SIAM J. CONTROL OPTIM.
Vol. 48, No. 1, pp. 206–233

CONTROL OF MINIMALLY PERSISTENT FORMATIONS IN THE PLANE*

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Abstract. This paper studies the problem of controlling the shape of a formation of point agents in the plane. A model is considered where the distance between certain agent pairs is maintained by one of the agents making up the pair; if enough appropriately chosen distances are maintained, with the number growing linearly with the number of agents, then the shape of the formation will be maintained. The detailed question examined in the paper is how one may construct decentralized *nonlinear* control laws to be operated at each agent that will restore the shape of the formation in the presence of *small distortions* from the nominal shape. Using the theory of rigid and persistent graphs, the question is answered. As it turns out, a certain submatrix of a matrix known as the rigidity matrix can be proved to have nonzero leading principal minors, which allows the determination of a stabilizing control law.

Key words. multi-agent system, directed formations, distributed control, coordinated motion

AMS subject classifications. 93C10, 68M14

DOI. 10.1137/060678592

1. Introduction and problem description.

The problem of controlling agent formations is gaining more and more attention, as witnessed by an increasing number of contributions in recent years. Among the older contributions, we note, e.g., [1, 2, 3, 4, 5, 6, 7, 8]. Roughly speaking, a collection of agents is prescribed, to move in two- or three-dimensional space, and it is envisaged that they will move as a formation from point A to point B, possibly executing some mission, possibly avoiding obstacles, etc. An agent can be possibly but not necessarily treated as a massless point agent. The words "move as a formation" have the following meaning which a layman might ascribe to them: the formation at one instant of time is congruent to the formation at another instant of time, or equivalently, interagent distances are preserved over all time. Many early contributions deal with the question of just what interagent distances or other constraints are needed to assure this property; see, e.g., [1, 4, 9, 10].

Exactly how motion is achieved in a stable way is an issue of great interest, and recent papers have tended to focus more on the control laws required [11, 12, 13, 14, 15, 16]. It has been observed that if some interagent distances are preserved, for example $2n-3$ well-chosen distances where n is the number of agents in a two-dimensional formation of point agents, then all interagent distances could be possibly

*Received by the editors December 26, 2006; accepted for publication (in revised form) August 1, 2008; published electronically February 11, 2009. The first author was supported by the Australian Research Council through an Australian Postdoctoral Fellowship under grant DP-0877562. The first, second, and fourth authors were supported by National ICT Australia-NICTA; NICTA is funded by the Australian Government as represented by the Department of Broadband, Communications and the Digital Economy and the Australian Research Council through the ICT Centre of Excellence program. The third author was supported by National Science Foundation grants CCF-0729025 and ECS-0622017 and by NICTA.
http://www.siam.org/journals/sicon/48-1/67859.html
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preserved as a consequence, and a scalable and even distributed control algorithm can be envisaged. Other schemes for control of formation shape can be envisaged too; for example, some angles can be preserved instead of just the distances.

In this paper, like many predecessors, we consider control of formation shape based on interagent distance preservation. What distinguishes this work, however, from most, but not all, work up to this point is that we assign the task of controlling the distance between two agents to a set-point value to only one of the two agents, hence the task is a *directed* one.

Among papers dealing with what one might term *directed* formation control, we note those of [8, 2, 1, 11, 17]. Directed formation control is straightforward if the underlying directed graph is acyclic, since it engenders a triangular coupling, based on a partial ordering of the agents due to an absence of cycles. Thus the challenging problems lie with cyclic graphs. Tabuada, Pappas, and Lima [8] emphasize cyclic graphs, while maintaining a great degree of generality about the nature of the constraints linking the agents. Bailleul and Suri [1] raise the possibility of considering cyclic structures, where there are distance measurements used to achieve control, and argue that such structures are inherently flawed, at least in the presence of noise/bias errors, etc. Lee and Spong [11] consider directed structures, but with the requirement that the underlying graph be balanced (i.e., each node has the same number of inwardly and outwardly directed edges, though a variation is possible with a concept called weighted balancing), and in fact their work is aimed at a different problem (flocking) than preservation of the shape of a two-dimensional formation. Nevertheless, preliminary work of this paper suggests the notion that balanced graphs might also allow efficacious treatment of distance-based formation shape preserving problems, differently to the scheme of this paper.

Earlier on in this work, it was identified that the concept of *graph rigidity* could helpfully underpin much of the control law development. In the undirected graph case, where two agents work together to maintain the correct separation between them, a distributed control law that stabilizes a formation exists only if the underlying graph is rigid, a point specially emphasized in the contributions of Olfati-Saber and colleagues; see, e.g., [5, 2, 3, 4]. In the directed graph case, rigidity is not enough. One needs a further concept, termed *persistence*; see [9, 10]. This concept is reviewed in the next section. It includes rigidity, but overlays this with a further condition that rules out certain information-flow or sensing patterns that are otherwise consistent with the rigidity property. In a persistent graph, it remains possible to have cycles.

The purpose of this work is to demonstrate in detail how control based on distance preservation can be achieved in a directed formation even when cycles are present. We particularly consider a class of such formations termed *minimally persistent formations*; the word *minimal* naturally reflects a certain optimality. *For this class we present a distributed, nonlinear control law and demonstrate its local stability.*

In section 2, we review some background concepts, such as (minimal) rigidity and (minimal) persistence, and introduce different ways of arranging degrees of freedom (DOFs), which eventually affects the control structure. In fact, we focus on formations that have one particular type of DOF distribution. As will be explained later, the specific formations that we address are called “leader-first follower” formations. We then set up the equations of motion for various types of agents, characterized in terms of the number of constraints that each agent maintains. The main results are provided in section 4 and presented by way of verifying a principal minor condition. The structure of eigenvalues is examined in section 5. The paper ends with some concluding remarks in section 6.

2. Background concepts. In this section, we recall a number of graph theoretical concepts related to the maintenance of the shape of a formation. Consider a set of n point agents in the plane. Suppose that they are required to undergo a continuous motion so that the distance between any pair of agents remains constant. As is customary, we will model the formations as graphs: nodes will correspond to the agent positions. An edge will exist between two agents if the distance between these agents is specified as part of the formation specification. We will require that the formation can translate and rotate as a whole, but not flex within itself. This requirement for the formation and its *underlying graph*, viz., the graph modeling it, is called *rigidity*, which is formally defined together with some other fundamental notions of *rigid graph theory* in the following; see, e.g., [18].

DEFINITION 2.1. *In $\Re^2$, a* representation *of an undirected graph $G = (V, E)$ with vertex set V and edge set E is a function $\pi : V \to \Re^2$. We say that $\pi(i) \in \Re^2$ is the* position *of the vertex i and define the distance between two representations π_1 and π_2 of the same graph by $d(\pi_1, \pi_2) = \max_{i\in V} \|\pi_1(i) - \pi_2(i)\|$. A* distance set *$\bar{d}$ for G is a set of distances $d_{ij} > 0$, defined for all edges $(i, j) \in E$. A distance set is* realizable *if there exists a representation π of the graph for which $\|\pi(i) - \pi(j)\| = d_{ij}$ for all $(i, j) \in E$. Such a representation is then called a* realization.

A representation π is rigid *if there exists $\epsilon > 0$ such that for all realizations π' of the distance set induced by π and satisfying $d(\pi, \pi') < \epsilon$, there holds $\|\pi'(i) - \pi'(j)\| = \|\pi(i) - \pi(j)\|$ for all $i, j \in V$ (we say in this case that π and π' are* congruent*). A graph is said to be* generically rigid *(or simply* rigid*) if almost all its representations are rigid.*[1] *A rigid graph is further called* minimally rigid *if no single edge can be removed without losing rigidity.*

Consider a formation F in $\Re^2$ with agents in generic positions and with defined agent pairs having the interagent distances maintained, and let $G = (V, E)$ be the associated graph. Then the formation F is called rigid *if G is rigid. If G is minimally rigid, then F is also called* minimally rigid.

In a rigid formation, it is evident that if enough interagent distances are maintained, then the remainder will be consequentially maintained. For example, see Figure 1; if the distances between the agent pairs (1, 2), (2, 3), (3, 4), (4, 1), and (1, 3) are maintained, then the distance between the pair (2, 4) will be consequentially maintained. For a graph with n agents, it turns out that it is normally enough to maintain $2n-3$ well-distributed distances that are constant in order that all distances are constant. The adverb "normally" connotes that there are some exceptional cases associated with exceptional agent positions. For example, if agents are collinear, or occupy the same position, the conclusion may fail. This conclusion is a standard result of the *rigid graph theory* and is due to [19, 18]. Figure 2 shows an example of two graphs, (a) with well-distributed edges, and (b) without.

THEOREM 2.1 (Laman's theorem [19]). *Consider a formation F in $\Re^2$ with agents in generic positions and with defined agent pairs having the interagent distances maintained, and let $G = (V, E)$ be the associated graph. Then F is rigid if and only if there exists a subgraph of G, call it G', with $G' = (V, E')$, $E' \subset E$ such that $|E'| = 2|V| - 3$ and for any $V' \subset V$ defining an induced subgraph $G'' = (V', E'')$ of*

[1] In the graph rigidity literature, the vertex positions in a representation (or the agent positions of a formation) are termed *generic* if the set corresponding to the coordinates of the vertex (or agent) positions is independent over the rationals. An obvious example of nongenericity is when three or more vertices (agents) are collinear. Some discussions on the need for using "generic" and "almost all" can be found in [9, 18].

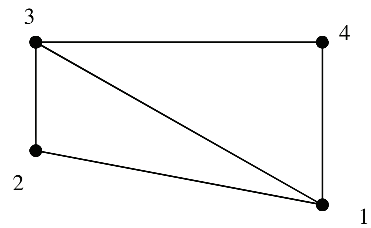

FIG. 1. *A rigid formation in the plane.*

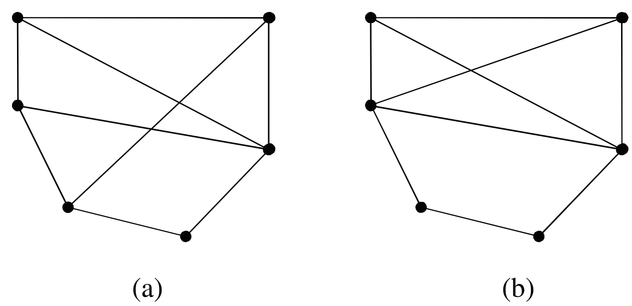

FIG. 2. (a) *Well-distributed edges guarantee rigidity.* (b) *Ill-distributed edges result in nonrigid formation.*

G', there holds $|E''| \leq 2|V'| - 3$.

As a particular immediate corollary of Theorem 2.1, we see that a minimally rigid graph $G = (V, E)$ obeys $|E| = 2|V| - 3$. An alternative characterization of rigidity is provided by the *rigidity matrix*, which we now define.

DEFINITION 2.2. *Order the agents of the formation. Let $p_i \in \Re^2$ be the position of agent i. Each edge has a length $||p_i - p_j||$ for some i, j. Order these edges as $e_1, e_2, \dots,$ then define the edge function $f(p_1, p_2, \dots) = \frac{1}{2}(||e_1||^2, ||e_2||^2, \dots)$. The Jacobian of f is called the* rigidity matrix.

The detailed structure of the rigidity matrix will be important in what follows, and so we record it now. Suppose that the formation has $|V|$ agents and $|E|$ agent pairs with maintained distances (and so there are $|V|$ vertices and $|E|$ edges in the corresponding graph). The rigidity matrix has $|E|$ rows and $2|V|$ columns, with columns $2i-1$ and $2i$ corresponding to vertex i. When edge i joins vertices j and k, the ith row of the rigidity matrix has four nonzero entries, in columns $2j-1, 2j, 2k-1$, and $2k$ (corresponding to vertices j and k). These entries are, respectively, $x_j - x_k$, $y_j - y_k$, $x_k - x_j$, $y_k - y_j$, where the (x_j, y_j) denotes the coordinates of agent j. The main result is as follows; see [18].

THEOREM 2.2. *Consider a formation F with associated graph $G = (V, E)$, and let the $|E| \times 2|V|$ rigidity matrix be formed as just described. Then for generic[2] coordinate values, the formation is rigid if and only if the rigidity matrix has rank $2|V| - 3$.*

Evidently, a minimally rigid formation necessarily has $2|V| - 3$ edges and has a rigidity matrix with full row rank. Evidently also, the kernel of the rigidity matrix has a minimum dimension of 3; any $2n$ vector in the kernel defines both a set of

[2]For a discussion of "generic" in this context, see [18]. Agents would not be in generic positions if, for example, they coincided, or were all collinear.

infinitesimal displacements, which preserve the shape of the formation, and a set of velocity vectors for the agents such that motion along the vector field preserves the interagent distances. For a rigid formation, evidently the kernel has dimension precisely 3. The three independent displacements/motions that it permits correspond to translation in two directions and rotation.

2.1. Constraint consistence and persistence. A distance between two agents can be cooperatively maintained by the two agents, in which case the rigidity ideas can be directly applied. But one can also give the full responsibility for maintaining the constraint to one agent, which has to maintain its distance toward the other agent constant, this latter agent being unaware of that fact and therefore taking no specific action to help satisfy the distance constraint. This unilateral character can be imposed by technical limitations of the autonomous agents or in the interests of greater implementation efficiency. It is this unilateral distance-maintaining approach that is covered in this paper.

Accordingly, henceforth we will deal with directed graphs, where a directed outgoing edge exists from agent i to j if agent i is responsible for maintaining its specified distance from j. The directed edge in this case is denoted by $\{i, j\}$, and the directed graph representing the formation is called the *underlying directed graph* of the formation. As it turns out, a further development of the concept of rigidity is required to understand the idea. Rigidity says that if certain interagent distances are maintained, all other interagent distances are consequentially maintained. We require an additional concept, termed *constraint consistence*, which is equivalent to the requirement that it is possible to maintain the nominated interagent distances. We will require that a formation satisfies both the rigidity and constraint consistence conditions, and if these conditions are satisfied, we will call the formation *persistent*. The notions of *constraint consistence* and *persistence* are formally defined together with some other relevant notions below, following [9].

DEFINITION 2.3. *Consider a directed graph $G = (V, E)$, a set $\bar{d}$ of desired distances $d_{ij} > 0$ for all $\{i, j\} \in E$, and a representation π of G in $\Re^2$. We say that the edge $\{i, j\} \in E$ is* active *if $\|\pi(i) - \pi(j)\| = d_{ij}$. We say that the position of the vertex $i \in V$ is* fitting *for the distance set $\bar{d}$ if it is not possible to increase the set of active edges leaving i by modifying the position of i while keeping the positions of the other vertices unchanged, i.e., if there is no $\pi^* \in \Re^2$ for which $\{\{i, j\} \in E : \|\pi(i) - \pi(j)\| = d_{ij}\} \subset \{\{i, j\} \in E : \|\pi^* - \pi(j)\| = d_{ij}\}$. The realization π is called a* fitting *representation of G for $\bar{d}$ if all the vertices $v \in V$ are at fitting positions for $\bar{d}$. Note that any realization is a fitting representation for its distance set.*

A representation π is called constraint consistent *if there exists $\epsilon > 0$ such that any representation π' fitting for the distance set $\bar{d}$ induced by π and satisfying $d(\pi, \pi') < \epsilon$ is a realization of $\bar{d}$. π is called* persistent *if it is both constraint consistent and rigid (according to Definition* 2.1*, ignoring the edge directions). We say that G is* constraint consistent *(or* persistent*) if almost all its representations are constraint consistent (or persistent, respectively). A persistent graph G is further called* minimally persistent *if it is minimally rigid.*

Consider a formation F in $\Re^2$ with agents in generic positions and with defined agent pairs having the interagent distances maintained and a nominated agent in each pair to maintain the corresponding distance, and let $G = (V, E)$ be the underlying directed graph of F. Then the formation F is called constraint consistent *if G is constraint consistent,* persistent *if G is persistent, and* minimally persistent *if G is minimally persistent.*

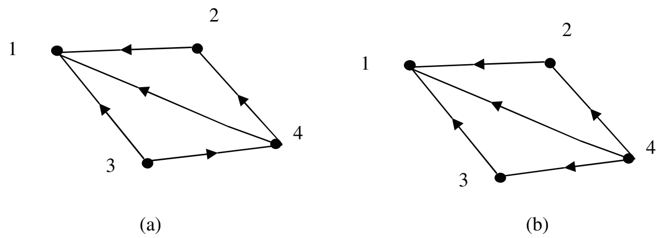

FIG. 3. *Constraint consistent and nonconsistent graphs with the same underlying undirected graph.*

An example of a nonconstraint consistent graph with the underlying undirected graph rigid is shown in Figure 3. If agents 2 and 3 are at their correct distances from agent 1, which allows them some choice of position, agent 4 may not be able to simultaneously achieve its three distance constraints; it cannot of course force agent 2 or 3 to move. In two dimensions, a graph in which the out-degree of every vertex is at most two is automatically constraint consistent, while if the out-degree exceeds two for one or more vertices, it may or may not be constraint consistent [9]. Below we state a particular result regarding minimally persistent graphs (and formations).

THEOREM 2.3 (see [9]). *Consider a directed graph with more than one vertex. Then it is minimally persistent if and only if the underlying undirected graph is minimally rigid and no vertex has more than two outgoing edges.*

It is clear from Theorem 2.3 that in a two-dimensional minimally persistent formation, all agents have zero, one, or two distance constraints to fulfill. Those that have zero distance constraints evidently have two DOFs in which to move, and those with one distance constraint have one DOF. Those with two distance constraints have no DOF. A further requirement for a minimally persistent formation is that the sum of the numbers of DOFs is precisely three, with this in fact corresponding to two translation motions and one rotation motion; then either there is exactly one vertex with two DOFs and another one with one DOF, or there are three vertices, each with one DOF; all other vertices have no DOF.

Nonminimally persistent formations are those where there are extra constraints that have to be fulfilled. Their presence is unnecessary, but may assist in securing robustness against communication or control failures, or they may assist in limiting control magnitudes—just as a linear system with two inputs may be controllable from either input alone but may be much more robustly controlled from both inputs. Not surprisingly, any nonminimally persistent formation contains at least one minimally persistent formation with the same set of vertices and a subset of the edges.

In this paper, we shall confine our attention to the control of minimally persistent formations. Obviously, understanding their control is a precursor to being able to control any formation. Our prime interest is in the following problem. Suppose that all agents in a formation are correctly positioned prior to $t = 0$. Just before $t = 0$, they undergo small displacements from their initial positions. After $t = 0$, those with a DOF are not allowed to exercise that DOF; those required to maintain distances are required to adjust their positions in order to restore any incorrect distances to the correct value. In so doing, they can use only relative position information between themselves and the agents from which they are required to maintain their distance. The whole process has to occur so that the closed loop is stable.

This is a form of zero-input stability. This stability then underpins the preservation of a formation shape when those agents with a positive number of DOFs actually move, so that the whole formation moves while maintaining its shape. In this paper, we do not work through the details of establishing formation shape stability when motion occurs. In this connection, one should recognize that unless the motions executed by the agents with a positive DOF are in some way regular, e.g., constant direction and speed are maintained, one must always expect some distortion of the formation shape away from the nominally correct shape. The analysis in the paper is also a small signal analysis, i.e., we postulate the applicability of linearized models.

2.2. Leader-first follower formations. In this subsection, we categorize minimally persistent formations by the various possible ways DOFs can be allocated. In the rest of the paper we will deal with formations that belong to one such category only, namely, that categorized below as having a leader and first follower structure.

As described in [20], the arrangement of vertices with a positive number of DOFs of a two-dimensional minimally persistent formation can have two possible structures: leader-follower and three-coleaders, and within these structures subtypes occur. The leader-follower structure has two possible subtypes: leader-first follower and leader-remote follower; also the three coleaders in a three-coleader structure may or may not be adjacent to each other, thus giving rise to four possible subtypes: cyclic coleaders, in-line coleaders, one-two coleaders, or distributed coleaders. One can define these terms in more detail as follows.

Definition 2.4. Leader-follower *is a formation structure in which there is an agent (called the* leader*) l that has two DOFs, and another agent f with one DOF. A leader-follower structure is further termed* leader-first follower *if l is a neighbor of f (f is then called the* first follower*) and named* leader-remote follower *if l is not a neighbor of f (f is called the* remote follower *in this case).*

Three-coleaders *is a formation structure in which there are three agents c_1, c_2, c_3 (called the* coleaders*) with one DOF each. A three-coleaders structure is further named* cyclic coleaders *if c_1, c_2, c_3 are adjacent to one another, thus creating a cycle (c_1, c_2, c_3); it is named* in-line coleaders *if the three agents are neighbors but do not form a cycle; it is named* one-two coleaders *if two (say, c_1, c_2) of the three agents are neighbors and the remaining one (c_3) is not a neighbor of c_1 or c_2; and it is named distributed coleaders, if none of c_1, c_2, c_3 are neighbors of one another.*

In this paper, the formations we study *are of a leader-first follower variety*. Figure 4 provides an example of such a formation. Note in particular that 5 is the first follower and 6 is the leader. All the other agents have exactly two outgoing edges (thus guaranteeing minimal persistence, as discussed in the previous subsection), and there is exactly one cycle {1,2,3}. There exists a formation with this same graph, but the coordinates are chosen such that an intuitively appealing control law presented later actually will lead to instability, as shown by an example later in section 5.

3. Equations of motion. In this section, we set up the equations related to the control of a minimally persistent formation in the plane. The formation has a leader and first follower. To motivate the derivation of the equations, we will first describe the approach using discrete time ideas. The derivation itself will, however, use continuous time.

With a leader-first follower structure, there are two vertices with a positive number of DOFs, and the remainder have no DOFs, having to maintain their distance from precisely two other vertices.

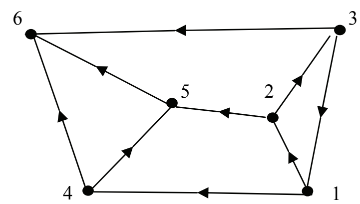

FIG. 4. *An example of a minimally persistent formation with leader-first follower structure and a cycle* $\{1, 2, 3\}$.

A discrete time view of the adjustment process is as follows. Prior to time 0, all distances are correct. At time 0, all agents are moved a small amount, so that distances are no longer correct. Between time 0 and 1 all agents determine the position of the agents from which they are required to maintain a correct distance, and then determine the point to which they would have to move to correct any distance errors. The leader itself makes no determination and remains permanently stationary after its initial move at time 0. Between time 0 and 1, the first follower determines the point on a straight line joining it to the leader which is at the correct distance from the leader. There are two possible points to which all other agents could move; each chooses the point closest to their position just after time 0.

At time 1, all agents then move to the positions they determined between 0 and 1. Between time 1 and 2, all agents review whether their positions are correct by checking the current distances to their neighbors, and determine the correction that would be required to re-establish the required distances, assuming their neighbors do not move. At time 2 the mispositioned agents actually execute a move. This process can clearly be repeated.

In the case of formations with an acyclic graph, it is clear that after a finite number of steps, all agents will become stationary. Indeed, after r steps, those agents with a path of length at most r steps to the first follower and the leader will cease to move. In the case of a formation with a cyclic graph, the question arises as to whether the process of adjustment is of infinite duration (it is, in general), and if so, whether the adjustments get smaller and smaller so that the agent positions converge, or whether there is continuous oscillation in the position. Our focus is on this case.

3.1. General approaches to control law derivation for formations. As is common, though not universal [1, 12, 13, 21, 22, 23], we shall adopt a simple kinematic velocity control model for each agent:

$$\dot{p}_i = u_i. \tag{3.1}$$

Suppose that agent i is tasked with maintaining distances d_{ij}^* and d_{ik}^* from two other agents j and k. The closed-loop control laws will typically be of the form

$$u_i = u_i(p_j - p_i, p_k - p_i, d_{ij}^*, d_{ik}^*). \tag{3.2}$$

There are several key points we need to make about this law. First, the law uses *relative positions* of agents j and k, and not just the current distances of agent i from agents j and k. Thus *more needs to be sensed than is controlled*, a not uncommon situation in modern control. Relative positions can be sensed if agent i is equipped with distance and direction sensors; alternatively, if agents are able to sense distances not just of their neighbors in the graph we are using to define the formation, but of their two-hop neighbors in the same graph, and if they are able to pass those distances to their neighbors, relative positions can be sensed. Thus if agent j can sense its distance from agent k and inform agent i, then agent i can determine the angle between the lines joining it to agents j and k. If agent i has its own coordinate basis, then knowledge of this angle is equivalent to knowledge of relative positions.

Agent i needs also to know not just the values of d_{ij}^* and d_{ik}^* but also the orientation of the triangle $i - j - k$ when agents are in their correct positions, and it is assumed that the departures from equilibrium conditions are sufficiently small that the orientation of this triangle is not disturbed.

Second, the law has to have a rotational invariance property. If R_i is a rotation matrix, it is clear that we need the property

$$u_i(R_i(p_j - p_i), R_i(p_k - p_i), d_{ij}^*, d_{ik}^*) = R_i u_i(p_j - p_i, p_k - p_i, d_{ij}^*, d_{ik}^*).$$

This expresses the fact that if the coordinate system in which positions are measured is rotated, the same rotation needs to apply to the controls, which determine derivatives of position. A consequence of this fact is that each agent can compute using its own local coordinate system, and agents do not need to share a common understanding of the direction of north. To see this, let q_{ij} denote the position of agent j in some local coordinate system maintained by agent i. Then there exists a rotation matrix R_i and a translation vector τ_i such that

$$q_{ij} = R_i p_j + \tau_i.$$

If the control is computed using the local coordinate basis, it will be

$$\begin{aligned} u_i(q_{ij} - q_{ii}, q_{ik} - q_{ii}, d_{ij}^*, d_{ik}^*) &= u_i(R_i(p_j - p_i), R_i(p_k - p_i), d_{ij}^*, d_{ik}^*) \\ &= R_i u_i(p_j - p_i, p_k - p_i, d_{ij}^*, d_{ik}^*) \end{aligned} \tag{3.3}$$

and in the global coordinate basis, this is the same as (3.2).

Third, the control law is decentralized. Obviously, its implementation just uses sensed data local to agent i. It could also have been decentralized in a second sense, specifically if the design of the law for agent i apparently took no account of the design for other agents. As it turns out, to control a persistent formation, we are able to propose a law which is decentralized in its operation, but *not* decentralized in its design. Put another way, the control law we end up proposing for agent i will depend on more data defining the desired formation shape (but not the current formation shape) than just the two distances d_{ij}^* and d_{ik}^*, even though the only data from the current formation shape are the relative positions of its neighbors. We will in fact present an approach which fixes the control laws for each agent in sequence, and the law for any one agent depends on the parameters of the laws for the preceding agents of the sequence. The sensed data, however, for the law at each agent is unchanged.

As for the determination of the actual law, it is common [1, 2, 3, 4, 5, 6, 7, 12, 21] when distance constraints are bidirectional to select a type of Lyapunov function, typically reflecting the distance errors, and to choose the control laws to ensure that the derivative is nonpositive. (It is generally not possible to ensure that the derivative is negative). Given that the Lyapunov function converges to a limit, it is a separate issue to show that this convergence implies correct convergence of the formation shape. This approach, however, does not really extend easily to the case where the distance

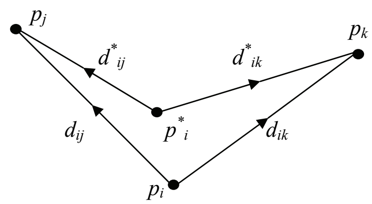

FIG. 5. *Illustration of position adjustment for point agent i at p_i with respect to agent j at p_j and agent k at p_k. The desired position is at p_i^*.*

constraints are unidirectional, which is the reason motivating the alternative approach outlined at the start of the section.

Last, we note that it is common to distinguish between results applicable to linearized closed-loop models, and results which offer convergence starting from a very wide range of initial conditions. Very few "almost global" results are actually available [21, 22, 23], serving as examples of exceptions. In this paper, we set up a nonlinear law and prove convergence of a linearized version in which adjustable parameters have to be set at certain values. Of course, if for a nonlinear law, convergence of a linearized version is proved, it may be that convergence for the nonlinear law will occur for a wide range of initial conditions, but in general this has to be established with an investigation of a particular case.

3.2. A nonlinear law. Suppose the agents are numbered from 1 to n, with the first follower and leader $n-1$ and n, respectively. Suppose that the initial (prior to time 0) position of all vertices is given (in a global coordinate basis) by $p_{i0} = [x_{i0}\ y_{i0}]'$, $i = 1, 2, \ldots, n$, with distance constraints all satisfied. Suppose that at time 0, all agents are displaced from their initial positions. In a moment, we shall impose a bound on the magnitude of the displacement.

At time t, agent i uses the relative position information of its neighbors (suppose they are agents j and k) to determine the point $p_i^*(t)$ which is at the correct distances d_{ij}^* and d_{jk}^* from agents j and k, respectively, and (noting that there are two such points) is the closer of the two possible points to p_i. (See Figure 5.) In order to do this, it is required that the displacements of p_j and p_k from their initial positions not be so great as to mean there is no possible point p_i^*. If the agents p_j and p_k were to remain at their initial predisplacement positions, p_i^* would coincide with the initial predisplacement position p_i of agent i. If they were to move so that their separation exceeded $d_{ij}^* + d_{jk}^*$, no p_i^* could be found. Obviously then, a nonzero upper bound on the displacements can be found, assuring the existence of p_i^*. Observe that we can write

$$p_i^* - p_i = f(p_j - p_i, p_k - p_i, d_{ij}^*, d_{ik}^*) \tag{3.4}$$

for some function f which is independent of i. The control law to be used is one which moves p_i closer to p_i^*, but it makes no allowance for the fact that because p_j and p_k are likely to be moving, p_i^* will also be changing. Thus we suppose that for some K_i,

$$\dot{p}_i = K_i(p_i^* - p_i) = K_i f(p_j - p_i, p_k - p_i, d_{ij}^*, d_{ik}^*). \tag{3.5}$$

If p_i^* were to be constant, any K_i with positive real part eigenvalues could be used, including $K_i = I$. However, p_i^* will not be constant in general, and so a more sophisticated way of choosing K_i is needed. It will be shown in section 5 by way of an example that the choice $K_i = I$ for all i may actually be destabilizing, and indeed the example displays stabilizing gains for which at least one of the K_i's does not even have positive real part eigenvalues. A significant part of the paper from subsection 3.5 will deal with the basis for choosing K_i. As indicated in the previous subsection, the law in question will be decentralized in operation, i.e., only local sensed data are used, but in design, it will not be decentralized; for, as it turns out, the choice of the 2×2 gain matrices K_i to assure stability demands this.

Equation (3.5) covers agents 1 through $n-2$. Agent n, the leader, will be assumed stationary. The law for agent $n-1$, the first follower, is one which requires it to determine p^*_{n-1} as the point on the line joining p_{n-1} to p_n which is at the correct distance for the first follower from p_n, and then the first follower moves toward that point.

Thus we have

$$p^*_{n-1} - p_{n-1} = \frac{||p_n - p_{n-1}|| - d^*_{n-1,n}}{||p_n - p_{n-1}||}(p_n - p_{n-1}), \tag{3.6}$$

and for the control law, with some positive k_{n-1},

$$\dot{p}_{n-1} = k_{n-1}(p^*_{n-1} - p_{n-1}) = k_{n-1}\frac{||p_n - p_{n-1}|| - d^*_{n-1,n}}{||p_n - p_{n-1}||}(p_n - p_{n-1}). \tag{3.7}$$

Together with

$$\dot{p}_n = 0, \tag{3.8}$$

(3.5) and (3.7) define the nonlinear closed-loop system. Rather than giving a formal proof of existence of solutions, later in the paper we shall demonstrate that stability can be assured for the linearized equations through an appropriate choice of the K_i and k_{n-1}, which is an indirect proof of solution existence.

3.3. Linearized equations. Let us suppose henceforth that all displacements are small enough to allow first order approximation, and in particular that we can represent at all times the position of agent i by $p_i(t) = \delta p_i(t) + \bar{p}_i$, where the $\bar{p}_i$ correspond to agent positions for which all desired distance constraints are met, and $\delta p_i(t)$ is small. Let $p_i(t) = [x_i(t)\ y_i(t)]'$, $\bar{p}_i = [\bar{x}_i\ \bar{y}_i]'$, and $\delta p_i(t) = [\delta x_i(t)\ \delta y_i(t)]'$. Below, we will indicate more specifically how the $\bar{p}_i$ are determined. Note that $p^*_i(t) \neq \bar{p}_i$ in general. This is because p^*_i would denote an equilibrium position for p_i only if p_j and p_k never moved. In general they will move. We assume also that all quantities $||p_i(t) - p^*_i(t)||$ are small, which can be guaranteed if the initial displacements away from equilibrium are all small and the subsequent motion is stable.

We consider first agents 1 through $n-2$. Refer to Figure 5, and apply the cosine law to the triangle with corners p_i, p^*_i, and p_j. Because $||p_i - p^*_i||$ is small, there holds (neglecting the square of $||p_i - p^*_i||$)

$$||p_j - p_i||^2 - 2[p_j - p_i]^T[p^*_i - p_i] \approx ||p_j - p^*_i||^2, \tag{3.9}$$

which may be rewritten as

$$2[p_j - p_i]^T[p^*_i - p_i] \approx d^2_{ij} - d^{*2}_{ij}.$$

Noting that $p_j - p_i = \bar{p}_j - \bar{p}_i + \delta p_j - \delta p_i$, and again neglecting second order terms, we get the further approximation

$$2[\bar{p}_j - \bar{p}_i]^T [p_i^* - p_i] \approx d_{ij}^2 - d_{ij}^{*2}. \tag{3.10}$$

Accordingly, provided agents i, j, and k are not collinear (i.e., that the vector $p_i - p_j$ is not parallel to the vector $p_i - p_k$), we will have

$$p_i^* - p_i \approx \frac{1}{2} \begin{bmatrix} \bar{x}_j - \bar{x}_i & \bar{y}_j - \bar{y}_i \\ \bar{x}_k - \bar{x}_i & \bar{y}_k - \bar{y}_i \end{bmatrix}^{-1} \begin{bmatrix} d_{ij}^2 - d_{ij}^{*2} \\ d_{ik}^2 - d_{ik}^{*2} \end{bmatrix}. \tag{3.11}$$

It is normal in applying rigid graph theory to formations to assume that the formations are generic; a consequence of this assumption is that colinearities are excluded, and so the matrix inverse in (3.11) exists.

Next, it is straightforward to check that, again neglecting second order terms,

$$\begin{aligned} \frac{1}{2}[d_{ij}^2 - d_{ij}^{*2}] &= \frac{1}{2}[||p_j - p_i||^2 - ||\bar{p}_i - \bar{p}_j||^2] \\ &= \frac{1}{2}[||(\bar{p}_j - \bar{p}_i) + (\delta p_j - \delta p_i)||^2 - ||\bar{p}_j - \bar{p}_i||^2] \\ &\approx [\bar{p}_i - \bar{p}_j]^T \delta p_i - [\bar{p}_i - \bar{p}_j]^T \delta p_j, \\ \frac{1}{2}[d_{ik}^2 - d_{ik}^{*2}] &\approx [\bar{p}_i - \bar{p}_k]^T \delta p_i - [\bar{p}_i - \bar{p}_k]^T \delta p_k. \end{aligned}$$

Putting this together with (3.5), there results

$$\begin{bmatrix} \dot{\delta x_i} \\ \dot{\delta y_i} \end{bmatrix} = K_i \begin{bmatrix} \bar{x}_j - \bar{x}_i & \bar{y}_j - \bar{y}_i \\ \bar{x}_k - \bar{x}_i & \bar{y}_k - \bar{y}_i \end{bmatrix}^{-1} R_{(ij,ik)} \begin{bmatrix} \delta x_i \\ \delta y_i \\ \delta x_j \\ \delta y_j \\ \delta x_k \\ \delta y_k \end{bmatrix} \tag{3.12}$$

with $R_{(ij,ik)} = \begin{bmatrix} \bar{x}_i - \bar{x}_j & \bar{y}_i - \bar{y}_j & -\bar{x}_i + \bar{x}_j & \bar{y}_j - \bar{y}_i & 0 & 0 \\ \bar{x}_i - \bar{x}_k & \bar{y}_i - \bar{y}_k & 0 & 0 & -\bar{x}_i + \bar{x}_k & -\bar{y}_i + \bar{y}_k \end{bmatrix}$.

Observe for later reference that $R_{(ij,ik)}$ is a submatrix of the rigidity matrix, with rows corresponding to the edges $\{i, j\}$ and $\{i, k\}$ and columns corresponding to vertices i, j, and k.

We turn now to the equation governing the first follower. Consider first the motion defined by (3.7), and recall that the first follower moves along the line joining it to the leader. Further, the leader remains stationary, and it is logical then to take $\bar{p}_n$ as its position so that $p_n(t) = \bar{p}_n$. Then the instantaneous target point for the first follower remains constant, and it is natural to take this point as the desired equilibrium position, i.e., $p_{n-1}^*(t) = \bar{p}_{n-1}$.

For the linearized system, there results

$$\begin{bmatrix} \dot{\delta} x_{n-1} \\ \dot{\delta} y_{n-1} \end{bmatrix} = k_{n-1} I_2 \begin{bmatrix} \bar{x}_n - \bar{x}_{n-1} & \bar{y}_n - \bar{y}_{n-1} \\ -(\bar{y}_n - \bar{y}_{n-1}) & \bar{x}_n - \bar{x}_{n-1} \end{bmatrix}^{-1} R_{((n-1)n,00)} \begin{bmatrix} \delta x_{n-1} \\ \delta y_{n-1} \\ \delta x_n \\ \delta y_n \end{bmatrix} \tag{3.13}$$

with

$$R_{((n-1)n,00)} = \begin{bmatrix} \bar{x}_{n-1} - \bar{x}_n & \bar{y}_{n-1} - \bar{y}_n & -\bar{x}_{n-1} + \bar{x}_n & -\bar{y}_{n-1} + \bar{y}_n \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Of course, the equations for the leader are

$$\begin{bmatrix} \dot{\delta} x_n \\ \dot{\delta} y_n \end{bmatrix} = 0. \tag{3.14}$$

In the light of the above discussion, one could arrive at the following theorem which is the main result of this section.

THEOREM 3.1. *The linearization of* (3.5), (3.7), *and* (3.8) *under the first order approximation is*

$$\dot{\delta} p(t) = K R_e^{-1} \begin{bmatrix} R \\ 0 \end{bmatrix} \delta p(t), \tag{3.15}$$

where K and R_e are diagonal block matrices, with each block of size 2×2. *(The last block of R_e for convenience can be taken as the identity).*

Of course, $K = \operatorname{diag}[K_1, K_2, \ldots, K_{n-2}, k_{n-1} I_2, 0]$, and the first $(n-2)$ diagonal blocks of R_e are obtained as 2×2 submatrices of the rigidity matrix, selecting rows corresponding to edges $\{i, j\}, \{i, k\}$ and columns corresponding to vertex i.

3.4. Simplified dynamics. Our ultimate goal is to show that through a suitable choice of gains, the nonlinear system can be stabilized. This will be done by choosing gains to stabilize the system obtained by linearizing the nonlinear system around the equilibrium point. However, as is evident from the linearized equation (3.15), there will necessarily be three modes of the linearized system which are located at the origin; this apparently makes it much more difficult to establish a stability result for the nonlinear system than for the linear system. Nevertheless, a modest modification of the usual approach will work.

For the purpose of the theoretical analysis, without loss of generality let us choose the global coordinate basis so that the x-axis coincides with the line joining agents $n-1$ and n at the start of the motion.[3] Because agent $n-1$ moves solely on this line, it will stay on the x-axis, and so there will hold $y_{n-1}(t) = \bar{y}_{n-1} = \bar{y}_n$ for all t. Obviously then, for the nonlinear system, $\delta y_{n-1}(t) = y_{n-1}(t) - \bar{y}_{n-1}$ will be identically zero. Because the leader does not move, for the nonlinear system δx_n and δy_n are identically zero.

Examination of the linearized equation (3.13) shows also that

$$\begin{bmatrix} \dot{\delta} x_{n-1} \\ \dot{\delta} y_{n-1} \end{bmatrix} = \begin{bmatrix} -k_{n-1} \delta x_{n-1} \\ 0 \end{bmatrix}, \tag{3.16}$$

while the equations for the leader remain the same as (3.14). It is straightforward to check that the second entry of (3.16) is also true for the original nonlinear system.

Let $\hat{R}_e$ denote R_e with the last three rows and columns discarded; i.e., it is a $(2n-3) \times (2n-3)$ submatrix of R_e. Correspondingly, we have to use a $(2n-3) \times (2n-3)$ submatrix of K and consider $K = \operatorname{diag}[\hat{K}, 0_3]$, i.e.,

$$\hat{K} = \left(\bigoplus_{i=1}^{n-2} K_i \right) \bigoplus k_{n-1}, \tag{3.17}$$

[3]If the new global basis is obtained from the original one by a rotation and translation, the gains K will all differ in the two coordinate bases by the same orthogonal similarity transformation.

where $\hat{K}_i$ are each 2×2 and k_{n-1} is a scalar.

Recall also that

$$\begin{bmatrix} \delta y_{n-1} \\ \delta x_n \\ \delta y_n \end{bmatrix} = 0. \tag{3.18}$$

This means that we can replace (3.15) with the simplified equation

$$\begin{bmatrix} \dot{\delta x_1} \\ \dot{\delta y_1} \\ \dot{\delta x_2} \\ \dot{\delta y_2} \\ \vdots \\ \dot{\delta x_{n-2}} \\ \dot{\delta y_{n-2}} \\ \dot{\delta x_{n-1}} \end{bmatrix} = \hat{K}\hat{R_e}^{-1}\hat{R} \begin{bmatrix} \delta x_1 \\ \delta y_1 \\ \delta x_2 \\ \delta y_2 \\ \vdots \\ \delta x_{n-2} \\ \delta y_{n-2} \\ \delta x_{n-1} \end{bmatrix}, \tag{3.19}$$

where $\hat{R_e}$ and $\hat{R}$ are obtained by removing the last three columns from R_e and R, respectively.

Notice that (3.18) also holds for the nonlinear system. Equation (3.19) is actually the linearized version of the nonlinear equations governing the first $2n-3$ coordinates. Accordingly, if $\hat{K}$ in (3.19) is chosen to ensure that (3.19) is exponentially stable, then the nonlinear equations governing the first $2n-3$ coordinates will be exponentially stable for all initial conditions within some domain of attraction, and the motion of the last three coordinates is trivially defined by (3.18), i.e., there is no motion.

Observe in (3.19) that $\dot{\delta p_i}$ can depend on only $\delta p_i, \delta p_j$, and δp_k, where agents j and k are those from which agent i must maintain its distance. This forces $\hat{K}$ to have the structure of (3.17), but does not constrain the individual blocks to be, for example, individually diagonal or multiples of the identity. For this problem, $\hat{K}$ in fact serves as the controller. Of course, one could contemplate replacing $\hat{K}$ by some dynamics, in which $\dot{\delta p_i}$ was determined by dynamic processing of $\delta p_i, \delta p_j$, and δp_k, but this is beyond the scope of the paper.

It should be noted that in (3.19), the matrix $\hat{R}_e^{-1}\hat{R}$ is defined in terms of the target positions $\bar{p}_i$, which of course are not known a priori. Consider now, though, the setting where a previously intact formation was deformed by the "small" movements of multiple agents. *Then if under a given $\hat{K}$, $\hat{K}\hat{R}_e^{-1}\hat{R}$ is sufficiently Hurwitz, with $\bar{p}_i$ in $\hat{R}_e^{-1}\hat{R}$ replaced by the positions prior to the deformation, then under sufficiently small movements that cause the deformation,* (3.19) *will remain Hurwitz. Thus, in what follows, which is concerned with designing $\hat{K}$ to ensure local stability, to avoid notational complexities we will assume that the matrix $\hat{R}_e^{-1}\hat{R}$ is formed with the positions prior to deformation replacing $\bar{p}_i$, subject to the coordinate transformation described in this section.*

3.5. Choosing the block diagonal control multiplier. From the above analysis of a continuous time version of the formation shape maintenance problem, we have determined that the underlying dynamic equation is of the form

$$\dot{z} = \Lambda A z. \tag{3.20}$$

In this equation, Λ is a diagonal or possibly block diagonal matrix. Its entries correspond to gains associated with the control used by each agent. If each agent were to apply the same gain to the two distance constraints, then we would have Λ of the form $\lambda_1 I_2 \oplus \lambda_2 I_2 \oplus \dots$.

For the moment, in this section we will assume that Λ is diagonal and all the diagonal elements of Λ can be independently chosen. The key result, which will be established with a constructive procedure, is as follows.

THEOREM 3.2. *Suppose A is an $m \times m$ nonsingular matrix with every leading principal minor nonzero. Then there exists a diagonal Λ such that the real parts of the eigenvalues of ΛA are all negative.*

Proof. The proof of the theorem will proceed by induction on m. We shall ensure that $m-1$ eigenvalues of ΛA lie very far in the left half plane by selecting the first $m-1$ diagonal entries of Λ, and then show how the mth diagonal entry λ_m can be chosen to make the last eigenvalue of ΛA have negative real part. Suppose the theorem is true for $m = 1, 2, \dots, r-1$ (it is trivially true for $m = 1$). Consider the case $m = r$ and suppose A has nonzero leading principal minors. Write

$$A = \begin{bmatrix} A_{11} & a_{12} \\ a_{21}^T & a_{22} \end{bmatrix},$$

where A_{11} is $(r-1) \times (r-1)$ and nonsingular with nonsingular leading principal minors, $a_{12}, a_{21} \in \Re^{r-1}$, $a_{22} \in \Re$. Appealing to the induction hypothesis, choose Λ_1 diagonal so that $\Lambda_1 A_{11}$ has all eigenvalues with negative real parts. Now recall that if

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \epsilon^{-1}\Lambda_1 A_{11} z_1 + \epsilon^{-1}\Lambda_1 a_{12} z_2 \\ \lambda_2 a_{21}^T z_1 + \lambda_2 a_{22} z_2 \end{bmatrix}$$

and if the real parts of the eigenvalues of $\Lambda_1 A_{11}$ are all negative, then provided ϵ is small enough we can use singular perturbation theory to study stability [24]. The high order system is asymptotically stable if an associated lower order system is asymptotically stable. This low order system is obtained by replacing the differential equation for z_1 by the equation

$$\begin{aligned} z_1 &= -(\epsilon^{-1}\Lambda_1 A_{11})^{-1}(\epsilon^{-1}\Lambda_1 a_{12}) z_2 \\ &= -A_{11}^{-1} a_{12} z_2, \end{aligned}$$

and then the differential equation for z_2 becomes

$$\begin{aligned} \dot{z}_2 &= -\lambda_2 a_{21}^T A_{11}^{-1} a_{12} z_2 + \lambda_2 a_{22} z_2 \\ &= \lambda_2 [a_{22} - a_{21}^T A_{11}^{-1} a_{12}] z_2. \end{aligned}$$

Choosing λ_2 so that $\lambda_2[a_{22} - a_{21}^T A_{11}^{-1} a_{12}] < 0$ ensures stability of z_2 and then the whole of z. Also,

$$|A| = |A_{11}|(a_{22} - a_{21}^T A_{11}^{-1} a_{12}),$$

so there is no possibility that $a_{22} - a_{21}^T A_{11}^{-1} a_{12} = 0$. $\square$

Remark 3.1. Examining the literature for theorems of this variety, one can see that if A and Λ are permitted to be complex, then the same leading principal minor condition guarantees that a Λ can be found to produce any prescribed set of eigenvalues; see [25]. However, the method of proof cannot be carried over to the real case. Further, it is easy to show with a 2×2 counterexample that eigenvalue positionability in the real case cannot always be guaranteed, even with nonzero principal minors.

Remark 3.2. In order to apply this theorem to our situation, we will need to show that the matrix $\hat{R}$ defined in the previous subsection (which corresponds to A in the theorem) has all principal minors nonzero. This is not straightforward and will be done in the next section. Of course, $\hat{K}(\hat{R}_e)^{-1}$ will correspond to Λ, and we achieve a Λ by choice of $\hat{K}$ that in turn conforms to (3.17).

Remark 3.3. The condition of the theorem is a sufficiency condition. It would appear not to be necessary. However, in order that a stabilizing Λ exist, it is certainly necessary that A be nonsingular. For if A is singular, then so is ΛA and there is necessarily at least one zero eigenvalue. Also, suppose A is $m \times m$, and the characteristic polynomial is $|sI - A| = A^m + \alpha_1 A^{m-1} + \cdots + \alpha_m$; then $\alpha_i = (-1)^i \sum$ (all $i \times i$ principal minors of A). Hence if all $i \times i$ principal minors of A are zero, then $\alpha_i = 0$. Also, all $i \times i$ principal minors of A are zero if and only if all $i \times i$ principal minors of ΛA (for nonsingular Λ) are zero. Hence if all $i \times i$ principal minors of A are zero, the characteristic polynomial of ΛA will have a zero coefficient. Since for the characteristic polynomial to be stable, all coefficients must be positive, we see that a necessary condition for Λ to exist is that for all i, at least one $i \times i$ principal minor of A is nonzero.

Remark 3.4. There is a trivial extension to the theorem: if the rows and columns of A can be symmetrically reordered so that every leading principal minor is nonzero, then Λ can be chosen with the desired properties.

4. The principal minor condition. Recall from the previous sections that the key technical condition required for stabilizability is that a certain matrix have all its leading principal minors nonzero. This section addresses this issue. In particular, recall that with $V = \{1, \dots, n\}$, the directed graph $G = (V, E)$ has a leader-first follower structure with n and $n-1$ being the leader and the first follower, respectively. In what follows, suppose the rigidity matrix R is such that its last row corresponds to the only outgoing edge that the first follower has, i.e., the outgoing edge from $n-1$ to n.

Recall that $\hat{R}$ is the $(2n-3) \times (2n-3)$ submatrix of the rigidity matrix R of G, obtained by removing the last three columns of R. Further, $\hat{R}_e$ is the $(2n-3) \times (2n-3)$ matrix as below:

$$\left(\bigoplus_{i=1}^{n-2} B_i \right) \bigoplus (x_n - x_{n-1}), \tag{4.1}$$

where if node $i \in V' = \{1, \dots, n-2\}$ has outgoing edges to j and k, then

$$B_i = - \begin{bmatrix} x_i - x_j & y_i - y_j \\ x_i - x_k & y_i - y_k \end{bmatrix}. \tag{4.2}$$

This implicitly enforces the following ordering of the rows and columns of $\hat{R}$. Columns $2k-1$ and $2k$ correspond to node $k \in V'$. Rows $2k-1$ and $2k$ correspond to the two outgoing edges of node $k \in V'$. The last row of R corresponds to the edge from $n-1$ to n.

Our goal is to prove the following main result of this section.

THEOREM 4.1. *Consider an n-node minimally persistent formation F with agent set $P = \{1, \dots, n\}$ at generic positions, and n and $n-1$ the leader and the first follower, respectively. Suppose $\hat{R}$ is the $(2n-3) \times (2n-3)$ submatrix of the rigidity matrix R of F, obtained by removing the last three columns of R and obeying the row and column ordering noted above. Then there exists an ordering of the first $n-2$*

vertices of F and an ordering of the pair of outgoing edges for each of these vertices such that the leading principal minors of the associated $\hat{R}$ are generically nonzero.

In order to prove this theorem, we will first establish several lemmas and another theorem in this section. The proof is completed at the end of the section. The significance of this result is as follows. From Theorem 3.2, one can find a diagonal Λ such that $\Lambda\hat{R}$ has all eigenvalues in the open left half plane. Consequently, with

$$\hat{K} = \Lambda\hat{R}_e \tag{4.3}$$

the matrix $\hat{K}\hat{R}_e^{-1}\hat{R}$ can be made to have all eigenvalues in the open left half plane. For such a choice the control laws of the previous section stabilize the system. Further, the the block diagonal nature of $\hat{R}_e$ outlined in (4.1) and (4.2) ensures that $\hat{K}$ has the right structure, in that it is block diagonal with the first $n-2$ diagonal blocks being 2×2 matrices and the last diagonal element being scalar. As noted in the previous section, this ensures that to implement its individual control law each agent need only sense the relative positions of its neighbor agents.

To proceed with the proof of Theorem 4.1 we first prove that the matrix $\hat{R}$, obtained by removing the last three columns of the rigidity matrix R, is generically nonsingular.

LEMMA 4.2. *Under the hypothesis of Theorem* 4.1*, $\hat{R}$ is generically nonsingular.*

Proof. The null space of R has as its basis the following three $2n$-vectors:

$\eta_1 = [1, 0, 1, 0, 1, 0, \ldots]'$,

$\eta_2 = [0, 1, 0, 1, 0, 1, 0, \ldots]'$, and

$\eta_3 = [y_1, -x_1, y_2, -x_2, y_3, -x_3, \ldots, y_{n-1}, -x_{n-1}, y_n, -x_n]'$.

If the $(2n-3)\times(2n-3)$ matrix $\hat{R}$ is singular, it must have a $(2n-3)$-dimensional null vector $\eta \neq 0$. Then

$$R[\eta', 0, 0, 0]' = 0.$$

Thus as $[\eta', 0, 0, 0]'$ is in the space spanned by η_i, and $\eta \neq 0$, the matrix formed by the last three elements of each of η_i,

$$\begin{bmatrix} 1 & 0 & -x_{n-1} \\ 0 & 1 & y_n \\ 1 & 0 & -x_n \end{bmatrix},$$

must be generically singular. This is of course false. ∎

We require further notation. Recall that the set V' comprises the follower nodes. Consider now a subset of the follower nodes: $V_1 \subset V'$. Then we define $R(V_1)$ as the principal submatrix of $\hat{R}$ obtained by retaining the columns corresponding to the elements of V_1. Call $G_1 = (V_1, E_1)$ the subgraph of G induced by V_1 and conforming to the row and column ordering noted earlier. Note $R(V_1)$ is not the rigidity matrix of the induced subgraph G_1, as it may contain edge information regarding certain edges of G which are not in G_1.

First we have the following lemma.

LEMMA 4.3. *Under the hypothesis of Theorem* 4.1*, $R(V')$ is generically nonsingular.*

Proof. This follows by noting that with $\times$ a *don't care* vector, $\hat{R}$ can be partitioned as

$$\hat{R} = \begin{bmatrix} R(V') & \times \\ 0 & x_{n-1} - x_n \end{bmatrix}.$$

Then the result follows from Lemma 4.2. ∎

Next, we prove the following lemma.

LEMMA 4.4. *Consider a minimally persistent graph with a leader-first follower structure $G = (V, E)$, and any induced subgraph $G_1 = (V_1, E_1)$, such that V_1 contains neither the leader nor the first follower. Call $V_2 \subset V - V_1$ the set of vertices in $V - V_1$ that have incoming edges from V_1 in G. Suppose $|V_2| \geq 2$. Define E_{21} as the set of outgoing edges of nodes of V_1 in G and terminating in V_2. Construct a new graph $\bar{G} = (\bar{V}, \bar{E})$ with the following properties:*

(a) $\bar{V} = V_1 \cup V_2$.

(b) *Let $\widetilde{G}_2 = (V_2, \widetilde{E}_2)$ be any minimally persistent graph with a leader-first follower structure, with vertex set V_2, and with edge set $\widetilde{E}_2$ that is not required to be related in any way to the edges in G. Choose $\bar{E} = \widetilde{E}_2 \cup E_{21} \cup E_1$.*

Then $\bar{G}$ is a minimally persistent graph with a leader-first follower structure, and the leader and first follower belong to V_2.

Proof. By construction, while all outgoing edges of nodes in V_1 in the original graph G are in $\bar{E}$, no incoming edge to V_1 from V_2 in E also appears in $\bar{E}$. As $\widetilde{G}_2$ is minimally persistent, and hence by Theorem 2.3 minimally rigid, by Laman's theorem, $|\widetilde{E}_2| = 2|V_2| - 3$, and as no node of V_1 is either the leader or the first follower of G, $|E_{21}| + |E_1| = 2|V_1|$. Thus

$$|\bar{E}| = |\widetilde{E}_2| + |E_{21}| + |E_1| = 2|V_2| - 3 + 2|V_1| = 2|\bar{V}| - 3.$$

Further, again from Theorem 2.3, no vertex in $\bar{G}$ has more than two outgoing edges. Also by construction, $\bar{G}$ has a leader-first follower structure, with the leader and first follower the same as that for $\widetilde{G}_2$. Thus, in view of Theorem 2.3, to complete the proof of minimal persistence, we must demonstrate that $\bar{G}$ is rigid. Thus we must show that for any $\hat{V} \subset \bar{V}$, the subgraph of $\bar{G}$ induced by $\hat{V}$ has no more than $2|\hat{V}| - 3$ edges. Choose

$$\hat{V} = \hat{V}_1 \cup \hat{V}_2 \text{ with } \hat{V}_i \subset V_i.$$

We shall count the edges by counting outgoing edges. Suppose the nodes of $\hat{V}_1$ have m outgoing edges to the nodes of $\hat{V}_2$. Then observe that the number of edges in the graph induced by $\hat{V}_1$ is no greater than $2|\hat{V}_1| - m$. Finally, to count the outgoing edges associated with $\hat{V}_2$, note that by construction, V_2 has no outgoing edges to V_1 in $\bar{G}$, and hence all outgoing edges from $\hat{V}_2$ vertices must be edges of $\widetilde{G}_2$. Thus, as $\widetilde{G}_2$ is minimally persistent from Laman's theorem, the number of outgoing edges from $\hat{V}_2$ vertices is no greater than $2|\hat{V}_2| - 3$. Hence the number of edges in the graph induced by $\hat{V}$ is no greater than

$$m + 2|\hat{V}_1| - m + 2|\hat{V}_2| - 3 = 2|\hat{V}| - 3$$

and the result holds. ∎

Using this lemma we will prove the following theorem.

THEOREM 4.5. *Under the hypothesis of Theorem 4.1, $R(V_1)$ is generically nonsingular for every $V_1 \subset V'$.*

Proof. From Lemma 4.3 the result holds when $V_1 = V'$. Thus suppose $V_1 \neq V'$. Then we can argue that there are at least three outgoing edges from V_1 to $V - V_1$. If $g_1 = (V_1, E_1)$ is the induced subgraph, then by Laman's theorem, $|E_1| \leq 2|V_1| - 3$. Further, as V_1 does not contain either the leader or the first follower, every node in

V_1 has exactly two outgoing edges in G. Thus there must be at least three outgoing edges from V_1 to $V - V_1$.

Adopt now the notation of Lemma 4.4. Clearly $|V_2| \neq 0$. Suppose now, to obtain a contradiction, that $|V_2| = 1$. Then in the subgraph induced by $V_1 \cup V_2$ there are at least $2|V_1|$ edges, while $|V_1 \cup V_2| = |V_1| + 1$. This violates Laman's theorem as

$$2|V_1| > 2(|V_1| + 1) - 3.$$

Thus $|V_2| \geq 2$ and the conditions of Lemma 4.4 apply.

Using the notation and construction used in Lemma 4.4, call $\hat{\bar{R}}$ and $\hat{R}_2$ the matrices obtained by removing from the rigidity matrices of $\bar{G}$ and $\widetilde{G}_2$, respectively, the two columns corresponding to the common leader and one column corresponding to the common follower. By Lemma 4.4, both $\bar{G}$ and $\widetilde{G}_2$ are minimally persistent, and so by Lemma 4.2, $\hat{\bar{R}}$ and $\hat{R}_2$ are both generically nonsingular. Then the fact that no node in V_2 has an outgoing edge to the nodes of V_1 in $\bar{G}$, and that all the outgoing edges of V_1 in G are retained in $\bar{G}$, ensures that with $\times$ a *don't care* block,

$$\hat{\bar{R}} = \begin{bmatrix} R(V_1) & \times \\ 0 & \hat{R}_2 \end{bmatrix}.$$

Thus the result follows. □

Remark 4.1. Because of Theorem 4.5, we can only conclude at this point that every *even order* principal (rather than every leading principal) minor is nonzero.

We will now show that there is an ordering of vertices possible, and an ordering of the two outgoing rows associated with each vertex such that after this reordering, all leading principal minors of $\hat{R}$ are generically nonzero. We begin with the vertex reordering. Observe first the following fact, which will deal with the vertex reordering.

LEMMA 4.6. *Under the hypothesis of Theorem* 4.1*, there exists a sequence of nodes* $i_1, \ldots, i_j, \ldots, i_{n-2}$*, all in* V'*, that has the following property: for all* $1 < j \leq n-2$*,* i_j *has at most one outgoing edge in the subgraph of* G *induced by* $\{i_1, \ldots, i_j\}$*.*

Proof. Equivalently, one must show that the node i_j has at least one edge connecting to $\{i_{j+1}, i_{j+2}, \ldots, i_{n-2}, n-1, n\}$ in the original graph G. Because G is minimally persistent, by Theorem 2.3, the subgraph induced by V' has nodes which in the original graph G are the beginning vertex for at least three edges going outside the subgraph. This is because each vertex necessarily has two outgoing edges, meaning that in total there are three more than the induced subgraph is permitted to contain by Laman's theorem. Take node i_{n-2} to be the start node for any one of these edges.

Now consider the subgraph induced by the nodes in $V' - \{i_{n-2}\}$. Note that this subgraph has nodes which in the original graph G are the beginning vertices for at least three edges going outside the subgraph. Take node i_{n-3} to be the start node for any one of these edges. This procedure continues until all but two nodes have been assigned. Their assignment is trivial. □

Proof of Theorem 4.1. Select the ordering of the vertices guaranteed by Lemma 4.6. Now without loss of generality label $i_k = k$. Rows and columns $2j-1$ and $2j$ are associated with vertex j. In the ordering of rows of $R(V')$ select the penultimate row of $R(\{1, \ldots, j\})$ to be the outgoing edge of j that is not in the subgraph of G induced by $\{1, \ldots, j\}$. Consider now the $(2j-1)$th leading principal minor; i.e., under the relabeling above,

$$\det\left(\begin{bmatrix} R(\{1, \ldots, j-1\}) & \times \\ 0 & x_j - x_l \end{bmatrix}\right),$$

where $\times$ is a *don't care* vector and l is the node not in $\{1, \dots, j-1\}$ to which j's second outgoing edge goes. Then, as by Theorem 4.5 $R(\{1, \dots, j-1\})$ is nonsingular, this determinant is nonzero. Thus after the vertex reordering and ordering of outgoing edges at each vertex, all leading principal minors of $\hat{R}$ are generically nonzero.

Remark 4.2. We remark that our control law is distributed, but the design of the control laws of the agents requires a centralized view of the particular minimally persistent formation in question. There is in fact no agreed standard definition of decentralization for a multi-agent system. One needs to distinguish a *decentralized (local) controller* from a *decentralized controller design*. The key criterion for the former is that each agent has to be autonomous and local, such that it makes decisions entirely based on local sensing and without global knowledge. The latter, however, requires that the design process of the control laws be decentralized, which is to an extent counterintuitive. One might argue that, without knowing the desirable shape of the entire formation and the agent's role or relative position in this formation, there is no way that one could assign appropriate control laws to this agent such that it knows where to be and what to do.

5. Eigenstructure analysis and an example. In this section we explain in greater depth the need for premultiplication of the update kernel by a block diagonal matrix, with 2×2 diagonal blocks. In particular we ask: what happens if one selects the $\hat{K}$ in (4.3) as the identity matrix? In this case, stability would require that $\hat{R}_e^{-1}\hat{R}$ have all eigenvalues in the left half plane. The basic premise of this section is that while for acyclic graphs, these conditions always hold, for graphs with cycles there may be node coordinates that result in their violation. We present an example of such instability and show how a suitably selected $\hat{K}$ repairs this instability.

In subsection 5.1 we explore the structure of $\hat{R}_e^{-1}\hat{R}$. In subsection 5.2 we show that for acyclic graphs, all eigenvalues of $\hat{R}_e^{-1}\hat{R}$ are -1. Additionally, we provide certain conditions under which several eigenvalues of $\hat{R}_e^{-1}\hat{R}$ are -1 even in graphs with cycles. Subsection 5.3 considers graphs that may have nonoverlapping cycles, and subsection 5.4 uses the eigenstructure thus established to present the example noted above.

5.1. Structure of $\hat{R}_e^{-1}\hat{R}$. Consider any set $V_j = \{1, \dots, j\} \subset V' = \{1, \dots, n-2\}$, and recall the definition of $R(V_j)$ and the ordering enforced on the first $n-2$ rows of $\hat{R}$ presented in section 4. With the 2×2 matrices B_i defined in (4.2), further define

$$S(V_j) = \left(\bigoplus_{i=1}^{j} B_i \right)^{-1} R(V_j). \tag{5.1}$$

Observe that, with $\times$ denoting a *don't care* block element,

$$\hat{R}_e^{-1}\hat{R} = \begin{bmatrix} S(V') & \times \\ 0 & -1 \end{bmatrix}. \tag{5.2}$$

Thus we have the following obvious fact.

FACT 1. *At least one eigenvalue of $\hat{R}_e^{-1}\hat{R}$ is -1.*

Thus the interesting eigenvalues are those of $S(V')$. Further, $S(V')$ has the structure outlined below.

Define $e_1 = [1, 0]'$ and $e_2 = [0, 1]'$. Partition $S(V_j)$ into 2×2 blocks; then each matrix on the block diagonal is $-I_2$. Call X_{lr} the lrth off diagonal block element of $S(V_j)$. Then X_{lr} is nonzero if and only if there is an outgoing edge from l to r in the graph induced by V_j. There are thus at most two off diagonal nonzero block elements in each block row. If l has an outgoing edge to a node r in the subgraph induced by V_j, then if this edge information were in the $(2l-1)$th row of R,

$$X_{lr} = B_l^{-1} e_1 e_1' B_l. \tag{5.3}$$

If l has a second outgoing edge to a node s in the subgraph induced by V_j, and as this edge information must then be in the $2l$th row of R, then

$$X_{ls} = I - X_{lr} = B_l^{-1} e_2 e_2' B_l. \tag{5.4}$$

We will call X_{lr} the *edge weight* of the outgoing edge from l to r. Each edge weight has the following properties:

(a) It has rank 1.
(b) Its trace is 1.

5.2. Acyclic graphs. In this section we are concerned primarily with acyclic graphs. We first present a somewhat more general result.

THEOREM 5.1. *Suppose that q that vertices in the graph induced by $V' = \{1, 2, 3, \ldots, n-2\}$ defined above have no incoming edges and that m vertices have only one incoming edge each. Then there are at least $2q+m$ eigenvalues of $S(V')$ that are -1.*

Proof. Equivalently we need to show that $S(V')+I$ has at least $2q+m$ eigenvalues that are zero. This follows by noting that the first $2q$ columns of $S(V')+I$ are zero and the next m block columns of size $2(n-2)\times 2$ are each of rank 1. □

We next turn to graphs that are acyclic.

THEOREM 5.2. *Suppose the graph induced by V' defined above is acyclic. Then all eigenvalues of $S(V')$ are -1.*

Proof. Since the graph induced by V' is acyclic, there exists a sequence of nodes $i_1, \ldots, i_{n-2}$ such that for each $j > 1$, i_j has no outgoing edges to $\{i_1, \ldots, i_{j-1}\}$ in the graph induced by V'. Consequently under a symmetric permutation of its rows and columns $S(V')$ is upper triangular with diagonal elements all -1. This proves the result. □

COROLLARY 1. *Suppose the graph induced by V' defined above is acyclic. Then all eigenvalues of $\hat{R}_e^{-1}\hat{R}$ are -1. Therefore, with $K_i = I_2$ and $k_{n-1} = 1$, the linearized system* (3.19) *is asymptotically stable (and triangularly coupled).*

5.3. Graphs with nonoverlapping cycles. In what follows we call a graph $G'' = (V'', E'')$ a *pure cycle* if with $V'' = \{1, \ldots, k\}$, then $E'' = \{\{1, 2\}, \{2, 3\}, \ldots, \{k-1, k\}, \{k, 1\}\}$, where $\{i, j\}$ denotes an edge from i to j. If G'' is actually an induced subgraph of G, then we define its *cycle weight* to be the rank-1 matrix

$$X_{12} X_{23} \cdots X_{k-1,k} X_{k1}. \tag{5.5}$$

Again we recall that in Figure 4, $\{1, 2, 3\}$ constitutes a pure cycle, and because it is an induced subgraph, it will also have a cycle weight. We call the graph induced by V' *one with nonoverlapping cycles* if

$$V' = \bigcup_{i=1}^{r} V^i,$$

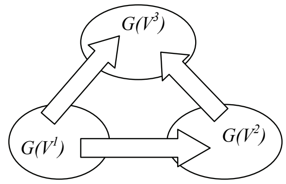

Fig. 6. *An abstracted formation showing a block of edges for a graph with pure cycles. Each $G(V^i)$ is a cycle or acyclic.*

where the graph induced by each V^i is either acyclic or a pure cycle, at least one such graph is a pure cycle, and no node of V^j has an outgoing edge to any node in

$$\bigcup_{i=1}^{j-1} V^i.$$

Then it is clear that for a graph with nonoverlapping cycles, such as in Figure 6, under a symmetric permutation of rows and columns, $S(V')$ has a block triangular structure, with $S(V^i)$ the diagonal blocks. Thus the set of eigenvalues of $S(V')$ is simply the union of the set of eigenvalues of these $S(V^i)$. In view of the results of subsection 5.2, we thus can just focus on one such V^i for which the induced subgraph is a pure cycle. Then we have the following result.

Theorem 5.3. *Suppose the subgraph induced by $V'' = \{1, \ldots, k\} \subset V'$ is a pure cycle. Define α to be the trace of the cycle weight. Then k eigenvalues of $S(V'')$ are at -1, and the remaining k are*

$$-1 + \alpha^{1/k} e^{j2\pi l/k}, \qquad l \in \{0, \ldots, k-1\}. \tag{5.6}$$

Proof. Observe

$$F = I + S(V'') = \begin{bmatrix} 0 & X_{12} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & X_{23} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & X_{k-1,k} \\ X_{k1} & 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix}. \tag{5.7}$$

As the X_{ij} have rank 1, F has at least k zero eigenvalues. Consequently, $S(V'')$ has k eigenvalues equal to -1. For $i \in \{1, \ldots, k-1\}$ call $X_{i,i+1} = a_i b_i'$ and $X_{k1} = a_k b_k'$, with a_i, b_i 2-vectors. Then

$$\alpha = b_k' a_1 \prod_{i=1}^{k-1} b_i' a_{i+1}. \tag{5.8}$$

Define

$$\gamma_1 = 1 \text{ and } \gamma_i = \frac{\alpha^{(i-1)/k}}{\prod_{j=1}^{i-1} b_j' a_{j+1}}, \quad i \in \{2, \ldots, k\}, \quad \Gamma = \bigoplus_{i=1}^{k} (\gamma_i I_2), \tag{5.9}$$

for $l \in \{0, \ldots, k-1\}$,

$$W_l = \bigoplus_{i=1}^{k} \left(e^{j2\pi l(i-1)/k} I_2 \right),$$

and $\eta = [a_1', \ldots, a_k']'$. Then we assert that for each $l \in \{0, \ldots, k-1\}$,

$$F W_l \Gamma \eta = \alpha^{1/k} e^{j2\pi l/k} W_l \Gamma \eta. \tag{5.10}$$

Indeed, observe from (5.9) that for $i \in \{2, \ldots, k\}$, $\gamma_i b_{i-1}' a_i = \alpha^{1/k} \gamma_{i-1}$, and from (5.8) and (5.9) that

$$\gamma_1 b_k' a_1 = \frac{\alpha}{\prod_{j=1}^{k-1} b_j' a_{j+1}} = \alpha^{1/k} \gamma_k.$$

Thus (5.10) follows because

$$F W_l \Gamma \eta = \begin{bmatrix} e^{j2\pi l/k} \gamma_2 (b_1' a_2) a_1 \\ e^{j4\pi l/k} \gamma_3 (b_2' a_3) a_2 \\ \vdots \\ e^{j2\pi l(k-1)/k} \gamma_k (b_{k-1}' a_k) a_{k-1} \\ \gamma_1 (b_k' a_1) a_k \end{bmatrix} = e^{j2\pi l/k} W_l \begin{bmatrix} \gamma_2 (b_1' a_2) a_1 \\ \gamma_3 (b_2' a_3) a_2 \\ \vdots \\ \gamma_k (b_{k-1}' a_k) a_{k-1} \\ \gamma_1 (b_k' a_1) a_k \end{bmatrix}.$$

Consequently, from (5.10) the result follows. ∎

Observe that the theorem characterizes the eigenvectors as well. As a result of the theorem, the eigenvalues of $\hat{R}_e^{-1} \hat{R}$ contributed by a k-node nonoverlapping cycle take the form

$$-1 + \alpha^{1/k} e^{j2\pi l/k}, \quad l \in \{0, \ldots, k-1\}.$$

As we show by example in the next subsection, for suitably selected node coordinates such eigenvalues may have positive real parts.

5.4. An example. Using the results of the previous subsection we provide an example that, for suitably placed vertex positions, leads to an unstable $\hat{R}_e^{-1} \hat{R}$, implying that K_i, k_{n-1}, other than the identity, must be chosen for there to be stability. Indeed, consider the graph in Figure 4 in subsection 2.2. Note in particular that in the example the subgraphs induced by {1, 2, 3} and {4, 5, 6} are a pure cycle and acyclic; respectively, 5 is the first follower and 6 is the leader. Further, none among {4, 5, 6} has an outgoing edge to any of {1, 2, 3}. Thus indeed this graph is one we have earlier characterized to be a graph with nonoverlapping cycles. Then $\hat{R}_e^{-1} \hat{R}$ has six eigenvalues at -1 and the remaining three are at

$$-1 + \alpha^{1/3} e^{j2\pi k/3}, \quad k \in \{0, 1, 2\}. \tag{5.11}$$

Observe in this case that, the trace of the cycle weight of the cycle $\{1, 2, 3\}$ is

$$\alpha = \frac{(x_{31} y_{14} - x_{14} y_{31})(x_{12} y_{25} - x_{25} y_{12})(x_{23} y_{36} - x_{36} y_{23})}{(x_{31} y_{36} - x_{36} y_{31})(x_{12} y_{14} - x_{14} y_{12})(x_{23} y_{25} - x_{25} y_{23})},$$

where $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$, and $i, j \in \{1, 2, 3, 4, 5, 6\}$.

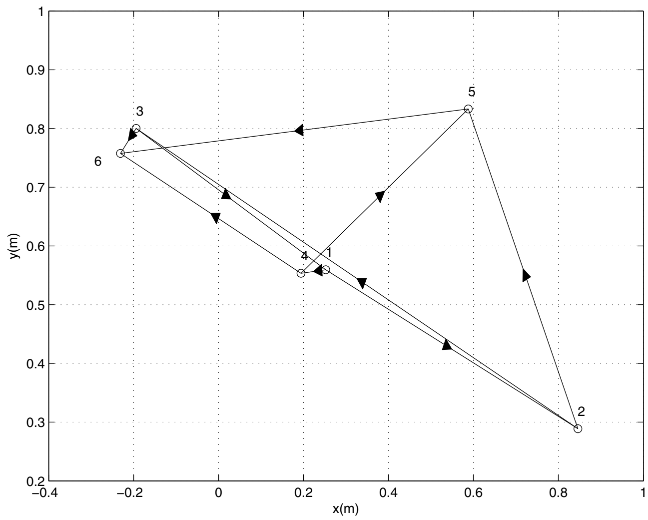

FIG. 7. *Initial setting before agent* 6 *is disturbed.*

Note that α can be alternatively expressed using angles

$$\alpha = \left(\frac{\sin \angle_{314}}{\sin \angle_{214}}\right)\left(\frac{\sin \angle_{125}}{\sin \angle_{325}}\right)\left(\frac{\sin \angle_{236}}{\sin \angle_{136}}\right), \tag{5.12}$$

where $\angle_{ijk}$ is the angle subtended by edges $\{i, j\}$ and $\{j, k\}$ at agent j, $i, j, k \in \{1, 2, 3, 4, 5, 6\}$. In general the cycle weight of a pure cycle is the product of ratios, with one ratio per node appearing in the cycle. The numerator of the ratio corresponding to a particular node in the cycle is the sine of the angle subtended by the incoming edge in the cycle to this node, and the outgoing edge from this node leaving the cycle. The denominator is the sine of the angle between the two outgoing edges of this node.

This expression provides some geometric clue for the formation stability, since a sine will be small if the angle approaches 0 or π, which means that a certain set of three agents is nearly collinear. Yet a large α and hence potential instability may not occur even if some nodes are near to being collinear. Thus, suppose in Figure 4 that if nodes 1, 2, and 4 are near collinear but $\angle_{314} < \angle_{214}$, then instability may well be avoided.

For an instantiation of the formation graph given in Figure 4, choose the six agent positions to be $p_1 = (0.2902, 0.5409)$, $p_2 = (0.8637, 0.2302)$, $p_3 = (-0.1388, 0.8117)$, $p_4 = (0.2316, 0.5387)$, $p_5 = (0.6438, 0.7909)$, and $p_6 = (-0.1784, 0.7716)$. An instantiation of this graph is shown in Figure 7.

Then $\alpha = 1.1407$ and (5.11) for $k = 0$ is real and positive, implying instability. Note in this particular example that α assumes an intermediate value, and none of the relevant angles appearing in either the numerator or the denominator of (5.12) is close to 0 or π. Thus, it is not relevant that agents 1, 2, and 3 are nearly collinear.

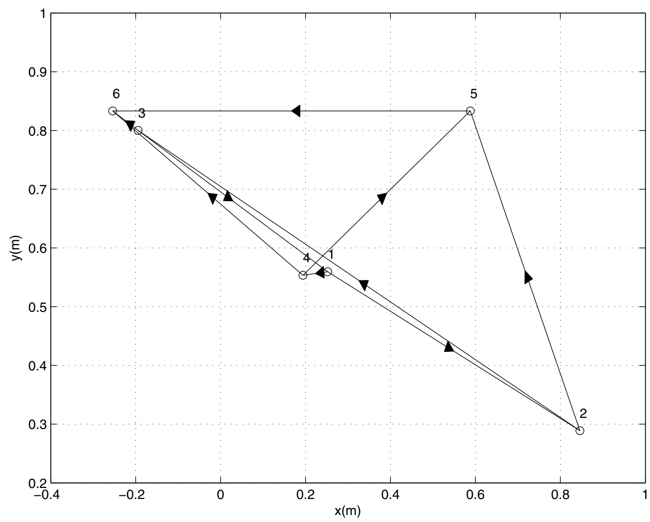

FIG. 8. *Initial setting after agent* 6 *is disturbed.*

On the other hand, it is readily checked that with the choice

$$\hat{K} = \begin{bmatrix} 5.4317 & -3.0234 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.9306 & -0.7167 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10.2598 & -6.9802 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2448 & 4.0662 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.4261 & 0.2535 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0634 & 0.1032 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

the eigenvalues of $\hat{K}\hat{R}_e^{-1}\hat{R}$ are $\{-0.0515 - 0.0473i,\ -0.0515 + 0.0473i,\ -0.4298,\ -1.0000,\ -1.0000,\ -1.0000,\ -3.2276,\ -4.7944,\ -10.1634\}$. Observe that this $\hat{K}$ has the structure imposed by (3.17) and provides a stable solution.

We next perturb the coordinates in the graph of Figure 7 to that in Figure 8. In the latter figure, agent 6 moves to $p_6 = (-0.1829, 0.7909)$. The nonlinear control law of (3.5), (3.7) is applied, with K_i as in the previous subsection. The simulation results are shown in Figures 9 and 10 and indicate rapid convergence.

6. Conclusion. This paper has only started on what is likely to be a fairly long road, developing efficacious control designs for maintaining formation shape. The methods of this paper do little more than demonstrate stabilizability. The particular stability theorem we are relying on, involving multiplying a matrix with nonzero leading principal minors by a diagonal matrix to make it stable, is almost certainly novel; however, it does not address the achieving of other control objectives apart from stability. In fact there is a broader list of issues that need to be addressed in the future, and we record some as follows:

(a) The control laws of this paper should really be regarded as nonlinear laws, with the rigidity matrix varying in the course of the motion. We have assumed small motions in order to justify an analysis using a linearized system. An

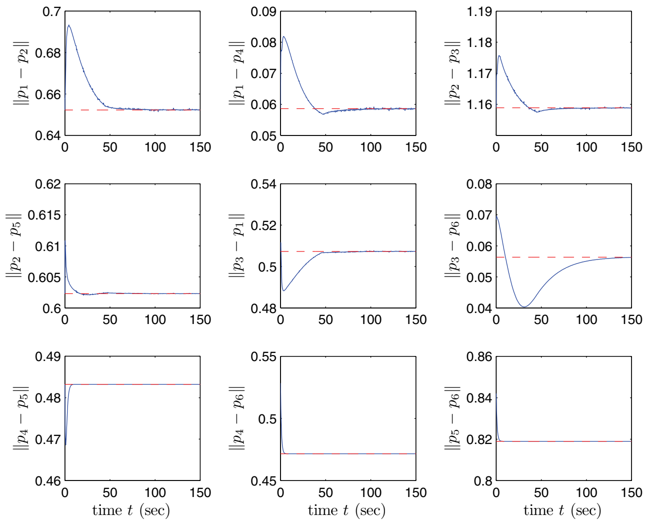

FIG. 9. *Distance keeping during the shape regulation process.*

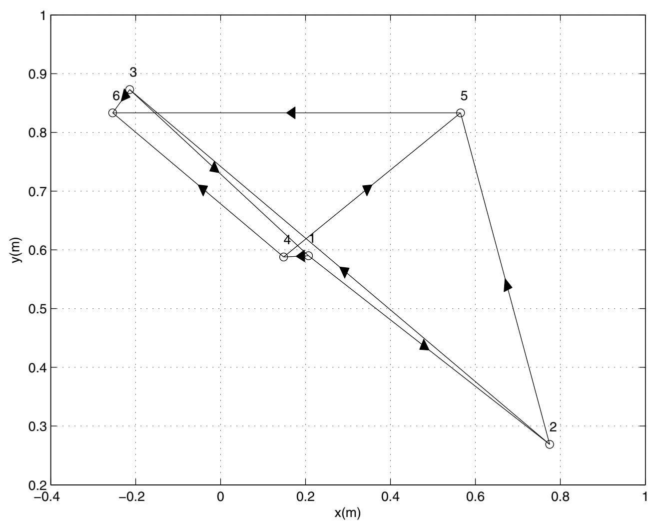

FIG. 10. *Final setting after shape regulation, i.e., at time $t = 150$ sec.*

immediate task would be to demonstrate stability of the nonlinear algorithm for a sizeable domain of attraction, and at the same time, or separately, construct a framework that embraces all minimally persistent formations, and not just ones of the leader-first follower type.

(b) We could have chosen different variables in which to describe the problem; for example, in another work [22], the dependent variables in the key differential equations were actual edge lengths rather than coordinate values. From a linearized point of view, it turns out that this makes no difference, but working with edge lengths may allow for a clearer understanding of the nonlinear behavior under big perturbations. Certainly in working with edge lengths in [22], we were able to obtain results that were almost global in their applicability.

(c) It may be important to study topologically balanced graphs (and then perhaps graphs with weights which can balance them) [11, 17] and see whether the results can be obtained far more easily. This is because for directed graphs and the consensus (flocking) problem, balanced graphs allow an easy solution [11], bringing the Laplacian into the picture; there is no guarantee that this will be so for the shape maintenance problem, which certainly needs to be viewed as a nonlinear problem, but it might be so. Note, though, that having a balanced graph precludes having a leader-follower structure, although it might be possible to separate these two vertices, much as we have done in this paper. To the extent that we are using weights here, which are the entries of the diagonal scaling matrix Λ, one might even wonder whether these could be put into a weighted balancing framework.

(d) Three-dimensional formations will not necessarily be a straightforward generalization, since the persistence concept is more difficult to generalize than might at first appear. Further, there is lacking a full graphical characterization of rigidity in three dimensions analogous to Laman's theorem, which we drew on heavily in this paper. On the other hand, it is reasonable to conjecture that the critical technical requirement, that a certain submatrix of the rigidity matrix will have all leading principal minors nonzero, will continue to hold.

(e) There are only limited insights here into which structures are difficult to control, in the sense that very large control signals will need to be used or noise will be a great problem. Such insights can be obtained through a more detailed investigation of formulas such as (5.12).

(f) Next, it is quite evident that it is desirable to include redundancy in formations to allow for loss of communication links or loss of an agent. This means we need to be able to handle nonminimally persistent problems, especially where there is noise in the measurements; for there is not then an obvious single point toward which a follower agent should aim.

These remarks of course do not exhaust the problems. One could imagine treating agents with mass, inertia, orientation; other control laws; maintenance of shape mixed with formation motion objectives, including obstacle avoidance; minimization of control energy; agents which move asynchronously; or formations with communications delays. These are all examples of issues which are relevant and have yet to be addressed.

REFERENCES

[1] J. Baillieul and A. Suri, *Information patterns and hedging Brockett's theorem in controlling vehicle formations*, in Proceedings of the 42nd IEEE Conference on Decision and Control, Maui, HI, 2003, pp. 556–563.

[2] R. Olfati-Saber and R. M. Murray, *Distributed cooperative control of multiple vehicle formations using structural potential functions*, in Proceedings of the 15th IFAC World Congress, Barcelona, Spain, 2002, pp. 1–7.

[3] R. Olfati-Saber and R. M. Murray, *Distributed structural stabilization and tracking for formations of dynamic multi-agents*, in Proceedings of the 41st IEEE Conference on Decision and Control, Las Vegas, NV, 2002, pp. 209–215.

[4] R. Olfati-Saber and R. M. Murray, *Graph rigidity and distributed formation stabilization of multivehicle systems*, in Proceedings of the 41st IEEE Conference on Decision and Control, Las Vegas, NV, 2002, pp. 2965–2971.

[5] R. Olfati-Saber, W. B. Dunbar, and R. M. Murray, *Cooperative control of multi-vehicle systems using cost graphs and optimization*, in Proceedings of the American Control Conference, Denver, CO, 2003, pp. 2217–2222.

[6] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, *Stable flocking of mobile agents, part* I*: Fixed topology*, in Proceedings of the 42nd IEEE Conference on Decision and Control, Maui, HI, 2003, pp. 2010–2015.

[7] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, *Stable flocking of mobile agents, part* II*: Dynamic topology*, in Proceedings of the 42nd IEEE Conference on Decision and Control, Maui, HI, 2003, pp. 2016–2021.

[8] P. Tabuada, G. J. Pappas, and P. Lima, *Cyclic directed formations of multi-agent systems*, in Proceedings of the 40th IEEE Conference on Decision and Control, Arlington, VA, 2001, pp. 56–61.

[9] J. M. Hendrickx, B. D. O. Anderson, J.-C. Delvenne, and V. D. Blondel, *Directed graphs for the analysis of rigidity and persistence in autonomous agent systems*, Internat. J. Robust Nonlinear Control, 17 (2007), pp. 960–981.

[10] C. Yu, J. M. Hendrickx, B. Fidan, B. D. O. Anderson, and V. D. Blondel, *Three and higher dimensional autonomous formations: Rigidity, persistence and structural persistence*, Automatica, 43 (2007), pp. 387–402.

[11] D. Lee and M. Spong, *Stable flocking of inertial agents on balanced communication graphs*, in Proceedings of the American Control Conference, Minneapolis, MN, 2006, pp. 2136–2141.

[12] Z. Lin, M. E. Broucke, and B. A. Francis, *Local control strategies for groups of mobile autonomous agents*, IEEE Trans. Automat. Control, 49 (2004), pp. 622–629.

[13] J. A. Marshall, M. E. Broucke, and B. A. Francis, *Formations of vehicles in cyclic pursuit*, IEEE Trans. Automat. Control, 49 (2004), pp. 1963–1974.

[14] R. Olfati-Saber, *Flocking for multi-agent dynamic systems: Algorithms and theory*, IEEE Trans. Automat. Control, 51 (2006), pp. 410–420.

[15] W. Ren and R. W. Beard, *A decentralized scheme for spacecraft formation flying via the virtual structure approach*, J. Guidance Control Dynam., 27 (2004), pp. 73–82.

[16] F. E. Schneider and D. Wildermuth, *Experimental comparison of a directed and a non-directed potential field approach to formation navigation*, in Proceedings of IEEE International Symposium on Computational Intelligence in Robotics and Automation, Espoo, Finland, 2005, pp. 21–26.

[17] J. Cortes, *Distributed algorithms for reaching consensus on general functions*, Automatica, 44 (2008), pp. 726–737.

[18] T. Tay and W. Whiteley, *Generating isostatic frameworks*, Structural Topology, No. 11, 1985, pp. 21–69.

[19] G. Laman, *On graphs and rigidity of plane skeletal structures*, J. Engrg. Math., 4 (1970), pp. 331–340.

[20] B. Fidan, C. Yu, and B. D. O. Anderson, *Acquiring and maintaining persistence of autonomous multi-vehicle formations*, IET Control Theory Appl., 1 (2007), pp. 452–460.

[21] S. L. Smith, M. E. Broucke, and B. A. Francis, *Stabilizing a multi-agent system to an equilibrium polygon formation*, in Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems, Kyoto, Japan, 2006, pp. 2415–2424.

[22] B. D. O. Anderson, C. Yu, S. Dasgupta, and A. S. Morse, *Control of a three coleaders persistent formation in the plane*, Systems Control Lett., 56 (2007), pp. 573–578.

[23] M. Cao, A. S. Morse, C. Yu, B. D. O. Anderson, and S. Dasgupta, *Controlling a triangular formation of mobile autonomous agents*, in Proceedings of the 46th IEEE Conference on Decision and Control, New Orleans, MA, 2007, pp. 3603–3608.

[24] R. S. Johnson, *Singular Perturbation Theory. Mathematical and Analytical Techniques with Applications to Engineering*, Springer, New York, 2005.

[25] S. Friedland, *On inverse multiplicative eigenvalue problems for matrices*, Linear Algebra Appl., 12 (1985), pp. 127–137.