Automatica 45 (2009) 1058–1065

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Minimization of the effect of noisy measurements on localization of multi-agent autonomous formations ☆

Iman Shames *, Barış Fidan, Brian D.O. Anderson

Australian National University, Canberra, ACT 0200, Australia
National ICT Australia, Canberra, ACT 0200, Australia

ARTICLE INFO

Article history:
Received 28 October 2007
Received in revised form
13 November 2008
Accepted 20 November 2008
Available online 8 January 2009

Keywords:
Localization
Multi-agent systems
Sensor networks

ABSTRACT

This paper considers the problem of reduction of self-localization errors in multi-agent autonomous formations when only distance measurements are available to the agents in a globally rigid formation. It is shown that there is a relationship between different selections of anchors, agents with exactly known positions, and the error induced by measurement error on localization solution. This fact is exploited to develop a mechanism to select anchors in order to minimize the effects of inter-agent distance measurement errors on localization solution. Finally, some simulation results are presented to demonstrate the optimal anchor selection for a particular general class of formations, the globally rigid formations.

1. Introduction

The topic of multi-agent formations has gained much attention in recent years. In order to be able to accomplish most of the tasks associated with multi-agent autonomous formations, such as reconnaissance and surveillance, the formation should be able to determine its position in a known global coordinate system. For example in surveying an unknown territory, the formation should be able to localize itself in a known global coordinate system in order to successfully record the acquired data in the mission (such as the location of emitters, sensed perhaps using angle of arrival or time-difference of arrival, see Applewhite (2002) and Schau and Robinson (1987) schemes) and make it possible for the data to be used when the formation has returned to the base. A trivial solution for this problem might be obtained by installing a Global Positioning System (GPS) sensor on each of the agents in the formation. But due to the fact that a precise GPS sensor is expensive and/or constitutes a weight burden, this solution may be impractical. In order to solve the aforementioned problem of determining position information of the agents within the formation, the tools used in the field of multi-agent system localization can be employed. Localization problems have been well studied in the context of wireless sensor networks. In sensor network localization, it is typically assumed that a small fraction of sensors, called *anchors*, have *a priori* information about their global coordinates (Mao, Fidan, & Anderson, 2006). Exploiting the fact that the position of these anchor nodes are known in a global coordinate system and that a number of inter-node distances are known, all the other nodes in the network can be localized under a condition which will be discussed later in this paper, i.e. global rigidity of the underlying graph of the network. We can carry over this idea to a formation of mobile agents. We designate some agents as anchor agents, and use them (together with inter-agent range measurements, typically obtained from timing information in inter-agent communications) to localize other agents in the formation. In this way one can perform localization with a smaller number of accurate GPS sensors.

While in principle, any choice of at least three noncollinear agents in a two-dimensional formation or four non-coplanar agents in a three-dimensional formation can be made for anchors (given also enough inter-agent distance measurements), in fact there is a nontrivial choice to be made. This is because, when agent distances are not exactly known but rather measured with some error, the localization algorithm for the non-anchor agents will inevitably give erroneous positions, with the error depending on the inter-agent distance errors *and also the choice of agents to serve as anchors*. While there exist many studies addressing noisy localization, e.g. Barooah and Hespanha (2007) and Olfati-Saber (2007), none has considered the effect of the different choices for selection of the anchors on localization accuracy. However, there exist

☆ The material in this paper was partially presented at IFAC World Congress, Seoul, South Korea, July 2008. This paper was recommended for publication in revised form by Associate Editor Andrey V. Savkin under the direction of Editor Ian R. Petersen.

* Corresponding address: Department of Information Engineering Research, School of Information Science and Engineering, Australian National University, Bldg 115, 0200 Canberra, Australia. Tel.: +61 2 61258685; fax: +61 2 61258660.

E-mail addresses: Iman.Shames@anu.edu.au (I. Shames), Baris.Fidan@anu.edu.au (B. Fidan), Brian.Anderson@anu.edu.au (B.D.O. Anderson).

0005-1098/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2008.11.018

other related studies which consider different types of sensor selection problem; for more information see Bian, Kempe, and Govindan (2006) and Joshi and Boyd (in press) and the references therein.

The main contributions of this paper are, firstly, to introduce a criterion to measure the effect of distance measurement error on the localization of agents, and, second, to introduce a methodology (including an algorithm) for selection of anchors among a formation of agents with a view to minimizing that error. For the time being, only planar formations are taken into consideration. In addition, the current study only deals with errors originating from inter-agent distance measurements.

The paper is organized as follows. In the next section graph theoretic preliminaries relevant to the localization problem are described. In the third section calculation of error statistics given a set of anchors and a criterion for selection of anchors are presented. The fourth section contains a method for choosing a pre-fixed number of anchors to minimize the errors; some simulation results are presented in this section as well. In Section 5 the application of convex optimization in anchor selection is presented and two numerical examples using the proposed algorithm are presented later in that section. In the last section some concluding remarks are presented.

2. Rigidity, global rigidity and sensor network localization

Use of the graph rigidity notions in problems of control of formations and network localization is well described in Anderson, Yu, Fidan, and Hendrickx (2008), Eren et al. (2004) and Mao et al. (2006). In these works, each agent of a formation is represented by a vertex in a graph. When localization is occurring, any agent pair (or sensor pair) between which the distance is known gives rise to an edge in the graph. In fact, it is shown in Eren et al. (2004) that in order to localize a two-dimensional formation, it is needed that its underlying graph is *generically globally rigid*, which is defined later in this section, and there must be at least three noncollinear anchors with exactly known positions in the formation.

To provide the necessary background, some of the key tools used in establishing the results and the related graph theory concepts are now briefly explained. For the sake of simplicity, we use sensor network terminology throughout this paper, noting that this mild abuse of language does not affect the results produced for formations of autonomous agents. A network N is represented by a graph coordinates set pair (G, Π) where the graph $G = (V, E)$ represents the inter-agent sensing topology of N, $V \triangleq \{v_i\}_{i=1}^{|V|}$ is the set of vertices and E is the set of edges in G, each vertex v_i representing a sensor node in N and $e_{ij} \in E$ denoting the edge connecting the vertices v_i and v_j, an edge being incident on vertices representing two nodes (agents) just when they can measure the Euclidean distance between each other or the distance between these two nodes is precisely known, $\Pi \triangleq \{\pi_i\}_{i=1}^{|V|}$ is the set of coordinates for the agents (nodes) in the network, each π_i being a 2-vector. The graph G is called the *underlying graph* of N.

Two networks (G, Π) and (G, Π'), where $G = (V, E)$, are *equivalent* if $\|\pi_i - \pi_j\| = \|\pi_i' - \pi_j'\|$ for any vertex pair $v_i, v_j \in V$, for which $e_{ij} \in E$. The two networks (G, Π) and (G, Π'), where $G = (V, E)$, are *congruent* if $\|\pi_i - \pi_j\| = \|\pi_i' - \pi_j'\|$ for any vertex pair $v_i, v_j \in V$, whether or not $e_{ij} \in E$. This means that if (G, Π') and (G, Π) are congruent then (G, Π') can be obtained from (G, Π) applying a combination of translation, rotation and reflection only. A network (G, Π) is called rigid if there exists a sufficiently small positive constant ε such that if (G, Π') is equivalent to (G, Π) and $\|\pi_i - \pi_i'\| < \varepsilon$ for all $v_i \in V$ then (G, Π') is congruent to (G, Π). A network (G, Π) is *globally rigid* if every network which is equivalent to (G, Π) is congruent to (G, Π). It is easy to see that if G is a complete graph then the network (G, Π) is necessarily globally rigid.

An approach to characterizing the rigidity and global rigidity of a formation uses the concept of the *rigidity matrix*. We will use this concept also to establish some of the main results of this paper.

Consider a network (G, Π) in $\mathbb{R}^2$ with the underlying graph $G = (V, E)$. Let the coordinates of vertex v_j be $\pi_j = (x_j, y_j)^\top$. The rigidity matrix is defined with an arbitrary ordering of the vertices and edges, and has $2|V|$ columns and $|E|$ rows. Each edge gives rise to a row, and if the edge links vertices v_j and v_k, the nonzero entries of the row of the matrix are in columns $2j - 1$, $2j$, $2k - 1$ and $2k$, and are respectively $x_j - x_k, y_j - y_k, x_k - x_j, y_k - y_j$.

We call a graph *generically rigid* if for any generic[1] set of vertex position coordinates, Π, the associated network is rigid. The rank of the matrix R contains information about the rigidity of the network, or the generic rigidity of the underlying graph G. The key result is:

Theorem 1 (Tay and Whiteley (1985)). *A graph $G = (V, E)$ modeling a network in $\mathbb{R}^2$ of $|V|$ vertices and $|E|$ edges is rigid if and only if for generic vertex positions, the rigidity matrix has rank $2|V| - 3$.*

In more detail, suppose that $e_{ij} \in E$, so that the coordinates of vertices i, j in the network obey for all time

$$\|\pi_i(t) - \pi_j(t)\|^2 = d_{ij}^2 \tag{1}$$

where d_{ij} is the actual (constant) distance between vertices i and j, and $\|.\|$ denotes the 2-norm of a vector. Assuming motion is smooth, it follows that

$$[\pi_i(t) - \pi_j(t)]^\top [\dot{\pi}_i(t) - \dot{\pi}_j(t)] = 0 \tag{2}$$

and by stacking together $|E|$ such equations there results

$$R \frac{\mathrm{d}}{\mathrm{d}t} \pi(t) = 0 \tag{3}$$

where $\pi(t)$ denotes the $2|V|$-vector obtained by stacking the $\pi_i(t)$. There is a second useful consequence of (1). Suppose that vertex positions are initially fixed and satisfy distance constraints, but then a small displacement $\delta\pi$ is made to the vector of vertex positions, in a way allowing edge lengths to change. *To first order*, the change in the lengths corresponding to the edges of the graph, is described by

$$R\delta\pi = \delta d/2 \tag{4}$$

where δd is the vector of changes in the squares of the lengths, ordered in the same way as the edges are ordered in defining R.

In general R is not square, let alone invertible. Therefore, it does not immediately make sense to contemplate the change in vertex positions that would flow from a change in lengths, at least without some kind of constraint. One can however contemplate constraining some of the vertices not to move, and some of the lengths not to change. Then a submatrix of R will map small changes in some of the vertex positions to small changes in some of the lengths, and the inverse of this submatrix (assuming that it is square and nonsingular) will map small changes in some of the lengths to small changes in some of the vertex positions. In the next section however, we shall consider a situation where this submatrix is not square, and the resulting effect on the posing and solution of a localization problem.

3. Anchor agents and localization accuracy in autonomous formations

For the rest of this paper we consider a formation as described in the following assumption.

[1] Here, "generic" positions correspond to "almost all" arbitrary selections of positions. Some discussions on the need for using the qualifiers "generic" and "almost all" can be found in Tay and Whiteley (1985).

Assumption 1. Consider a formation, $\mathcal{F} = (G, \Pi)$, with the underlying sensing graph $G = (V, E)$. Suppose a subset of the agents are anchor agents, i.e. we know their global position. Denote the corresponding subset of V by V'. Adopt the standard convention that two anchor agents know their inter-agent distance, and let $E' \subset E$ denote the set of edges in G joining vertices in V'. In addition, suppose that the graph G is generically globally rigid.

The noiseless *formation localization problem* as defined in the following is fully addressed in Eren et al. (2004). In this paper we will address a follow-on problem postulating noisy measurements that will be introduced in Section 4.

Problem 1 (*Formation Localization Problem*). Let $\mathcal{F}$ be a network in $\mathbb{R}^2$, consisting of $m \geq 3$ noncollinear anchor agents located at known positions $\pi_1, \pi_2, \ldots, \pi_m$ and $n - m > 0$ ordinary agents located at unknown positions $\pi_{m+1}, \ldots, \pi_n$, and let $G = (V, E)$ be the underlying graph of $\mathcal{F}$. For each $e_{ij} \in E$, let the distance between agents i and j be given as $\|\pi_i - \pi_j\| = d_{ij}$. Find locations $p_1, \ldots, p_n \in \mathbb{R}^2$ satisfying $\|p_i - p_j\| = d_{ij}, \forall e_{ij} \in E$ subject to the constraint that $p_k = \pi_k$ for $k \in \{1, \ldots, m\}$.

Remark 1. The full determination of G requires nomination of anchors; before choice of anchors, not all agents which end up being nominated as anchors may be able to sense one another.

We would like to characterize errors in localization ($\delta p_i = p_i - \pi_i, i = 1, \ldots, |V|$) which occur when noise perturbs inter-agent distance measurements (d_{ij}), apart from distances between anchor agent pairs.

For a noiseless situation the equations which apply to the formation after using anchor agent information include distance information and coordinate information, and are of the form

$$\begin{aligned} &\|p_i - p_j\| = d_{ij}, \quad \forall_{e_{ij}} \in E \setminus E' \\ &p_i = \pi_i, \quad \forall v_i \in V'. \end{aligned} \tag{5}$$

In the presence of the typically small noise, δd_{ij}, perturbing the squares of the true distances (presumably due to measurement error), (5) would formally become

$$\begin{aligned} &\|p_i - p_j\|^2 = d_{ij}^2 + \delta d_{ij}, \quad \forall_{e_{ij}} \in E \setminus E' \\ &p_i = \pi_i, \quad \forall v_i \in V' \end{aligned} \tag{6}$$

which results in an overdetermined system of simultaneous equations, and therefore now there will generally be no solution (There are at least $2|V \setminus V'| + 1$ equations and exactly $2|V \setminus V'|$ unknowns.). Nevertheless, the notion of approximate localization makes sense. Instead of solving (6), we seek those position values for p_i, call them π_i^*, for $v_i \in V \setminus V'$ which solve the following minimization problem:

$$\begin{aligned} &\min_{p_i, v_i \in V \setminus V'} \quad \sum_{e_{ij} \in E \setminus E'} [\|p_i - p_j\|^2 - (d_{ij}^2 + \delta d_{ij})]^2 \\ &\text{subject to} \quad p_i = \pi_i, \quad \forall v_i \in V'. \end{aligned} \tag{7}$$

It is intuitively reasonable that if the noises are bounded by a suitably small constant, the solution of the minimization problem stated in (7), π_i^*, for $v_i \in V \setminus V'$, will be close to the solution of (5), and in fact the error in the position will depend continuously on the error in the vector of squared distance. We can use this fact and rewrite the minimization problem in (7) in terms of the rigidity matrix. The following theorem deals with this issue.

Theorem 2. *Consider a formation $\mathcal{F}(G, \Pi)$ with the underlying graph $G = G(V, E)$ generically globally rigid. Suppose that a subset V' of V corresponds to vertices whose coordinates are precisely known. Let $E' \subset E$ correspond to edges joining vertices in V', so that the corresponding edge lengths in the formation are precisely known. Suppose that the measurement of the edge length d_{ij} corresponding to each edge in $E \setminus E'$ is available with a measurement error bounded in magnitude by some fixed $\Delta > 0$. Let δd denote the vector of errors in the squares of the edge length measurements, ordered in the same way as the edges are ordered in defining R, the rigidity matrix of G evaluated at Π. Note that π_i is the correct position of the ith agent. Let π_i^* for $v_i \in V \setminus V'$ solve the approximate localization problem (7). Moreover, suppose that Δ is sufficiently small that there is a unique solution to (7) whose distance from π for $v_i \in V \setminus V'$ depends continuously on the δd_{ij}. Define $\delta\pi^* = \pi^* - \pi$. Then the vector $\delta\pi^*$ is the solution to the following minimization problem neglecting higher order terms in δd:*

$$\begin{aligned} &\min_{\delta p} \quad \|R\delta p - \delta d/2\|^2 \\ &\textit{subject to} \quad \delta p_i = 0 \quad \forall v_i \in V'. \end{aligned} \tag{8}$$

Proof. We only need to show that (7) and (8) are the same for the particular summand in (7) and the corresponding summand in the above minimization problem. (Obviously the result then holds after summation of all of these summands as well.) Replacing p_i and p_j by $\pi_i + \delta p_i$ and $\pi_j + \delta p_j$, where δp_i is the perturbation of the position of ith node from its real positions, in (7) respectively, for the summand associated with the edge e_{ij}, s_{ij}, we obtain,

$$s_{ij} = \|\pi_i - \pi_j + \delta p_i - \delta p_j\|^2 - (d_{ij}^2 + \delta d_{ij}). \tag{9}$$

Replacing $\pi_i, \pi_j, \delta p_i$, and δp_j with $(x_i, y_i)^\top$, $(x_j, y_j)^\top$, $(\delta x_i, \delta y_i)^\top$ and $(\delta x_j, \delta y_j)^\top$ in (9) respectively, we obtain,

$$\begin{aligned} s_{ij} = {} & (x_i - x_j)^2 + (y_i - y_j)^2 - d_{ij}^2 + (\delta x_i - \delta x_j)^2 + (\delta y_i - \delta y_j)^2 \\ & - \delta d_{ij} + 2(x_i - x_j)(\delta x_i - \delta x_j) + 2(y_i - y_j)(\delta y_i - \delta y_j). \end{aligned} \tag{10}$$

On the other hand, one can rewrite (8) as

$$\begin{aligned} &\min_{\delta p_i, v_i \in V \setminus V'} \quad \sum_{k=1}^{|E|} (R_k \delta p - \delta d_k/2)^2 \\ &\text{subject to} \quad \delta p_i = 0 \quad \forall v_i \in V' \end{aligned} \tag{11}$$

where R_k and δd_k are the kth rows of R and δd, respectively. Considering the summand associated with the edge e_{ij}, s'_{ij} we obtain,

$$s'_{ij} = (x_i - x_j)(\delta x_i - \delta x_j) + (y_i - y_j)(\delta y_i - \delta y_j) - \delta d_{ij}/2. \tag{12}$$

The two summands (10) and (12) are the same for a particular edge (i, j) (neglecting the factor of 2 and the second order term $(\delta x_i - \delta x_j)^2 + (\delta y_i - \delta y_j)^2$). □

For considering the norm presented in (8), viz. $\|R\delta p - \delta d/2\|^2$, one can delete those columns from R relating to anchors to study the effect of perturbation, since these columns correspond to positions of anchors that are already known so no perturbation may happen. We can also delete any row corresponding to an edge between two anchors (such an edge may or may not be present before designation of certain nodes as anchors, but in any case the corresponding entry of $R\delta p - \delta d/2$ will be zero.). After doing this deletion process, the norm presented in (8) will transform to

$$\min_{\delta p_r} \|R_{r,(|E \setminus E'|) \times (2|V \setminus V'|)} \delta p_r - \delta d_r/2\|^2. \tag{13}$$

Here R_r, the *reduced rigidity matrix*, is constructed by deletion of columns of R corresponding to anchor positions and rows of R corresponding to edges between two anchors, respectively. Furthermore, δp_r is the perturbation in non-anchor positions and δd_r is the error vector in the square of the length of edges connecting those edges with at least one non-anchor end.

The minimization problem stated in (13) can be solved for deterministic values of δd. For deterministic values of δd, $\delta\pi^*$ is computed by,

$$\delta\pi^* = R_r^{\dagger}\delta d_r/2 \tag{14}$$

where $R_r^{\dagger} = (R_r^{\top}R_r)^{-1}R_r^{\top}$ is the Penrose pseudoinverse of R_r, see Horn and Johnson (1991). However, in general the error in the square of the length measurement, δd_r, is not known and it is not possible to accurately compute the error in the localization. The measurement error can be more realistically modeled by random variables, with specific covariance and mean value. One is much more likely to be interested in translating statistics of δd_r to statistics of the agent position errors, δp_r, when δd_r is a random variable. This can be done using the following equation:

$$\mathrm{cov}(\delta\pi^*) = R_r^{\dagger}\,\mathrm{cov}(\delta d_r)(R_r^{\dagger})^{\top}/4 \tag{15}$$

where $\delta\pi^*$ is the solution to the minimization problem in (13), and cov(.) denotes the covariance matrix of its argument. If δd_r is a normal random variable with zero mean and I as its covariance we have

$$\mathrm{cov}(\delta\pi^*) = R_r^{\dagger}(R_r^{\dagger})^{\top}/4. \tag{16}$$

One might also reasonably postulate that each distance measurement, rather than its square, is subject to additive zero mean Gaussian noise, of variance σ^2, say. Then δd_r will be zero mean, with covariance matrix $\mathrm{diag}(d_{ij}^2)\sigma^2$, and (16) will be replaced by;

$$\mathrm{cov}(\delta\pi^*) = R_r^{\dagger}(\mathrm{diag}(d_{ij}^2)\sigma^2)(R_r^{\dagger})^{\top}/4. \tag{17}$$

Eqs. (16) and (17) address the task of characterizing localization errors posed in the beginning of Section 3. Note that, as is common in characterizing mean square errors arising out of algorithms using noisy data, the characterization involves the solution of the corresponding noiseless problem—in this case R_r involves the true agent positions. We comment on this further in Section 4.2.

The following theorem guarantees that R_r is of full column rank and the minimization problem stated in (13) has a unique solution.

Theorem 3. *Assume that R is the rigidity matrix of a formation such that there is at least one selection of $|V'| = m$, $m \geq 3$, noncollinear anchors with the property that the formation obtained by adding inter-anchors edges is globally rigid. Then the reduced rigidity matrix obtained by deletion of columns of R corresponding to the anchor positions and each row of R corresponding to the edge between an anchor pair is of full column rank.*

Proof. The matrix R_r has dimensions of $(|E \setminus E'| \times 2|V| - 2|V'|)$. (Note that $|V'|$ is the set of vertices selected as anchors.) We assume that R has an ordering such that the last $2|V'|$ columns correspond to the anchors and the last E' rows correspond to the edges connecting the anchors in the original graph.

Suppose to obtain a contradiction to the assertion of the theorem that, $\exists\alpha \neq 0$ such that $R_r\alpha = 0$. Define $\beta = \begin{bmatrix}\alpha^{\top} & 0_{2|V'|}^{\top}\end{bmatrix}^{\top}$; then $R\beta = 0$. Now it is known from Tay and Whiteley (1985) that $\nu_1 = [1, 0, 1, 0, \ldots]^{\top}$, $\nu_2 = [0, 1, 0, 1, \ldots]^{\top}$, and $\nu_3 = [y_1, -x_1, y_2, -x_2, \ldots]^{\top}$ form a basis for the three-dimensional null-space of R. Hence β must be a linear combination of ν_1, ν_2, and ν_3, i.e. for some scalar a, b, and c, not all zero, there holds $\beta = a\nu_1 + b\nu_2 + c\nu_3$. The last $2|V'|$ rows give the equation $a\bar{\nu}_1 + b\bar{\nu}_2 + c\bar{\nu}_3 = 0$, where $\bar{\nu}_1$, $\bar{\nu}_2$, and $\bar{\nu}_3$ are the last $2|V'|$ rows of ν_1, ν_2, and ν_3, respectively. Inspection of these vectors show that they are independent. Hence nonzero a, b, and c cannot exist, i.e. there is no nonzero α that satisfies the equation $R_r\alpha = 0$. Therefore, R_r is of full column rank. □

In the next section we discuss which nodes are the best candidates for being selected as anchors.

4. Selection of anchors in the formation

In this section, we present a performance index to be used in selection of a pre-specified number ($m \geq 3$) of anchor agents for globally rigid formations in order to minimize the overall localization error $\delta\pi^*$ introduced in Section 3 in the sense of minimizing a scalar measure associated with it.

Remark 2. Although having more than three anchors in a formation is not necessary for accomplishing the localization task, it may prove to be useful. First, it provides redundancy in the case of another anchor failure, second, by having more anchors the number of unknown variables decreases and as a result the effect of measurement errors reduces.

In order to formulate a well-defined minimization problem, we modify Assumption 1 as follows.

Assumption 2. Consider a formation $\bar{\mathcal{F}} = (\bar{G}, \Pi)$, with the underlying sensing graph $\bar{G} = (V, \bar{E})$, where no agent is an anchor. For any subset $\mathcal{I} \subset \{1, 2, \ldots, n\}$ with $|\mathcal{I}| = m \geq 3$, let $\mathcal{F}(\mathcal{I}) = (G(\mathcal{I}), \Pi)$ with underlying sensing graph $G(\mathcal{I}) = (V, E(\mathcal{I}))$ denote the formation obtained from $\bar{\mathcal{F}}$ by assigning all the agents $i \in \mathcal{I}$ as anchors. Let $E'(\mathcal{I}) \subset E(\mathcal{I})$ denote the set of edges connecting these anchor agents. It is assumed that $\mathcal{F}(\mathcal{I})$ is globally rigid for any $\mathcal{I} \subset \{1, 2, \ldots, n\}$ with $|\mathcal{I}| = m \geq 3$.

4.1. Problem definition and a selection metric

We first adopt a scalar measure associated with $\delta\pi^*$ and then define the problem of localization error minimization formally based on this measure. Considering the analysis and discussions in Section 3, we select this measure as $\lambda_{\max}(\mathrm{cov}(\delta\pi^*(\mathcal{I})))$, and formulate the minimization problem as follows:

Problem 2. Consider a formation $\bar{\mathcal{F}} = (\bar{G}, \Pi)$ and the set of all formations $\mathcal{F}(\mathcal{I}) = (G(\mathcal{I}), \Pi)$ obtained from $\bar{\mathcal{F}}$ via anchor assignment as described in Assumption 2. For each anchor index set $\mathcal{I}$, let $R_r(\mathcal{I})$ denote the reduced rigidity matrix, and $\delta\pi^*(\mathcal{I})$ denote the localization error vector, as introduced in Section 3, for the formation $\mathcal{F}(\mathcal{I})$. For what selection of the index set $\mathcal{I}$, is $\lambda_{\max}(\mathrm{cov}(\delta\pi^*(\mathcal{I})))$ minimum?

Next we present a metric, the minimization of which will provide an answer to Problem 2.

Remark 3. When (16) applies, minimizing $\lambda_{\max}(\mathrm{cov}(\delta\pi^*))$ can be considered as minimizing $\lambda_{\max}(R_r^{\dagger}(\mathcal{I})(R_r^{\dagger}(\mathcal{I}))^{\top})$. Furthermore, the problem of minimizing $\lambda_{\max}(R_r^{\dagger}(\mathcal{I})(R_r^{\dagger}(\mathcal{I}))^{\top})$ is equivalent to the problem of maximizing $\lambda_{\min}(R_r(\mathcal{I})R_r(\mathcal{I})^{\top})$.

Based on Remark 3 we define a performance index γ to quantify the performance of each anchor selection $\mathcal{I} \subset \{1, \ldots, n\}$ with $|\mathcal{I}| = m \geq 3$ as follows:

$$\gamma(\mathcal{I}) = \lambda_{\min}(\mathcal{I})(R_r(\mathcal{I})R_r(\mathcal{I})^{\top}). \tag{18}$$

Remark 4. Note that $\lambda_{\max}(R_r(\mathcal{I})^{\dagger}(R_r(\mathcal{I})^{\dagger})^{\top}) = \lambda_{\min}^{-1}(R_r(\mathcal{I})R_r(\mathcal{I})^{\top}) = \sigma_{\min}^{-2}(R_r(\mathcal{I}))$. So these values can be used interchangeably.

Here we formally define *optimal anchor selection*. Consider the setting of Problem 2. The selection $\mathcal{I} \subset \{1, \ldots, n\}$ with $|\mathcal{I}| = m \geq 3$ minimizing the performance index $\gamma(\mathcal{I})$, i.e. $\arg\min\{\gamma(\mathcal{I}) | \mathcal{I} \subset \{1, \ldots, n\}$ and $|\mathcal{I}| = m\}$ is termed *γ-optimal selection of m anchors.*[2]

[2] Optimal anchor selection definition should be modified when one wants to use (17).

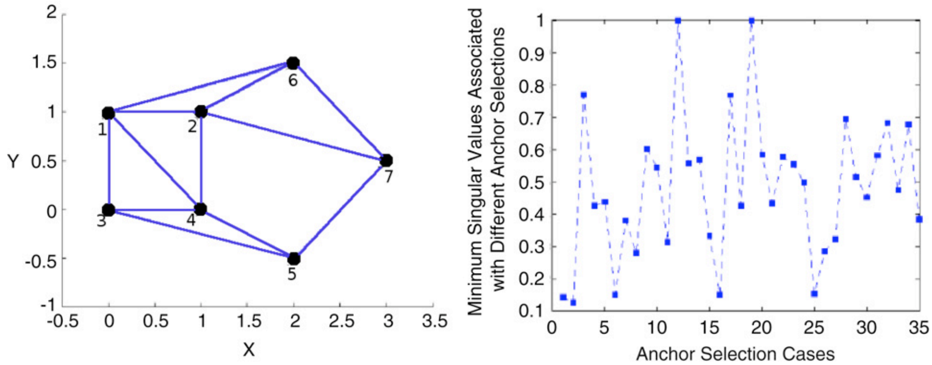

Fig. 1. The globally rigid formation used in the first simulation and $\sigma_{\min}$ value of reduced rigidity matrix for all possible 35 anchor selections. The selection index S, denotes the Sth ordered combination of $\{k, l, m\}$ when $k, l, m \in \{1, \ldots, 7\}$. The selection index S associated with the selection of the triple $\{k, l, m\}$ as anchors is calculated S and, $S = \binom{7}{3} - \binom{8-k}{3} + \binom{7-k}{2} - \binom{8-l}{2} + m - l$.

Remark 5. When (16) applies, the solution of Problem 2 is the γ-optimal selection of m anchors.

Based on Remark 5, to solve Problem 2 for a particular setting, we can use an exhaustive search through all possible selections of m anchors to find the largest possible $\gamma(\mathcal{I})$, or use another optimization method. Simulation results based on the exhaustive search appear in the next subsection. Obviously there will be a practical limit on the size of the network for which this is feasible. In Section 5 we discuss another method for maximizing γ. This alternative method allows consideration of far larger formations.

4.2. Simulation results

In this section, we introduce two different examples discussing the problem of finding the best (γ-optimal) selection of anchors in different scenarios. The solutions are obtained by exhaustive search, and therefore may not be considered interesting. Nevertheless, the results motivate the formulation of some qualitative guidelines for selecting anchors in a formation later in this section.

Example 1. A formation of 7 agents with a globally rigid underlying graph of the formation is studied, where 3 anchors are to be selected. Fig. 1 shows the formation and the minimum singular value of the reduced rigidity matrix associated with each of the possible 35 anchor selections $\lambda_{\max}(\mathrm{cov}(\delta\pi^*(\mathcal{I})))$ is presented.

It is evident that selection of nodes $\{1, 4, 7\}$ and $\{2, 3, 7\}$ results in having the largest minimum singular value. Selections $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{2, 6, 7\}$ yield the 5 lowest singular values.

Example 2. Consider the same graph that was used in Example 1 but with different geometric positions for the agents; the formation is depicted in Fig. 2. The best choice of anchors in this case is $\mathcal{I} = \{1, 4, 7\}$. The minimum singular value associated with each selection is presented in Fig. 2 as well.

Simulations for Examples 1 and 2 suggest the following heuristically reasonable graphical and geometrical guidelines for maximizing $\gamma(\mathcal{I})$. From a graphical point of view, one can say that a triple of agents that already form a triangle in the formation prior to anchor selection are not good candidates for being selected as anchors. As can be seen from the simulation results, such selections are never among those selections that yield larger minimum singular values for the reduced rigidity matrix. Furthermore, in some other cases γ-optimal selection of 3 anchors happens when the smallest loop connecting the three anchors is the longest. From a geometrical point of view, in many cases (that cannot be presented here due to the lack of space) the γ-optimal selection of 3 anchors is the selection of those anchors such that the triangle formed by these anchors has the largest area among all other possible triangles defined by any other three nodes in the graph.

4.3. Remark on computational complexity

The computational complexity for calculating singular values of a matrix $A_{n\times m}$ is $4nm^2 + 8m^3$. For checking the singular values for all the possible selections of three anchors, singular values should be calculated for $n(n-1)(n-2)/6$ times. Hence the computational complexity of the method for n nodes is $O(n^6)$.

For larger number of agents and anchors the exhaustive search is not capable of addressing the problem of optimal anchor selection. In the next section we propose a method for dealing with the problem of anchor selection in larger formations.

5. Selection of anchors using convex optimization

In this section we address the anchor selection problem by developing a method analogous to one proposed in Joshi and Boyd (in press), for a different style of selection problem, namely sensor selection problem. The method uses the idea of relaxing an integer problem and solving it using convex optimization methods for continuous values and then applying the solution to the original integer case.

As a notational convention, for a symmetric positive semidefinite matrix $A_{M\times M}$, $M \in \mathbb{N}$, the eigenvalues are ordered in a way that

$$\lambda_M(A) \le \lambda_{M-1}(A) \le \cdots \le \lambda_1(A), \tag{19}$$

where $\lambda_i(A)$ is the ith eigenvalue of A.

Consider R, the rigidity matrix introduced earlier. By selecting $|V'| = m$ agents out of $|V| = n$ agents as anchors, we are discarding $2m$ columns of this matrix, in other words we are selecting the remaining $2n - 2m$ columns. To continue, first we state the following lemma.

Lemma 1. *Let r_i denote the ith column of the rigidity matrix, $R_{|E|\times 2|V|}$, for a given globally rigid formation. Then the following relationship holds:*

$$\mathrm{rank}\left(\sum_{i=1}^{n} s_i r_i r_i^\top\right) = 2n - 2m, \tag{20}$$

where $\mathbf{1}^\top s = 2n - 2m$, $s_i \in \{0, 1\}$, $i = 1, \ldots, 2n$, $s_{2i-1} = s_{2i}$, and $\mathbf{1}$ is a vector with 1 entries.

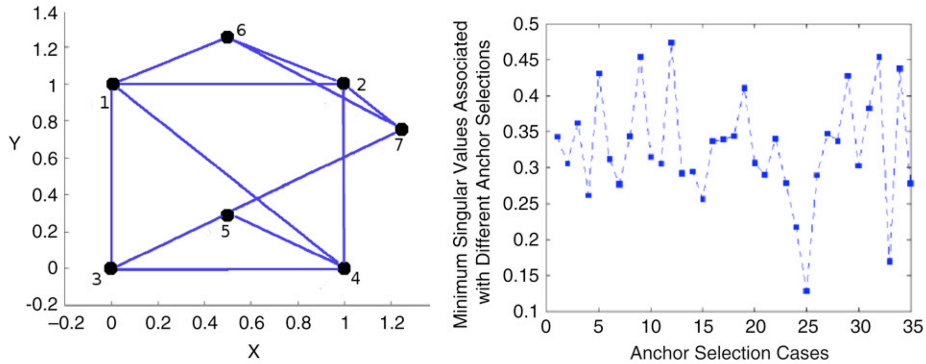

Fig. 2. The globally rigid formation used in the second simulation and $\sigma_{\min}$ value of reduced rigidity matrix for all possible 35 anchor selections. The selection index is the same as the one used in Fig. 1.

Proof. Let s_i obey the conditions of the lemma. Delete from the rigidity matrix those columns whose indices are the same as the indices of the s_i assuming a value zero. Because $\mathbf{1}^\top s = 2n - 2m$, this implies that $2m$ columns are deleted. Because $s_{2i-1} = s_{2i}$, this implies that columns corresponding to m vertices are deleted. Let $\bar{R}_r$ be the matrix so obtained. It is easy to show that $\bar{R}_r$ is the same as R_r with the possible addition of extra all-zero rows. With the same argument used as in the proof of Theorem 3, we can show that $\operatorname{rank}(\bar{R}_r) = 2n - 2m$. This, since $\sum_{i=1}^n s_i r_i r_i^\top = \bar{R}_r \bar{R}_r^\top$, implies that $\operatorname{rank}(\bar{R}_r \bar{R}_r^\top) = \operatorname{rank}\left(\sum_{i=1}^n s_i r_i r_i^\top\right) = 2n - 2m$. □

Corollary 1. *For a given rigid formation with rigidity matrix $R_{|E|\times 2|V|}$,*

(1) $\lambda_{|E|}\left(\sum_{i=1}^n s_i r_i r_i^\top\right) = \lambda_{|E|-1}\left(\sum_{i=1}^n s_i r_i r_i^\top\right) = \cdots = \lambda_{(2n-2m)+1}\left(\sum_{i=1}^n s_i r_i r_i^\top\right) = 0$, *for $s_i \in \{0, 1\}$.*
(2) *The smallest nonzero eigenvalue of $\sum_{i=1}^n s_i r_i r_i^\top$ is equal to $\sum_{j=2n-2m}^{|E|} \lambda_j\left(\sum_{i=1}^n s_i r_i r_i^\top\right)$.*

Remark 6. The smallest eigenvalue of $R_r R_r^\top$ is equal to the smallest nonzero eigenvalue of $\bar{R}_r \bar{R}_r^\top = \sum_{i=1}^n s_i r_i r_i^\top$.

In the light of Lemma 1, Corollary 1, and Remark 6 we can write the problem of maximizing γ as

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j=2n-2m}^{|E|} \lambda_j\left(\sum_{i=1}^{2n} s_i r_i r_i^\top\right) \\
\text{subject to} \quad & \mathbf{1}^\top s = 2n - 2m \\
& s_i \in \{0, 1\}, \quad i = 1, \ldots, 2n \\
& s_{2i-1} = s_{2i}.
\end{aligned} \tag{21}
$$

The expression in (21) is not altogether standard; nevertheless, it enjoys a useful standard property:

Proposition 1. *For any integer $k \le M$, the sum of the k smallest eigenvalues of a symmetric matrix $A_{M\times M}$ is concave.*

Proof. See Appendix A. □

5.1. Relaxation

Because the constraint $s_i \in \{0, 1\}$ is Boolean, (21) is not solvable using convex optimization techniques. However one can relax this constraint and replace it with a convex constraint. We have the following relaxed version of (21):

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j=2n-2m}^{|E|} \lambda_j\left(\sum_{i=1}^{n} s_i r_i r_i^\top\right) \\
\text{subject to} \quad & \mathbf{1}^\top s = 2n - 2m \\
& 0 \le s_i \le 1, \quad i = 1, \ldots, 2n \\
& s_{2i-1} = s_{2i}.
\end{aligned} \tag{22}
$$

In the light of Proposition 1 we know that the objective function is concave; hence one can solve the maximization problem with the convex constraints using standard convex optimization techniques. Assume s^* is the solution to this problem; it is not necessarily a solution to the original problem, since s_i^* can take a non-integer value. However, one can say that the value of the objective function for s^* is an upper bound for the value of the objective function at a solution of the original problem, because the feasible set for the relaxed problem contains the solution to the original problem as well. Call this upper bound U_s. To generate a possibly suboptimal solution to the nonrelaxed problem we can proceed as follows: Let $s_{i_1}^*, \ldots, s_{i_{2n}}^*$ denote the elements of s^* rearranged in descending order; compose the $2n$-vector $\hat{s}$ with entries $\hat{s}_i \in \{0, 1\}$ such that $\hat{s}_{i_k} = 1$ for $k \in \{1, \ldots, 2n - 2m\}$ and $\hat{s}_{i_k} = 0$ for $k \in \{2n - 2m + 1, \ldots, 2n\}$. This way the entries of $\hat{s}$ with indices corresponding to the $2n - 2m$ largest elements of s^* are assigned to be unity and the rest to be zero. The associated objective value with $\hat{s}$, L_s, is a lower bound for the optimal objective function. We define a gap between the upper bound and the lower bound as, $\delta_s = U_s - L_s$. For the cases that δ_s is negligible, $\hat{s}$ is the solution of the original problem. However, if δ_s is not negligible a local optimization method can be used if desired to seek other better solutions.

Standard convex optimization may appear not immediately applicable to solving the convex optimization problem (22); however, if we cast the problem in the Semidefinite Programming (SDP) framework it can be solved easily. Consider the affine matrix function $F(s) = F_0 + s_1 F_1 + \cdots + s_\mu F_\mu$, where $\mu \in \mathbb{N}$, $s = [s_1, \ldots, s_\mu]^\top$, and $F_0, F_1, \ldots, F_\mu$ are fixed η-by-η symmetric matrices, and $\eta \in \mathbb{N}$. Note that to maximize the sum of the k smallest eigenvalues of $F(s)$ one can minimize the sum of the k largest eigenvalues of $-F(s)$. To do so we solve the following semidefinite programming problem, in t, X, s:

$$
\begin{aligned}
\text{minimize} \quad & kt + \operatorname{tr}(X) \\
\text{subject to} \quad & tI + X - (-F(s)) \ge 0 \\
& X = X^\top \ge 0
\end{aligned} \tag{23}
$$

where $X \in \mathbb{R}^{\eta\times\eta}$. For further information see Nesterov and Nemirovsky (1994) and Vandenberghe and Boyd (1996) and the references therein. To establish that (23) minimizes the k largest eigenvalues of $F'(s) = -F(s)$, we use the following lemma.

Lemma 2. *Let t and X satisfy $tI + X - F'(s) \ge 0$, $X = X^\top \ge 0$. Then*

$$
kt + \operatorname{tr}(X) \ge \sum_{i=1}^{k} \lambda_i(F'(s)),
$$

and there exist t^* and X^* attaining the lower bound.

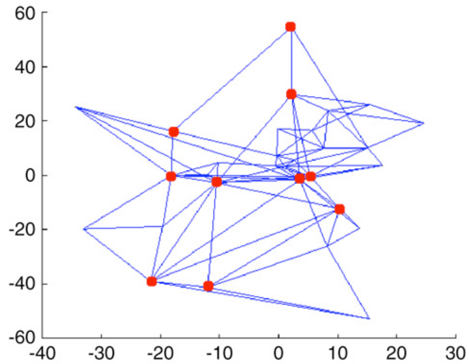

Fig. 3. A formation with 30 agents. The agents represented by red circles correspond to optimal selection of anchors.

Proof. See Appendix B. □

In what follows we bring two numerical examples to show the applicability of the method. The software package used to solve them is CVX, for more information see Grant and Boyd (2008).

Example 3. Consider the formation of Examples 1 and 3 depicted in Fig. 1. Solving the optimization problem (22), we obtain, the γ-optimal anchor selection to be the selection $\{1, 5, 7\}$. (See Fig. 1 for comparison.) In this example $\delta_s = 2.1440$, while $U_s = 2.4680$.

Example 4. In this last numerical example, we study the optimal anchor selection of 10 anchors in a formation with 30 agents. It took 25 s on a laptop computer with CPU 2.0 GHz. The result is depicted in Fig. 3, where the agents represented by red circles are selected as anchors. The values of δ_s and L_s for this case are 707.1169 and 8.1732 respectively, which implies a big gap. Consequently, one may consider running a local optimization algorithm to obtain a better selection.

6. Conclusions and future works

In this paper, we have studied how to minimize the effects of noises arising in self-localization of mobile formations. The paper has postulated a statistical measure for the effect of these noises, and derived an intuitively pleasing result that the degree to which the noises will be a problem is captured by a certain rigidity matrix associated with the formation (but not the usual one, rather a 'reduced one'). The critical issue is the size of a performance index, namely γ associated with different selections. We proposed a method to tackle the problem of optimal anchor selection using convex optimization tools. We presented some numerical examples considering formations with reasonably large number of agents.

Acknowledgements

This work is supported by NICTA, which is funded by the Australian Government as represented by the Department of Broadband, Communications and the Digital Economy and the Australian Research Council through the ICT Centre of Excellence program.

Appendix A. Proof of Proposition 1

First we prove that sum of $M - k$ largest eigenvalues of $A_{M\times M}$ is convex. From Horn and Johnson (1991) we know,

$$\sum_{i=1}^{\mu} \lambda_i(A) = \sup\left\{\mathrm{tr}(Z^\top AZ) | Z \in \mathbb{R}^{M\times\mu}, Z^\top Z = I\right\}$$

where tr(.) is the trace function and $\mu = M - k$. For arbitrary matrices A and B, and constant $0 \le \theta \le 1$, we have

$$\begin{aligned}
\sum_{i=1}^{\mu} \lambda_i(\theta A + (1-\theta)B) &= \sup_{Z^\top Z = I} \left[\mathrm{tr}(Z^\top(\theta A + (1-\theta)B)Z)\right] \\
&= \sup_{Z^\top Z=I} \left[\mathrm{tr}(Z^\top \theta AZ) + \mathrm{tr}(Z^\top(1-\theta)BZ)\right] \\
&\le \sup_{Z^\top Z=I} \left[\mathrm{tr}(Z^\top \theta AZ)\right] + \sup_{Z^\top Z=I} \left[\mathrm{tr}(Z^\top(1-\theta)BZ)\right] \\
&\le \theta \sup_{Z^\top Z=I} \left[\mathrm{tr}(Z^\top AZ)\right] + (1-\theta) \sup_{Z^\top Z=I} \left[\mathrm{tr}(Z^\top BZ)\right].
\end{aligned}$$

Hence the sum of $\mu = M - k$ largest eigenvalues is convex. The sum of k smallest eigenvalues is calculated by

$$\sum_{i=\mu+1}^{n} \lambda_i(A) = \sum_{i=1}^{M} \lambda_i(A) - \sum_{i=1}^{\mu} \lambda_i(A) = \mathrm{tr}(A) - \sum_{i=1}^{\mu} \lambda_i(A).$$

The sum of all eigenvalues (trace) is linear (convex and concave), and the negative of the sum of the $M - k$ largest eigenvalues is concave, and from Boyd and Vandenberghe (2004) we know that the addition of two concave functions is concave.

Appendix B. Proof of Lemma 2

Let $s^* = \arg\min_s \sum_{i=1}^{k} \lambda_i(F'(s))$, where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_\eta$. Without loss of generality, using diagonalization by an orthogonal matrix if necessary, suppose

$$F'(s^*) = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

Let $E = [I_k \ 0_{n-k}]$. Then $tI + S - F'(s^*) \ge 0$, and hence $E(tI + X)E^\top - EF'(s^*)E^\top \ge 0$. This further implies that $tI_k + X_{11} - \mathrm{diag}(\lambda_1, \ldots, \lambda_k) \ge 0$, where X_{11} is the first k-by-k diagonal block of X. Hence, $kt + \mathrm{tr}(X_{11}) - \sum_{i=1}^{k} \lambda_i(F(s)) \ge 0$. Since $X \ge 0$, $\mathrm{tr}(X_{11}) \le \mathrm{tr}(X)$, we have

$$kt + \mathrm{tr}(X) \ge \sum_{i=1}^{k} \lambda_i(F'(s)).$$

Next, let $X^* = \mathrm{diag}(\lambda_1 - \lambda_{k+1}, \ldots, \lambda_k - \lambda_{k+1}, 0, \ldots, 0)$ and $t^* = \lambda_{k+1}$. Then we see that

$$\begin{aligned}
&t^*I + X^* - F'(s^*) \\
&\quad = \mathrm{diag}(0, \ldots, 0, \lambda_{k+1} - \lambda_{k+2}, \ldots, \lambda_{k+1} - \lambda_n) \ge 0.
\end{aligned}$$

Further $kt^* + \mathrm{tr}(X^*) = \sum_{i=1}^{k} \lambda_i(F'(s))$, i.e. the lower bound is attained.

References

Anderson, B. D. O., Yu, C., Fidan, B., & Hendrickx, J. M. (2008). Rigid graph control architectures for autonomous formations. *IEEE Control Systems Magazine*, *28*(6), 48–63.

Applewhite, A. (2002). What knows where you are? *IEEE Pervasive Computing*, *1*, 4–8.

Barooah, P., & Hespanha, J. P. (2007). Estimation on graphs from relative measurements: Distributed algorithms and fundamental limits. *IEEE Control Systems Magazine*, *27*(August), 57–74.

Bian, F., Kempe, D., & Govindan, R. (2006). Utility based sensor selection. In *Proc. 5th int. conference on information processing in sensor networks* (pp. 11–18).

Boyd, S. P., & Vandenberghe, L. (2004). *Convex optimization*. New York, NY, USA: Cambridge University Press.

Eren, T., Goldenberg, D., Whiteley, W., Yang, Y., Morse, A., & Anderson, B. D. O. (2004). Computation and randomization in network localization. In *Proc. network localization, joint conference of ieee computer and communication societies*.

Grant, M., & Boyd, S. P. (2008). CVX: Matlab software for disciplined convex programming (web page and software). http://stanford.edu/~boyd/cvx.

Horn, R. A., & Johnson, C. P. (1991). *Topics in matrix analysis*. Cambridge University Press, reprinted 1999.

Joshi, S., & Boyd, S. P. (2008). Sensor selection via convex optimization. *Transactions on Signal Processing*. doi:10.1109/TSP.2008.2007095.

Mao, G., Fidan, B., & Anderson, B. D. O. (2006). Localization. In N. P. Mahalik (Ed.), *Sensor networks and configuration: Fundamentals, techniques, platforms and experiments* (pp. 281–315) Springer-Verlag (Chapter 13).

Nesterov, Y., & Nemirovsky, A. (1994). *Interior-point polynomial methods in convex programming*. Philadelphia, PA: Society for Industrial and Applied Mathematics.
Olfati-Saber, R. (2007). Distributed kalman filtering for sensor networks. In *Proc. 46th IEEE conference on decision and control* (pp. 5492–5498).
Schau, H. C., & Robinson, A. Z. (1987). Passive source location emp1oying intersecting spherical surfaces from time-of-arrival difference. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, *35*, 1223–1225.
Tay, T., & Whiteley, W. (1985). Generating isostatic frameworks. *Structural Topology*, *11*, 21–69.
Vandenberghe, L., & Boyd, S. (1996). Semidefinite programming. *SIAM Review*, *38*, 49–95.

Iman Shames is a NICTA endorsed Ph.D. candidate at the Australian National University, Canberra, Australia, under the supervision of Prof. Brian D.O. Anderson, and holds an EIPRS (Endeavour International Postgraduate Research Scholarship). He received his B.S. degree in Electrical Engineering from Shiraz University, Iran in 2006. He has been a visiting researcher at the University of Tokyo in 2008, and at the University of Newcastle in 2005. His current research interests include multi-agent systems and sensor networks.

Barış Fidan received the B.S. degrees in electrical engineering and mathematics from Middle East Technical University, Turkey in 1996, the M.S. degree in electrical engineering from Bilkent University, Turkey in 1998, and the Ph.D. degree in electrical engineering at the University of Southern California, USA in 2003. He has been with National ICT Australia and the Research School of Information Sciences and Engineering of the Australian National University, Canberra, Australia since 2005, where he is currently a senior researcher. His research interests include autonomous formations, sensor networks, cooperative localization, adaptive and nonlinear control, switching and hybrid systems, mechatronics, and various control applications.

Brian D.O. Anderson is now Distinguished Professor at the Australian National University. In 2002–3, he was interim CEO of National ICT Australia, and then served as Chief Scientist until mid 2006.

Professor Anderson received the Quazza Medal at the IFAC 14th World Congress, Beijing, China in 1999 and an Automatica Prize Paper Award at the same time. He is an IFAC Fellow and has held a number of offices in IFAC, including the Presidency from 1990 to 1993.