can be extended to time-varying systems.

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