

Agreeing Asynchronously

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Abstract—This paper formulates and solves a version of the widely studied Vicsek consensus problem in which each member of a group of $n > 1$ agents independently updates its heading at times determined by its own clock. It is not assumed that the agents' clocks are synchronized or that the "event" times between which any one agent updates its heading are evenly spaced. Nor is it assumed that heading updates must occur instantaneously. Using the concept of "analytic synchronization" together with several key results concerned with properties of "compositions" of directed graphs, it is shown that the conditions under which a consensus is achieved are essentially the same as those applicable in the synchronous case provided the notion of an agent's neighbor between its event times is appropriately defined. However, in sharp contrast with the synchronous case where for analysis an n dimensional state space model is adequate, for the asynchronous version of the problem a $2n$ -dimensional state space model is required. It is explained how to analyze this model despite the fact that, unlike the synchronous case, the stochastic matrices involved do not have all positive diagonal entries.

Index Terms—Asynchronous systems, cooperative control, multi-agent systems, switched systems.

I. INTRODUCTION

IN a recent paper Vicsek and co-authors [1] consider a simple model consisting of n autonomous agents or particles all moving in the plane with the same speed but with different headings. Each agent's heading is updated using a local rule based on the average of the headings of its "neighbors." Agent i 's current neighbors are itself together with those agents which are either inside or on a circle of pre-specified radius centered at agent i 's current position. In their paper, Vicsek *et al.* provide a variety of interesting simulation results which demonstrate that the nearest neighbor rule they are studying can cause all agents to eventually move in the same direction despite the absence of centralized coordination and despite the fact that each agent's set of nearest neighbors can change with time. In the recent literature [2], Vicsek's problem

is often referred to as a "flocking problem." Mathematically the problem is what in statistics and computer science is called a "consensus problem" [3] or an "agreement problem" [4] although in computer science the issues tend to be concerned more with fault tolerance [5] rather than convergence. Roughly speaking, one has a group of agents which are all trying to agree on a specific value of some quantity. Each agent initially has only limited information available. The agents then try to reach a consensus by communicating what they know to their neighbors either just once or repeatedly, depending on the specific problem of interest. For the Vicsek problem, each agent always knows only its own heading and the headings of its neighbors. One feature of the Vicsek problem which sharply distinguishes it from other consensus problems, is that each agent's neighbors change with time, because all agents are in motion. Various mathematically similar versions of Vicsek's problem have been addressed in the literature [6]–[11] some it turns out well before Vicsek's own paper was published [3], [12]–[15]. Additionally, some readers may find [16] relevant to the problem at hand.

There are a small number of publications [13]–[15], [17], [18] dating back at least as far as the doctoral thesis of John Tsitsiklis [13], which consider "asynchronous" versions of the Vicsek problem in which each agent independently updates its heading at times determined by its own clock. What makes these problems asynchronous is that it is *not* assumed that the agents' clocks are synchronized or that the "event times" at which any one agent updates its heading are evenly spaced. There is a subtle issue associated with consensus problems, whether synchronous or not, which becomes especially apparent in the asynchronous case. Note that if τ is an event time of agent i at which agent j is a neighbor, then to completely describe the overall asynchronous process one must account for agent j 's heading at time τ . In other words, since τ may not be an event time of agent j , to have a complete description of the asynchronous system one must make sure that the values of each agent's headings are defined at the event times of each other agent in the group. This is automatically taken care of in [13] and subsequent work [14], [15] by explicitly assuming that each agent updates its heading *only* at its event times. This of course means that at all times between any two successive event times of a given agent, say τ_i and τ_{i+1} , the agent's heading is fixed at the value it had at time τ_i . A consequence of this assumption is therefore that each agent's heading must be able to undergo *discontinuous* changes at its event times. While this may make sense within the context of distributed computing considered in [13]–[15] where headings would be synonymous with computing variables, it is clearly not possible to justify this assumption in a flocking application where a heading would correspond to the direction of motion of a mobile autonomous agent with mass and often inertia. The central aim of this paper is to formulate and solve an

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asynchronous consensus problem without the assumption that headings must be able to change discontinuously.

In Section II we define the asynchronous system of interest. To analyze its behavior, we first convert the system into a finite family of asynchronously interacting systems which together we call a “way point model” in Section IV. Next in Section V-A we embed the way point model in a suitably defined *synchronous* discrete-time, dynamical system \mathcal{S} using the concept of *analytic synchronization* outlined previously in [19], [20]. This enables us to bring to bear key results derived in [21] to characterize a rich class of system trajectories under which consensus is achieved. In particular, we prove that the conditions under which a consensus is achieved are essentially the same as those in the synchronous case derived in [7], [8] provided the notion of an agent’s neighbor between its event times is appropriately defined. However, in sharp contrast with the synchronous case where for analysis an n dimensional state space model is adequate, for the asynchronous version of the problem a $2n$ -dimensional model is required. In Section VI it is explained how to analyze this model despite the fact that, unlike the synchronous case, the stochastic matrices involved do not have all positive diagonal entries.

Both the way point model discussed in Section IV and the synchronous state-space system \mathcal{S} derived in Section V-A are quite different from the update model upon which the formulation of the asynchronous problem addressed in [13]–[15], [18] depends. Nonetheless it is shown in Section VII-A that it is possible to derive a model which is similar to the one studied in [13]–[15], [18]. Because of this, one might be tempted to approach the consensus problem we are considering using this model and existing ideas from [13]–[15]. There are a number of reasons why we’ve not done this. First and perhaps most important, the model derived in Section VII-A is not the same as the model used in [13]–[15], [18], and one of the distinguishing features of the former precludes the applicability to it, of the techniques from [13]–[15] as they stand; in particular, the proofs in [13]–[15] rely heavily on the assumption that there is a positive number α underbounding the nonzero entries which appear in the model considered—were such an assumption added to the problem posed in this paper, it would not be possible to claim results for continuous heading updates; this is discussed in greater detail in Section VII-A. Second, even if the approach used in [13]–[15] could be modified to handle the model in Section VII, the expected results would probably be a good deal more restrictive than those derived in this paper; this conjecture stems from the comparison made in Section VII-B of the existing results on asynchronous consensus in [13], [14] against the existing results for the same problem from [17]. Third, this paper provides a good example of the use of the idea of analytic synchronization, a technique potentially applicable to a variety of asynchronous convergence problems, not just those involving consensus; this point is discussed further in Section VIII. Fourth, the state-space approach in this paper is the natural extension of the state-space approach used in deriving analogous results for the synchronous case [6], [8]. Fifth, the state space system we are considering would be a convenient model to work with, were one to consider a generalization of the flocking problem in which Kalman filtering were introduced to handle noisy heading measurements.

II. ASYNCHRONOUS SYSTEM

The system to be studied consists of n autonomous agents, labelled 1 through n , all moving in the plane with different headings. Each agent’s desired heading or “next way-point” is computed at its current “event time” using a simple local rule based on the average of its own current heading plus the current headings of its “neighbors.” Agent i ’s *neighbors* at real time $t \geq 0$, are those agents, including itself, which are in a closed disk of pre-specified radius r_i centered at agent i ’s position at time t . In the sequel $\mathcal{N}_i(t)$ denotes the set of labels of those agents which are neighbors of agent i at time t . In contrast to earlier work [6]–[10], this paper considers a version of the flocking problem in which each agent independently updates its desired headings at times determined by its own clock. We do not assume that the agents’ clocks are synchronized or that the event times any one agent updates its way-points are evenly spaced. We assume for $i \in \{1, 2, \dots, n\}$ that agent i ’s *event times* $t_{i0}, t_{i1}, \dots, t_{ik}, \dots$ satisfy the constraints

$$\bar{T}_i \geq t_{i(k+1)} - t_{ik} \geq T_i, \quad k \geq 0 \quad (1)$$

where $t_{i0} = 0$ and \bar{T}_i and T_i are positive numbers. Agent i ’s event times could be any pre-specified sequence of times satisfying the preceding; alternatively, agent i ’s event time sequence could be determined in real time, where $t_{i(k+1)}$ might be defined to be the time at which agent i ’s actual heading first reaches the value of agent i ’s k th way-point.

Note that (1) implies that for each $i \in \{1, 2, \dots, n\}$, agent i ’s event time sequence is strictly monotone increasing, unbounded, and with no finite accumulation points. On the other hand, the assumption does not preclude arbitrary closeness of event times from different agent sequences. In fact, two agents could have an identical event time.

Updating of agent i ’s heading is done as follows. At its k th event time t_{ik} , agent i senses the headings $\theta_j(t_{ik}), j \in \mathcal{N}_i(t_{ik})$ of its current neighbors and from this data computes its k th way-point $w_i(t_{ik})$. We will consider way-point rules based on averaging. In particular

$$w_i(t_{ik}) = \frac{1}{n_i(t_{ik})} \left(\sum_{j \in \mathcal{N}_i(t_{ik})} \theta_j(t_{ik}) \right) \\ i \in \{1, 2, \dots, n\}, \quad k \geq 0 \quad (2)$$

where $n_i(t_{ik})$ is the number of indices in $\mathcal{N}_i(t_{ik})$. Agent i then changes its heading from $\theta_i(t_{ik})$ to $w_i(t_{ik})$ on the continuous-time interval $(t_{ik}, t_{i(k+1)})$. Thus

$$\theta_i(t_{i(k+1)}) = w_i(t_{ik}), \quad i \in \{1, 2, \dots, n\}, \quad k \geq 0. \quad (3)$$

Although we will not be concerned about the precise manner in which the value of each θ_i changes between successive way-points, we will assume that for each $i \in \{1, 2, \dots, n\}$, the change is monotonic and at least piecewise-continuous¹. Actu-

¹Practically speaking, one would expect headings to change continuously. However, to encompass existing versions of the problem such as [17] and prior work, one requires piece-wise continuity.

ally all we shall require, in addition to piecewise continuity, is that the θ_i satisfy

$$|\theta_i(t) - \theta_i(t_{ik})| \leq |w_i(t_{ik}) - \theta_i(t_{ik})|, t \in (t_{ik}, t_{i(k+1)}], k \geq 0. \quad (4)$$

This requirement is of course implied by monotonicity. Even though (2)–(4) cannot generally be modelled as a *dynamical* system or even as a set of unsynchronized dynamical systems, we shall nonetheless refer to (2)–(4) as an *asynchronous system* and shall call the set $\{[\theta_1(t) \ \theta_2(t) \ \cdots, \theta_n(t)]' : t \geq 0\}$ one of its *trajectories*.

It is worth pointing out that a great many real systems admit the description just given. For example, provided the θ_i satisfies the monotone and continuity requirements, each could be the heading of a realistically modelled robot moving in the plane; in this case the continuity of the θ_i would be automatic and with minor effort, each robot could be programmed to change its headings monotonically between its event times.

A. Extended Neighbor Graphs

The way-point (2) depend on the relationships between neighbors which exist at each agent's event times. It is possible to describe all neighbor relationships at any time t using a directed graph $\mathbb{N}(t)$ with vertex set $\mathcal{V} = \{1, 2, \dots, n\}$ and arc set $\mathcal{A}(\mathbb{N}) \subset \mathcal{V} \times \mathcal{V}$ which is defined in such a way so that (i, j) is an *arc* or directed edge from i to j just in case agent i is a neighbor of agent j at time t . Thus $\mathbb{N}(t)$ is a directed graph on n vertices with at most one arc between each ordered pair of vertices and with exactly one self-arc at each vertex. We write \mathcal{G}_{sa} for the set of all such graphs. It is natural to call a vertex i a *neighbor* of vertex j in any graph \mathbb{G} in \mathcal{G}_{sa} if (i, j) is an arc in \mathbb{G} .

Although the neighbors of each agent i are well defined at event times of other agents, what's important for computing agent i 's way-points are the headings of neighboring agents *only* at agent i 's own event times. Between agent i 's event times, it turns out to be useful to re-define agent i 's set of neighbors to consist only of itself. Said differently, since the only times agent i can sense its neighbor's headings are at its event times, we may as well take the definition of a neighbor of agent i at real time t to be the agents whose headings it can sense at time t . Our reason for doing this will become clear later when, for purposes of analysis, we use analytic synchronization to embed the salient features of the n agent asynchronous model defined by (2)–(4) in a synchronous dynamical system.

To proceed, let \mathcal{T} denote the set of all event times of all n agents. Relabel the elements of \mathcal{T} as t_0, t_1, t_2, \dots in such a way so that $t_0 = 0$ and $t_\tau < t_{\tau+1}, \tau \in \{0, 1, 2, \dots\}$. For $i \in \{1, 2, \dots, n\}$, let \mathcal{T}_i denote the set of $t_\tau \in \mathcal{T}$ which are event times of agent i . For each $i \in \{1, 2, \dots, n\}$ define

$$\bar{\mathcal{N}}_i(\tau) = \begin{cases} \mathcal{N}_i(t_\tau) & \text{if } t_\tau \in \mathcal{T}_i \\ i & \text{if } t_\tau \notin \mathcal{T}_i. \end{cases} \quad (5)$$

Thus $\bar{\mathcal{N}}_i(\tau)$ coincides with $\mathcal{N}_i(t_\tau)$ whenever t_τ is an event time of agent i and is simply the single index i otherwise.

Much like $\mathbb{N}(t)$ which describes the original neighbor relations of system (2)–(4) at time t , we describe all re-defined neighbor relationships at time $\tau \in \{0, 1, \dots\}$ to be the directed

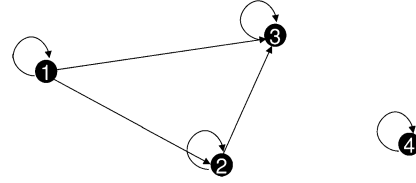


Fig. 1. $\bar{\mathbb{N}}$ for $\bar{\mathcal{N}}_1 = \{1\}, \bar{\mathcal{N}}_2 = \{1, 2\}, \bar{\mathcal{N}}_3 = \{1, 2, 3\}$, and $\bar{\mathcal{N}}_4 = \{4\}$.

graph $\bar{\mathbb{N}}(\tau)$ with vertex set \mathcal{V} and arc set $\mathcal{A}(\bar{\mathbb{N}}(\tau)) \subset \mathcal{V} \times \mathcal{V}$ which is defined so that (i, j) is an arc from i to j just in case agent i is in the neighbor set $\bar{\mathcal{N}}_j(\tau)$. Thus like the neighbor graphs $\mathbb{N}(t)$, each $\bar{\mathbb{N}}(\tau)$ is a directed graph on n vertices with at most one arc between each ordered pair of vertices and with exactly one self-arc at each vertex. We call $\bar{\mathbb{N}}(\tau)$ the *extended neighbor graph* of the asynchronous system (2)–(4) at time τ . Fig. 1 shows an extended neighbor graph $\bar{\mathbb{N}}(\tau)$ for a time τ for which t_τ is an event time of agents 2, 3, and 4.

B. Objective

A complete description of the asynchronous system defined by (2)–(4) would have to include a model which explains how the $\theta_i(t)$ and $\mathcal{N}_i(t)$ change over time as functions of the positions of the n agents in the plane. While such a model is easy to derive and is essential for simulation purposes, it would be difficult to take into account in a convergence analysis. To avoid this difficulty, we shall adopt a more conservative approach which ignores how the $\mathcal{N}_i(t)$ and the $\theta_i(t)$ depend on the agent positions in the plane and assumes instead that each might be any function in some suitably defined set of interest.

Our ultimate objective is to show for a large, interesting class of trajectories satisfying (2)–(4), that the headings of all n agents will converge to the same steady state value θ_{ss} . Naturally there are situations where convergence to a common heading cannot occur. The most obvious of these is when one agent—say the i th—starts so far away from the rest that it never acquires any neighbors. Mathematically this would mean not only that $\bar{\mathbb{N}}(\tau)$ is never strongly connected² at any event time index τ , but also that vertex i remains an isolated vertex of $\bar{\mathbb{N}}(\tau)$ for all τ in the sense that within each $\bar{\mathbb{N}}(\tau)$, vertex i has no neighbors other than itself. This situation is likely to be encountered if the r_i are very small. At the other extreme, which is likely if the r_i are very large, each agent might have all n agents as its neighbors at each of its own event times. But even in this extreme case, the extended neighbor graphs encountered along a typical trajectory would contain vertices whose only neighbor is itself except in the very special case which t_τ turned out to be an event time for all n agents. We will return to this issue in the next section.

III. MAIN RESULTS

To state our main result, we need a few ideas from [22]. We call a vertex i of a directed graph \mathbb{G} , a *root* of \mathbb{G} if for each other vertex j of \mathbb{G} , there is a path from i to j . Thus i is a root of \mathbb{G} , if

²A directed graph with arc set \mathcal{A} is *strongly connected* if it has a “path” between each distinct pair of its vertices i and j ; by a *path* {of length m } between vertices i and j is meant a sequence of arcs in \mathcal{A} of the form $(i, k_1), (k_1, k_2), \dots, (k_{m-1}, k_m)$ where $k_m = j$ and, if $m > 1$, i, k_1, \dots, k_{m-1} are distinct vertices. Such a graph is *complete* if it has a path of length one {i.e., an arc} between each distinct pair of its vertices.

it is the root of a directed spanning tree of \mathbb{G} . We will say that \mathbb{G} is *rooted at i* if i is in fact a root. Thus \mathbb{G} is rooted at i just in case each other vertex of \mathbb{G} is *reachable* from vertex i along a path within the graph. \mathbb{G} is *strongly rooted at i* if each other vertex of \mathbb{G} is reachable from vertex i along a path of length 1. Thus \mathbb{G} is strongly rooted at i if i is a neighbor of every other vertex in the graph. By a *rooted graph* \mathbb{G} is meant a graph which possesses at least one root. Finally, a *strongly rooted graph* is a graph which has at least one vertex at which it is strongly rooted. In other words, a strongly rooted graph is a directed graph containing a star graph [23] as a subgraph.

By the *composition* of two directed graphs $\mathbb{G}_p, \mathbb{G}_q$ with the same vertex set we mean that graph $\mathbb{G}_q \circ \mathbb{G}_p$ with the same vertex set and arc set defined such that (i, j) is an arc of $\mathbb{G}_q \circ \mathbb{G}_p$ if for some vertex k , (i, k) is an arc of \mathbb{G}_p and (k, j) is an arc of \mathbb{G}_q . Let us agree to say that a finite sequence of directed graphs $\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \dots, \mathbb{G}_{p_m}$ with the same vertex set is *jointly rooted* if the composition $\mathbb{G}_{p_m} \circ \mathbb{G}_{p_{m-1}} \circ \dots \circ \mathbb{G}_{p_1}$ is rooted. An infinite sequence of graphs $\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \dots$, with the same vertex set is *repeatedly jointly rooted* if there is a positive integer m for which each finite sequence $\mathbb{G}_{p_{m(k+1)}}, \dots, \mathbb{G}_{p_{m(k+m)}}$, $k \geq 0$, is jointly rooted.

Equations (2) and (3) can be combined. What results is a {partial} description of the evolution of θ_i on agent i 's event time set

$$\theta_i(t_{i(k+1)}) = \frac{1}{n_i(t_{ik})} \left(\sum_{j \in \mathcal{N}_i(t_{ik})} \theta_j(t_{ik}) \right), i \in \{1, 2, \dots, n\}. (6)$$

The description is only partial, because (6) does not model the evolution of the headings of agent i 's neighbors at agent i 's event times. However in the synchronous version of the problem treated previously in [6]–[10], for each $k \geq 0$, the k th event times $t_{1k}, t_{2k}, \dots, t_{nk}$ of all n agents are the same. Thus in this case (6) is a complete description and each agent's heading update equation at event times can be written as

$$\theta_i(t_{k+1}) = \frac{1}{n_i(t_k)} \left(\sum_{j \in \mathcal{N}_i(t_k)} \theta_j(t_k) \right), k \geq 0, i \in \{1, 2, \dots, n\} (7)$$

where $t_0 = 0$ and $t_k = t_{ik}$; this of course is a conventional discrete-time system. The most complete result for this version of the problem was given in [7], [8]. An equivalent result can be found in [22] and is as follows.

Theorem 1: For any trajectory of the synchronous system determined by (7) along which the sequence of neighbor graphs $\mathbb{N}(0), \mathbb{N}(1), \dots$ is repeatedly jointly rooted, there is a constant θ_{ss} for which

$$\lim_{t \rightarrow \infty} \theta_i(t) = \theta_{ss} (8)$$

where the limit is approached exponentially fast.

The aim of this paper is to prove that essentially the same result holds in the face of asynchronous updating.

Theorem 2: For any trajectory of the asynchronous system defined by (2)–(4) whose associated sequence of extended

neighbor graphs $\bar{\mathbb{N}}(0), \bar{\mathbb{N}}(1), \dots$ is repeatedly jointly rooted, there is a constant θ_{ss} for which

$$\lim_{t \rightarrow \infty} \theta_i(t) = \theta_{ss} (9)$$

where the limit is approached exponentially fast.

It is worth noting that the validity of this theorem depends critically on the fact that there are finite positive numbers, namely $T_{\max} = \max\{\bar{T}_1, \bar{T}_2, \dots, \bar{T}_n\}$ and $T_{\min} = \{T_1, T_2, \dots, T_n\}$, which uniformly bound from above and below respectively, the time between any two successive event times of any agent. This is a consequence of the assumption that inequality (1) holds.

As noted in the last section, for the asynchronous problem under consideration, the only vertices of $\bar{\mathbb{N}}(\tau)$ which can have more than one neighbor, are those corresponding to agents for whom t_τ is an event time. Thus in the most likely situation when distinct agents have only distinct event times, there will be at most one vertex in each graph $\bar{\mathbb{N}}(\tau)$ which has more than one neighbor. It is this situation we want to explore further. Toward this end, let $\mathcal{G}_{sa}^* \subset \mathcal{G}_{sa}$ denote the subclass of all graphs which have at most one vertex with more than one neighbor. Note that for $n > 2$, there is no rooted graph in \mathcal{G}_{sa}^* . Nonetheless, in the light of Theorem 2 it is clear that convergence to a common steady state heading will occur if the infinite sequence of graphs $\bar{\mathbb{N}}(0), \bar{\mathbb{N}}(1), \dots$ is repeatedly jointly rooted. This of course would require that there exist jointly rooted sequences of graphs from \mathcal{G}_{sa}^* . We will now explain why such sequences do in fact exist.

Let us agree to call a graph $\mathbb{G} \in \mathcal{G}_{sa}$ an *all neighbor graph centered at v* if every vertex of \mathbb{G} is a neighbor of v . Note that all neighbor graphs are maximal in \mathcal{G}_{sa}^* with respect to the partial ordering of \mathcal{G}_{sa}^* by inclusion, where in this context $\mathbb{G}_p \in \mathcal{G}_{sa}^*$ is contained in $\mathbb{G}_q \in \mathcal{G}_{sa}^*$ if $\mathcal{A}(\mathbb{G}_p) \subset \mathcal{A}(\mathbb{G}_q)$. Note also the composition of any all neighbor graph with itself is itself. On the other hand, because the arcs of any two graphs in \mathcal{G}_{sa} are arcs in their composition, the composition of n all neighbor graphs with distinct centers must clearly be a graph in which each vertex is a neighbor of every other; i.e., the complete graph. Thus the composition of n all neighbor graphs from \mathcal{G}_{sa}^* with distinct centers is strongly rooted. In summary, the hypothesis of Theorem 2 is not at all vacuous for the asynchronous problem under consideration. When that hypothesis is satisfied, convergence to a common steady state heading will occur.

IV. WAY-POINT MODEL

For purposes of analysis it is helpful to characterize the system described by (2)–(4) in a slightly different way. We claim that there is a piece-wise continuous signal $\mu_i : [0, \infty) \rightarrow [0, 1]$ such that

$$\theta_i(t) = \theta_i(t_{ik}) + \mu_i(t)(w_i(t_{ik}) - \theta_i(t_{ik})) t \in (t_{ik}, t_{i(k+1)}], k \geq 0, i \in \{1, 2, \dots, n\}. (10)$$

Moreover $\mu_i \in \mathcal{M}_i$ where for $i \in \{1, 2, \dots, n\}$, \mathcal{M}_i denotes the class of all piecewise continuous signals $\rho : [0, \infty) \rightarrow [0, 1]$

satisfying $\lim_{t \downarrow t_{ik}} \rho(t) = 0$ and $\rho(t_{ik}) = 1$ for all $k \geq 0$. In particular, for a given trajectory of (2)–(4), μ_i is defined on $(t_{ik}, t_{i(k+1)})$ as

$$\mu_i(t) = \begin{cases} \frac{\theta_i(t) - \theta_i(t_{ik})}{w_i(t_{ik}) - \theta_i(t_{ik})}, & \text{if } w_i(t_{ik}) \neq \theta_i(t_{ik}) \\ 1, & \text{if } w_i(t_{ik}) = \theta_i(t_{ik}). \end{cases}$$

Note that (4) guarantees that $[0, 1]$ is the co-domain of μ_i .

For μ_i to be in \mathcal{M}_i means that μ_i could be constant at the value 1 on each open interval $(t_{ik}, t_{i(k+1)})$; this would mean that just after t_{ik} , θ_i would jump discontinuously from its value at t_{ik} to $w(t_{ik})$ and remain constant at this value until just after $t_{i(k+1)}$ [17]. More realistically, μ_i might change continuously from 0 to 1 on $(t_{ik}, t_{i(k+1)})$ which would imply that θ_i is continuous on $[0, \infty)$. Under any conditions (2) and (10) completely describe the temporal evolution of the relevant part the n agent asynchronous system of interest. We call the system defined by (2) and (10) the original system's *way-point model*.

Note that each trajectory of the original system uniquely determines a way-point model. On the other hand it is easy to see that each trajectory of the way-point model {with the μ_i fixed} determines a family of trajectories of the original system. In the sequel we will fix the μ_i and study the behavior of the trajectories in this family.

V. ANALYTIC SYNCHRONIZATION

To prove Theorem 2 requires the analysis of the asymptotic behavior of the n mutually unsynchronized processes $\mathbb{P}_1, \mathbb{P}_2, \dots, \mathbb{P}_n$ which the n pairs of heading (2), (10) define. Despite the apparent complexity of the resulting *asynchronous* system which these n interacting processes determine, it is possible to capture its salient features using a suitably defined *synchronous* discrete-time, hybrid dynamical system \mathbb{S} . The sequence of steps involved in defining \mathbb{S} has been discussed before and is called *analytic synchronization* [19], [20]. First, all n event time sequences are merged into a single ordered sequence of event times \mathcal{T} , as we've already done. This clever idea has been used before in [15] to study the convergence of totally asynchronous iterative algorithms. Second, between event times each agent's neighbor set is defined to have exactly one neighbor, namely itself; this we have also already done. Third, the "synchronized" state of \mathbb{P}_i is then defined to be the original state of \mathbb{P}_i at \mathbb{P}_i 's event times $\{t_{i1}, t_{i2}, \dots\}$ plus possibly some additional state variables; at values of $t \in \mathcal{T}$ between event times t_{ik} and $t_{i(k+1)}$, the synchronized state of \mathbb{P}_i is taken to be the same at the value of its state at time t_{ik} . Although it is not always possible to carry out all of these steps, in this case it is. What ultimately results is a synchronous dynamical system \mathbb{S} evolving on the index set of \mathcal{T} , with state composed of the synchronized states of the n individual processes under consideration. We now use these ideas to develop such a synchronous system \mathbb{S} for the asynchronous process under consideration.

A. Definition of \mathbb{S}

For each such i and each $t_q \in \mathcal{T}_i$ define

$$\bar{\theta}_i(\tau) = \theta_i(t_q), \quad q \leq \tau < q' \quad (11)$$

$$\bar{w}_i(\tau) = w_i(t_q), \quad q \leq \tau < q' \quad (12)$$

where $t_{q'}$ is the first event time of agent i after t_q . Note that for any $t_q \in \mathcal{T}_i$ there is always such a q' because we've assumed via (1) that the time between any two successive event times of agent i is bounded above. We claim that for $i \in \{1, 2, \dots, n\}$ and $\tau > 0$

$$\bar{\theta}_i(\tau) = \bar{w}_i(\tau - 1), \quad t_\tau \in \mathcal{T}_i \quad (13)$$

$$\bar{\theta}_i(\tau) = \bar{\theta}_i(\tau - 1), \quad t_\tau \notin \mathcal{T}_i \quad (14)$$

$$\bar{w}_i(\tau) = \frac{1}{\bar{n}_i(\tau)} \sum_{j \in \bar{\mathcal{N}}_i(\tau)} \{(1 - \bar{\mu}_j(\tau))\bar{\theta}_j(\tau - 1) + \bar{\mu}_j(\tau)\bar{w}_j(\tau - 1)\}, t_\tau \in \mathcal{T}_i \quad (15)$$

$$\bar{w}_i(\tau) = \bar{w}_i(\tau - 1), \quad t_\tau \notin \mathcal{T}_i \quad (16)$$

where for $\tau \in \{0, 1, \dots\}$, $\bar{\mu}_j(\tau) = \mu_j(t_\tau)$ for $j \in \{1, 2, \dots, n\}$, and $\bar{n}_i(\tau)$ is the number of indices in $\bar{\mathcal{N}}_i(\tau)$. This set of equations constitute the synchronous system \mathbb{S} we intend to analyze. First we justify the claim that (13)–(16) hold.

Observe first that for $i \in \{1, 2, \dots, n\}$, (10) implies that $\theta_i(t_{q'}) = w_i(t_q), t_q \in \mathcal{T}_i$. Thus

$$\bar{\theta}_i(q') = \bar{w}_i(q), \quad t_q \in \mathcal{T}_i. \quad (17)$$

Moreover $q < q'$ because we've assumed via (1) that the time between any two successive event times of agent i is bounded away from zero. Thus $q \leq q' - 1 < q'$. In view of (12), $\bar{w}_i(\tau)$ is constant for $q \leq \tau < q'$ so $\bar{w}_i(q) = \bar{w}_i(q' - 1)$. Therefore (17) can be written as $\bar{\theta}_i(q') = \bar{w}_i(q' - 1)$. Clearly this holds for all $i \in \{1, 2, \dots, n\}$ and all $t_{q'} \in \mathcal{T}_i$. Therefore (13) holds for all positive $t_\tau \in \mathcal{T}_i$. In addition, (11) also implies that for $i \in \{1, 2, \dots, n\}$, $\bar{\theta}_i(\tau)$ is constant for $q \leq \tau < q'$; this in turn implies that (14) is true.

To justify (15), fix $i \in \{1, 2, \dots, n\}$ and let t_q be any positive time in \mathcal{T}_i . Note from (2), (11), and (12) that

$$\bar{w}_i(q) = \frac{1}{\bar{n}_i(q)} \left(\bar{\theta}_i(q) + \sum_{j \in (\bar{\mathcal{N}}_i(q) - i)} \theta_j(t_q) \right) \quad (18)$$

where $\bar{\mathcal{N}}_i(q) - i$ is the complement of i in $\bar{\mathcal{N}}_i(q)$. Moreover because of (10), for each $j \in (\bar{\mathcal{N}}_i(q) - i)$

$$\theta_j(t_q) = (1 - \bar{\mu}_j(q))\theta_j(t_r) + \bar{\mu}_j(q)w_j(t_r)$$

where t_r is the largest time in \mathcal{T}_j such that $t_r < t_q$. Using (11) and (12), this can be written as

$$\theta_j(t_q) = (1 - \bar{\mu}_j(q))\bar{\theta}_j(r) + \bar{\mu}_j(q)\bar{w}_j(r). \quad (19)$$

Since t_r is the largest time in \mathcal{T}_j less than t_q , it must be true that $r < q \leq r'$ where $t_{r'}$ is the next largest time in \mathcal{T}_j after t_r . Thus $r \leq q - 1 < r'$. Now (11) and (12) imply that both $\bar{\theta}_j(\tau)$ and

$\bar{w}_j(\tau)$ are constant for $r \leq \tau < r'$. Therefore $\bar{\theta}_j(r) = \bar{\theta}_j(q-1)$ and $\bar{w}_j(r) = \bar{w}_j(q-1)$. Thus (19) becomes

$$\theta_j(t_q) = (1 - \bar{\mu}_j(q))\bar{\theta}_j(q-1) + \bar{\mu}_j(q)\bar{w}_j(q-1). \quad (20)$$

Substitution in (18) gives

$$\bar{w}_i(q) = \frac{1}{\bar{n}_i(q)} \left(\bar{\theta}_i(q) + \sum_{j \in (\bar{\mathcal{N}}_i(q)-i)} \{(1 - \bar{\mu}_j(q))\bar{\theta}_j(q-1) + \bar{\mu}_j(q)\bar{w}_j(q-1)\} \right).$$

But $\bar{\theta}_i(q) = (1 - \bar{\mu}_i(q))\bar{\theta}_i(q) + \bar{\mu}_i(q)\bar{w}_i(q-1)$ because of (13) and the fact that $\bar{\mu}_i(q) = 1$. Therefore

$$\bar{w}_i(q) = \frac{1}{\bar{n}_i(q)} \sum_{j \in \bar{\mathcal{N}}_i(q)} \{(1 - \bar{\mu}_j(q))\bar{\theta}_j(q-1) + \bar{\mu}_j(q)\bar{w}_j(q-1)\}.$$

Since this is true for any positive time $t_q \in \mathcal{T}_i$, (15) is valid for any positive $t_\tau \in \mathcal{T}_i$.

Now suppose that t_τ is any positive time not in \mathcal{T}_i , assuming of course that such a time exists. Observe that (12) implies $\bar{w}_i(\tau)$ is constant for $q \leq \tau - 1 < \tau < q'$ where t_q is the largest time in \mathcal{T}_i such that $t_q < t_\tau$. Thus $\bar{w}_i(\tau) = \bar{w}_i(\tau - 1)$ so (16) is true. This completes our justification that the $\bar{\theta}_i$ and \bar{w}_i satisfy (13)–(16).

B. State Space Model

The equations defining \mathbb{S} , namely (13)–(16), determine a state space system of the form

$$x(\tau + 1) = F(\tau)x(\tau), \quad \tau \in \{1, 2, \dots\} \quad (21)$$

where

$$x(\tau) = [\bar{\theta}_1(\tau - 1) \cdots \bar{\theta}_n(\tau - 1) \bar{w}_1(\tau - 1) \cdots \bar{w}_n(\tau - 1)]'. \quad (22)$$

Each $F(\tau)$ is a $2n \times 2n$ stochastic matrix which can be described as follows.

Let \mathcal{R} denote the set of all lists of n numbers $\bar{\mu} = \{\bar{\mu}_1, \bar{\mu}_2, \dots, \bar{\mu}_n\}$ with each $\bar{\mu}_i$ taking a value in the real closed interval $[0, 1]$. Let \mathcal{B} denote the set of all lists of n integers $b = \{b_1, b_2, \dots, b_n\}$ with each b_i taking a value in the binary integer set $\{0, 1\}$. Each such triple $(\bar{\mathbb{N}}, \bar{\mu}, b) \in \mathcal{G}_{sa} \times \mathcal{R} \times \mathcal{B}$ determines a $2n \times 2n$ stochastic matrix $\mathbf{F}(\bar{\mathbb{N}}, \bar{\mu}, b)$ whose entries for $i \in \{1, 2, \dots, n\}$ are

$$f_{ij} = \delta_{(i+n)j}$$

and

$$f_{(i+n)j} = \begin{cases} \frac{1}{\bar{n}_i}(1 - \bar{\mu}_j) & j \in (\bar{\mathcal{N}}_i - i) \\ \frac{1}{\bar{n}_i}\bar{\mu}_j & j \in (\bar{\mathcal{N}}_i - i) + \{n\} \\ \frac{1}{\bar{n}_i}\delta_{(i+n)j} & j \notin (\bar{\mathcal{N}}_i - i) \cup ((\bar{\mathcal{N}}_i - i) + \{n\}) \end{cases}$$

if $b_i = 1$ and

$$f_{ij} = \delta_{ij}$$

and

$$f_{(i+n)j} = \delta_{(i+n)j}$$

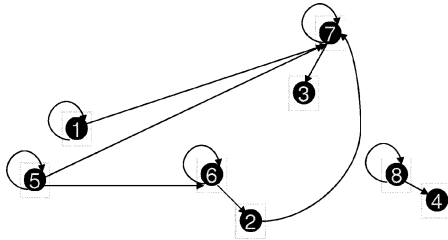
if $b_i = 0$. Here $\bar{\mathcal{N}}_i$ is the set of neighbors of vertex i in $\bar{\mathbb{N}}$, \bar{n}_i is the number of elements in $\bar{\mathcal{N}}_i$, $\bar{\mathcal{N}}_i - i$ is the complement of i in $\bar{\mathcal{N}}_i$, δ_{ij} is the Kronecker delta, and for any set of integers \mathcal{I} , $\mathcal{I} + \{n\}$ is the set $\mathcal{I} + \{n\} = \{i+n : i \in \mathcal{I}\}$. We call any such matrix F an *asynchronous flocking matrix*. Thus, the image of \mathbf{F} is the set of all possible asynchronous flocking matrices.

It is easy to verify that the matrix $F(\tau)$ in (21) is of the form $\mathbf{F}(\bar{\mathbb{N}}(\tau), \bar{\mu}(\tau), b(\tau))$ where $\bar{\mathbb{N}}(\tau)$ is that graph in \mathcal{G}_{sa} with neighbor sets $\bar{\mathcal{N}}_1(\tau), \bar{\mathcal{N}}_2(\tau), \dots, \bar{\mathcal{N}}_n(\tau)$, $\bar{\mu}(\tau)$ is that list in \mathcal{R} whose i th element is $\bar{\mu}_i(\tau)$, and $b(\tau)$ is that list in \mathcal{B} whose i th element is $b_i(\tau) = 1$ if $t_\tau \in \mathcal{T}_i$ or $b_i(\tau) = 0$ if $t_\tau \notin \mathcal{T}_i$. An example of an asynchronous flocking matrix which could arise in conjunction with the extended neighbor graph shown in Fig. 1 is

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1-\rho}{3} & \frac{1}{3} & 0 & 0 & \frac{\rho}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (23)$$

Here ρ can be any real number in the closed interval $[0, 1]$.

Note that the diagonal entries of a typical asynchronous flocking matrix $F(\tau)$ can sometimes be zero which is very different than what arises in the synchronous case treated in [3], [6], [7]–[10], [12], [14], [15], and even the discrete-time asynchronous case treated in [14], [15], [17], and [18]. Note in addition that unlike the other flocking problems considered in the past where the $F(\tau)$ were matrices from a finite set, the set of all asynchronous flocking matrices which arise here, namely image \mathbf{F} , is not a finite set because \mathcal{R} is not a finite set. Nonetheless image \mathbf{F} is a closed and therefore compact subset of the set of all $2n \times 2n$ stochastic matrices \mathcal{S} . To understand why this is so, note first that for each fixed $b \in \mathcal{B}$ and $\bar{\mathbb{N}} \in \mathcal{G}_{sa}$, the mapping $\mathcal{R} \rightarrow \mathcal{S}, \mu \mapsto \mathbf{F}(\bar{\mathbb{N}}, \mu, b)$ is continuous on \mathcal{R} . Therefore its image must be compact because \mathcal{R} is. Next note that \mathcal{G}_{sa} and \mathcal{B} are each finite sets. Since the union of a finite

Fig. 2. $\gamma(F)$.

number of compact sets is compact, it must therefore be true that the image of \mathbf{F} is compact as claimed.

VI. ANALYSIS

The ultimate aim of this section is to give a Proof of Theorem 2. We begin with the notion of the graph of a stochastic matrix.

Any $2n \times 2n$ stochastic matrix S such as those in image \mathbf{F} , determines a directed graph $\gamma(S)$ with vertex set $\{1, 2, \dots, n, n+1, n+2, \dots, 2n\}$ and arc set defined in such a way so that (i, j) is an arc of $\gamma(S)$ from i to j just in case the j th entry of S is non-zero. It is easy to verify that for any two such matrices S_1 and S_2

$$\gamma(S_2 S_1) = \gamma(S_2) \circ \gamma(S_1). \quad (24)$$

Assuming that ρ is in the open interval $(0,1)$, the graph of the asynchronous flocking matrix F in (23) would be as shown in Fig. 2.

A major technical difference between the synchronous and discrete-time asynchronous flocking problems addressed previously in [6]–[10], [14], [15], [17], [18] and the problem under consideration here is that the graphs of the stochastic matrices encountered in [6]–[10], [14], [15], [17], and [18] have self-arcs at all vertices whereas the stochastic matrices which arise here do not. What this means is that the technical tools used to establish exponential convergence in [6]–[10], [14], [15], [17], and [18] are not sufficient to establish convergence here.

A. Graphs and Their Properties

We now define a set of directed graphs \mathcal{G} on vertex set $\{1, 2, \dots, n, n+1, n+2, \dots, 2n\}$ which contains all $\gamma(F)$, $F \in \text{image } \mathbf{F}$, and which is large enough to be closed under composition. For this purpose it is convenient to adopt the notation $[v]$ for the subset $\{v, v+n\}$ whenever $v \in \mathcal{V}$, and to say that $([v], u)$ is an arc of a graph \mathbb{G} in \mathcal{G} if either (v, u) or $(v+n, u)$ is. Similarly we say that $(v, [u])$ is an arc of \mathbb{G} if either (v, u) or $(v, u+n)$ is and $([v], [u])$ is an arc of \mathbb{G} if either $(v, [u])$ or $(v+n, [u])$ is.

We define \mathcal{G} to be the set of all directed graphs with vertex set $\{1, 2, \dots, 2n\}$ whose graphs have the following properties. For each $\mathbb{G} \in \mathcal{G}$ and each pair of vertices $u \in \{1, 2, \dots, 2n\}$ and $v \in \mathcal{V}$:

- p1: $v+n$ has a self-arc in \mathbb{G} .
- p2: $([v], v)$ is an arc in \mathbb{G} .

p3: If (u, v) is an arc in \mathbb{G} and $u \neq v$, then $(u, v+n)$ is an arc in \mathbb{G} .

p4: If $(u, [v])$ is an arc in \mathbb{G} and $u \neq v$, then $(v+n, v)$ is an arc in \mathbb{G} .

It is straightforward to verify that for each $F \in \text{image } \mathbf{F}$, $\gamma(F)$ as a graph in \mathcal{G} . In view of the structure of the matrices in image \mathbf{F} it is natural to call a graph $\mathbb{G} \in \mathcal{G}$ an *event graph of agent $i \in \mathcal{V}$* if $(i+n, i)$ is the *only* incoming arc to vertex i . Note that the graph of every matrix $\mathbf{F}(\mathbb{N}, \bar{\mu}, b)$ for which $b_i = 1$ is an event graph of agent i . Thus $\gamma(F(\tau))$ is an event graph of agent i if t_τ is an event time of agent i . It is easy to see that there are graphs in \mathcal{G} which are not the graphs of any matrix in image \mathbf{F} . Let us agree to say that $\mathbb{G} \in \mathcal{G}$ is *attached at $i \in \mathcal{V}$* if vertex i has *at least* $(i+n, i)$ as an incoming arc. A graph $\mathbb{G} \in \mathcal{G}$ is *attached* if it is attached at every vertex in \mathcal{V} . Thus $\gamma(F(\tau))$ would be attached if and only if t_τ were an event time of every agent. Note that the graph shown in Fig. 2 is an event graph for agents 2, 3 and 4 and consequently is attached at vertices 2, 3 and 4. Note that the definition \mathcal{G} allows this set to contain graphs which are attached at i which are not event graphs of agent i . In other words, an event graph of agent i must be attached at i , but the converse is not necessarily so.

We begin our analysis with the following observation.

Proposition 1: The set of graphs \mathcal{G} is closed under composition.

The proof of this and subsequent assertions can be found at the end of this section.

To prove that all θ_i converge to a common heading, it is clearly necessary to prove that $\bar{\theta}_i$ also converge to a common heading. On the other hand, if *both* the $\bar{\theta}_i$ and \bar{w}_i also converge to a common heading—say θ_{ss} —then both θ_i and w_i converge to θ_{ss} at each event time of agent i . Because of this and (10), it is clear that each θ_i will also converge to θ_{ss} between event times if both θ_i and w_i converge to θ_{ss} at each event time of agent i . In other words, to prove Theorem 2 it is enough to prove that the state x of \mathcal{S} converges to a vector of the form $\theta_{ss} \mathbf{1}$ where $\mathbf{1}$ is the $2n \times 1$ vector of 1's. It is clear from (21) that x will converge to such a vector just in case as $\tau \rightarrow \infty$, the matrix product $F(\tau)F(\tau-1) \cdots F(1)$ converges to a rank one matrix of the form $\mathbf{1}c$ for some $2n \times 1$ row vector c . The following easy to prove result from [21] is key to establishing this convergence.

Proposition 2: Let \mathcal{S}_{sr} be any closed set of stochastic matrices which are all of the same size and whose graphs $\gamma(S)$, $S \in \mathcal{S}_{sr}$ are all strongly rooted. As $j \rightarrow \infty$, any product $S_j \cdots S_1$ of matrices from \mathcal{S}_{sr} converges exponentially fast to a matrix of the form $\mathbf{1}c$ at a rate no slower than λ , where c is a non-negative row vector depending on the sequence and λ is a non-negative constant less than 1 depending only on \mathcal{S}_{sr} .

In view of (24), this result can be applied to the problem at hand if there is an integer q for which each of the matrix products $F((k+1)q) \cdots F(kq+1)$, $k \geq 0$ is a member of a compact subset of stochastic matrices with strongly rooted graphs. For if such an integer exists, the infinite product $\cdots F(\tau) \cdots F(1)$ can be rewritten as an infinite product of the form $\cdots \bar{S}(k) \cdots \bar{S}(1)$ where $\bar{S}(k) = F((k+1)q) \cdots F(kq+1)$ is a matrix

from the set of all products of q matrices from \mathcal{S} . Since products of stochastic matrices are stochastic, every matrix $\bar{S}(k), k \geq 1$, is stochastic. Thus Proposition 2 can be applied if we can show that the $\bar{S}(k)$ come from a compact subset in \mathcal{S} whose members all have strongly rooted graphs. The following result from [22] plays a key role in [22] in dealing with this matter in the synchronous case.

Proposition 3: Suppose $n > 1$ and let $\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \dots, \mathbb{G}_{p_m}$ be a finite sequence of rooted graphs with the same vertex set. If each vertex of each graph has a self arc and $m \geq (n - 1)^2$, then $\mathbb{G}_{p_m} \circ \mathbb{G}_{p_{m-1}} \circ \dots \circ \mathbb{G}_{p_1}$ is strongly rooted.

Unfortunately the graphs of importance in the asynchronous case, namely the $\gamma(F(\tau))$, do not have self arcs at all vertices. Thus Proposition 3 cannot be directly applied.

To describe the analog of Proposition 3 appropriate to the asynchronous problem at hand we need another concept. Note that each $\mathbb{G} \in \mathcal{G}$ determines a *quotient graph* $Q(\mathbb{G}) \in \mathcal{G}_{sa}$ defined in such a way that $Q(\mathbb{G})$ has an arc from i to j just in case \mathbb{G} has an arc from at least one vertex in the set $[i]$ to at least one vertex in the set $[j]$. Note that $Q(\gamma(\mathbf{F}(\bar{\mathbb{N}}, \bar{\mu}, b))) = \bar{\mathbb{N}}$. Thus for example, the quotient graph of the graph shown in Fig. 2, is the extended neighbor graph shown in Fig. 1. The following is the analog of Proposition 3 which we just mentioned.

Proposition 4: Let $\mathbb{G}_{p_1}, \dots, \mathbb{G}_{p_{2m+1}}$ be a sequence of $2m+1$ attached graphs in \mathcal{G} whose quotient graphs are rooted. If $m \geq (n - 1)^2$ then $\mathbb{G}_{p_{2m+1}} \circ \dots \circ \mathbb{G}_{p_1}$ is strongly rooted.

To make use of Proposition 4, we need stochastic matrices with attached graphs whose quotients are rooted. Since individual asynchronous flocking matrices almost never have either of these properties, to make use of the proposition we need to show that under typical conditions, sufficiently long products of asynchronous flocking matrices do have attached graphs with rooted quotients. To accomplish this requires a more in depth study of the graphs in \mathcal{G} . We begin with the following observation.

Proposition 5: Let $\mathbb{G}_{p_1}, \dots, \mathbb{G}_{p_m}$ be a sequence of graphs from \mathcal{G} which for each $i \in \mathcal{V}$, contains a graph which is attached at i . Then $\mathbb{G}_{p_m} \circ \dots \circ \mathbb{G}_{p_1}$ is an attached graph.

The proposition implies that if $t_{\tau_1}, t_{\tau_2}, \dots, t_{\tau_m}$ is a sequence of event times containing at least one event time of each agent, then $\gamma(F(\tau_m) \cdots F(\tau_1))$ will be attached. Sequences for which this is true are guaranteed to occur repeatedly. To understand why, note that inequalities in (1) imply that there will be at least one event time of any given agent in a time interval of length at least $T_{\max} = \max\{\bar{T}_1, \bar{T}_2, \dots, \bar{T}_n\}$. Similarly, for any non-negative integer h , there will be at most h event times of any one agent in an interval of length at most hT_{\min} where $T_{\min} = \min\{T_1, T_2, \dots, T_n\}$. It follows that if h is the smallest positive integer such that $T_{\max} \leq hT_{\min}$, then there will be at least one event time of any one agent within a sequence of at most $h + 1$ consecutive event times of any other agent. We are led to the following conclusion.

Lemma 1: In any sequence of $(n - 1)h + 1$ or more consecutive event times, there will be at least one event time of each of the n agents.

The following proposition shows that for any sequence of graphs $\mathbb{G}_{p_1}, \dots, \mathbb{G}_{p_m}$ from \mathcal{G} whose quotients constitute a jointly rooted sequence, the quotient of the composition of the sequence is rooted.

Proposition 6: Let $\mathbb{G}_{p_1}, \dots, \mathbb{G}_{p_m}$ be a sequence of $m > 1$ graphs from \mathcal{G} for which $Q(\mathbb{G}_{p_m}) \circ \dots \circ Q(\mathbb{G}_{p_1})$ is a rooted graph. Then $Q(\mathbb{G}_{p_m} \circ \dots \circ \mathbb{G}_{p_1})$ is also rooted at the same vertex as $Q(\mathbb{G}_{p_m}) \circ \dots \circ Q(\mathbb{G}_{p_1})$.

Proposition 6 is more subtle than it might at first seem. While it is not difficult to show that any arc in the quotient of the composition of the \mathbb{G}_p is an arc in the composition of the quotients it is not true that every arc in the composition of the quotients is an *arc* in the quotient of the composition. For this reason it is not so obvious that Proposition 6 should be true. On the other hand it is possible to prove that for any arc (u, v) in the composition of the quotients there is a *path* in the quotient of the composition from u to v . It is this fact upon which the validity of Proposition 6 critically depends.

In proving Theorem 2, we will need to exploit the compactness of a particular subset of stochastic matrices in \mathcal{S} which can be described as follows. Let $p \geq n$ be any given positive integer. Write \mathcal{G}_{sa}^p for the subset of all sequences of p graphs in \mathcal{G}_{sa} which are jointly rooted and \mathcal{B}^p for the set of all lists of p binary vectors in \mathcal{B} with the property that for each $i \in \{1, 2, \dots, n\}$, each list $\{b_1, b_2, \dots, b_p\}$ contains at least one vector whose i th row is 1. Since $p \geq n$, \mathcal{B}^p is nonempty. Let \mathcal{R}^p be the Cartesian product of \mathcal{R} with itself p times. We claim that the image of the mapping $\mathbf{F}^p : \mathcal{G}_{sa}^p \times \mathcal{R}^p \times \mathcal{B}^p \rightarrow \mathcal{S}$ defined by

$$(\{\mathbb{N}_1, \mathbb{N}_2, \dots, \mathbb{N}_p\}, \{\mu_1, \mu_2, \dots, \mu_p\}, \{b_1, b_2, \dots, b_p\}) \mapsto \mathbf{F}(\mathbb{N}_p, \mu_p, b_p) \cdots \mathbf{F}(\mathbb{N}_2, \mu_2, b_2) \mathbf{F}(\mathbb{N}_1, \mu_1, b_1)$$

is compact. The reason for this is essentially the same as the reason image \mathbf{F} is compact. In particular, for any fixed $\{\mathbb{N}_1, \mathbb{N}_2, \dots, \mathbb{N}_p\} \in \mathcal{G}_{sa}^p$ and $\{b_1, b_2, \dots, b_p\} \in \mathcal{B}^p$, the restricted mapping $\{\mu_1, \mu_2, \dots, \mu_p\} \mapsto \mathbf{F}^p(\{\mathbb{N}_1, \mathbb{N}_2, \dots, \mathbb{N}_p\}, \{\mu_1, \mu_2, \dots, \mu_p\}, \{b_1, b_2, \dots, b_p\})$ is continuous so its image must be compact. Since \mathcal{B}^p and \mathcal{G}_{sa}^p are finite sets, the image of \mathbf{F}^p must therefore be compact as well.

Set $q = 2(n - 1)^2 + 1$ and let $\mathcal{F}^p(q)$ denote the set of all products of q matrices from image \mathbf{F}^p . Then $\mathcal{F}^p(q)$ is compact because image \mathbf{F}^p is. More is true.

Proposition 7: The graph of each matrix in $\mathcal{F}^p(q)$ is strongly rooted.

We are now finally in a position to prove our main result.

Proof of Theorem 2: As already noted, it is sufficient to prove that the matrix product $F(\tau) \cdots F(1)$ converges exponentially fast to a matrix of the form $\mathbf{1}c$ as $\tau \rightarrow \infty$. Observe first that there is a vector binary vector $b(\tau) \in \mathcal{B}$ and a vector $\mu(\tau) \in \mathcal{R}$ such that

$$F(\tau) = \mathbb{F}(\bar{\mathbb{N}}(\tau), \mu(\tau), b(\tau)), \tau \geq 0 \tag{25}$$

because each $F(\tau) \in \text{image } \mathbf{F}$.

By hypothesis, the sequence of extended neighbor graphs $\bar{\mathbb{N}}(0), \bar{\mathbb{N}}(1), \dots$, is repeatedly jointly rooted. This means that there is an integer m for which each of the sequences $\bar{\mathbb{N}}(km+1), \dots, \bar{\mathbb{N}}((k+1)m), k \geq 0$, is jointly rooted. Let h be as in Lemma 1 and define $p = rm$ where r is any positive integer large enough so that $p \geq (n-1)h+1$. Set $q = 2(n-1)^2+1$ and let $\mathcal{G}_{sa}^p, \mathcal{R}^p, \mathcal{B}^p, \mathbf{F}^p$, and $\mathcal{F}^p(q)$ be as defined just above Proposition 7.

Since each $\bar{\mathbb{N}}(km+1), \dots, \bar{\mathbb{N}}((k+1)m), k \geq 0$, is jointly rooted, each of the compositions $\bar{\mathbb{N}}((k+1)m) \circ \dots \circ \bar{\mathbb{N}}(km+1), k \geq 0$, is rooted. This implies that each graph $\bar{\mathbb{N}}((k+1)p) \circ \dots \circ \bar{\mathbb{N}}(kp+1), k \geq 0$, is rooted because $p = rm$ and because the composition of r rooted graphs is rooted. Therefore each sequence $\bar{\mathbb{N}}(kp+1), \dots, \bar{\mathbb{N}}((k+1)p), k \geq 0$, is jointly rooted. It follows:

$$\{\bar{\mathbb{N}}(kp+1), \dots, \bar{\mathbb{N}}((k+1)p)\} \in \mathcal{G}_{sa}^p, k \geq 0. \quad (26)$$

Note next that for each $i \in \{1, 2, \dots, n\}$ and each $k \geq 0$, at least one of the graphs in the sequence $\gamma(F(kp+1)), \dots, \gamma(F((k+1)p))$ must be attached at i because of Lemma 1 and the assumption that $p \geq (n-1)h+1$. This implies that for each $i \in \{1, 2, \dots, n\}$ there must be at least one vector in each list $\{b(kp+1), \dots, b((k+1)p)\}, k \geq 0$ whose i th row is 1. Therefore

$$\{b(kp+1), \dots, b((k+1)p)\} \in \mathcal{B}^p, k \geq 0. \quad (27)$$

For $k \geq 0$, define

$$S(k) = F((k+1)p) \cdots F(kp+1) \quad (28)$$

In view of (25)–(27) and the definition of \mathbf{F}^p , it must be true that $S(k) \in \text{image } \mathbf{F}^p, k \geq 0$. Thus if we define

$$\bar{S}(k) = S((k+1)q-1) \cdots S(kq), k \geq 0 \quad (29)$$

then each $\bar{S}(k)$ must be in $\mathcal{F}^p(q)$. Therefore by Proposition 7, the graph of each $\bar{S}(k)$ is strongly rooted. Therefore by Proposition 2, the matrix product $\bar{S}(k) \cdots \bar{S}(0)$ converges exponentially fast as $k \rightarrow \infty$ to a matrix of the form $1c$ as $k \rightarrow \infty$.

The definitions of $S(\cdot)$ and $\bar{S}(\cdot)$ in (28) and (29), respectively, imply that

$$\bar{S}(k) \cdots \bar{S}(0) = F((k+1)pq) \cdots F_1, \quad k \geq 0.$$

For $\tau \geq 0$, let $\kappa(\tau)$ and $\rho(\tau)$ denote respectively, the integer quotient and remainder of τ divided by pq . Then

$$F(\tau) \cdots F(1) = \hat{S}(\tau) \bar{S}(k(\tau)) \cdots \bar{S}(0)$$

where $k(\tau) = \kappa(\tau) - 1$, and $\hat{S}(\tau)$ is the bounded function

$$\hat{S}(\tau) = \begin{cases} F(\tau) \cdots F((k(\tau)+1)pq+1) & \text{if } \rho(\tau) \neq 0 \\ 1 & \text{if } \rho(\tau) = 0. \end{cases}$$

Since $k(\tau)$ is an unbounded monotone nondecreasing function and $\bar{S}(k) \cdots \bar{S}(0)$ converges exponentially fast as $k \rightarrow \infty$, it follows that $F(\tau) \cdots F(1)$ converges exponentially fast as $\tau \rightarrow \infty$ to a matrix of the form $1c$. ■

B. Proofs of Supporting Assertions

Proof of Proposition 1: Let \mathbb{G}_p and \mathbb{G}_q be two graphs in \mathcal{G} . Then both graphs have properties p1 through p4. Since both graphs have self-arcs at vertex $v+n, v \in \mathcal{V}$, so must their composition $\mathbb{G}_q \circ \mathbb{G}_p$; thus $\mathbb{G}_q \circ \mathbb{G}_p$ has property p1.

Fix $v \in \mathcal{V}$. In view of property p2, either $(v+n, v) \in \mathcal{A}(\mathbb{G}_q)$ or $(v, v) \in \mathcal{A}(\mathbb{G}_q)$. If the former is true then $(v+n, v) \in \mathcal{A}(\mathbb{G}_q \circ \mathbb{G}_p)$ because $v+n$ has a self-arc in \mathbb{G}_p ; on the other hand, if the latter holds, then $([v], v) \in \mathcal{A}(\mathbb{G}_q \circ \mathbb{G}_p)$ because $([v], v) \in \mathcal{A}(\mathbb{G}_p)$. In either case, $([v], v) \in \mathcal{A}(\mathbb{G}_q \circ \mathbb{G}_p)$. Since this is true for all $v \in \mathcal{V}$, $\mathbb{G}_q \circ \mathbb{G}_p$ has property p2.

To show that $\mathbb{G}_q \circ \mathbb{G}_p$ has property p3, fix $u \in \{1, 2, \dots, 2n\}, v \in \mathcal{V}$ and suppose that $u \neq v$ and $(u, v) \in \mathcal{A}(\mathbb{G}_q \circ \mathbb{G}_p)$. In view of the definition of composition, there must exist a vertex $w \in \mathcal{V}$ such that either (i) $(u, w) \in \mathcal{A}(\mathbb{G}_p)$ and $(w, v) \in \mathcal{A}(\mathbb{G}_q)$, or (ii) $(u, w+n) \in \mathcal{A}(\mathbb{G}_p)$ and $(w+n, v) \in \mathcal{A}(\mathbb{G}_q)$. If (i) is true and $w = v$, then $(u, v) \in \mathcal{A}(\mathbb{G}_p)$. This implies $(u, v+n) \in \mathcal{A}(\mathbb{G}_p)$ because of property p3; but $(v+n, v+n) \in \mathcal{A}(\mathbb{G}_q)$ because of property p1 so $(u, v+n) \in \mathcal{A}(\mathbb{G}_q \circ \mathbb{G}_p)$. On the other hand, if (i) is true and if $w \neq v$ then $(w, v+n) \in \mathcal{A}(\mathbb{G}_q)$ because of property p3, so in this case too $(u, v+n) \in \mathcal{A}(\mathbb{G}_q \circ \mathbb{G}_p)$.

Now suppose that (ii) holds. Then $(w+n, v+n) \in \mathcal{A}(\mathbb{G}_q)$ because of property p3. Therefore $(u, v+n) \in \mathcal{A}(\mathbb{G}_q \circ \mathbb{G}_p)$. This proves that $\mathbb{G}_q \circ \mathbb{G}_p$ has property p3.

To show that $\mathbb{G}_q \circ \mathbb{G}_p$ has property p4, again fix $u \in \{1, 2, \dots, 2n\}, v \in \mathcal{V}$ and suppose that $u \neq v$ and $(u, [v]) \in \mathcal{A}(\mathbb{G}_q \circ \mathbb{G}_p)$. Thus there must exist a vertex $w \in \mathcal{V}$ such that either (i) $(u, w) \in \mathcal{A}(\mathbb{G}_p)$ and $(w, [v]) \in \mathcal{A}(\mathbb{G}_q)$, or (ii) $(u, w+n) \in \mathcal{A}(\mathbb{G}_p)$ and $(w+n, [v]) \in \mathcal{A}(\mathbb{G}_q)$. If (i) is true and $w = v$, then $(v+n, v) \in \mathcal{A}(\mathbb{G}_p)$ because of property p4; but $(v+n, v+n) \in \mathcal{A}(\mathbb{G}_q)$ because of property p1 so $(v+n, v) \in \mathbb{G}_q \circ \mathbb{G}_p$. On the other hand, if $w = v$ and (ii) holds, then $(v+n, v) \in \mathcal{A}(\mathbb{G}_q)$ because of property p4; but $(v+n, v+n) \in \mathcal{A}(\mathbb{G}_p)$ because of property p1 so $(v+n, v) \in \mathbb{G}_q \circ \mathbb{G}_p$ in this case too. Therefore $(v+n, v) \in \mathbb{G}_q \circ \mathbb{G}_p$ if $w = v$.

Suppose finally that $w \neq v$. Then either $(w, [v]) \in \mathcal{A}(\mathbb{G}_q)$ or $(w+n, [v]) \in \mathcal{A}(\mathbb{G}_q)$; in either case then $(v+n, v) \in \mathcal{A}(\mathbb{G}_q)$ because of property p4; but $(v+n, v+n) \in \mathcal{A}(\mathbb{G}_p)$ because of property p1 so $(v+n, v) \in \mathbb{G}_q \circ \mathbb{G}_p$. ■

In the proofs which follow we use the symbol $I(\mathbb{G})$ to denote the subgraph of $\mathbb{G} \in \mathcal{G}$ induced by the vertex set $\mathcal{V} + \{n\} = \{n+1, n+2, \dots, 2n\}$ —and we write $\bar{\mathcal{G}}_{sa}$ for the set of all such graphs. We so denote $\bar{\mathcal{G}}_{sa}$, because each of its graphs has self-arcs at all vertices; this is a consequence of property p1 of \mathcal{G} . The Proof of Proposition 4 depends on the following lemma.

Lemma 2: Let \mathbb{G}_p and \mathbb{G}_q be graphs in \mathcal{G} and suppose that \mathbb{G}_p is attached. If $Q(\mathbb{G}_q)$ is rooted then $I(\mathbb{G}_q \circ \mathbb{G}_p)$ is rooted.

Proof: Suppose that $Q(\mathbb{G}_q)$ is rooted at $u \in \mathcal{V}$. It is enough to show that $I(\mathbb{G}_q \circ \mathbb{G}_p)$ is rooted at $u + n$. Let $v + n$ be any vertex in $\mathcal{V} + \{n\}$. It is enough to show that there is a path in $I(\mathbb{G}_q \circ \mathbb{G}_p)$ from $u + n$ to $v + n$. Note first that $\mathbb{G}_q \circ \mathbb{G}_p$ must have a self-arc about $u + n$ because of property p1; this implies that $I(\mathbb{G}_q \circ \mathbb{G}_p)$ must also have a self-arc about $u + n$; thus if $v = u$ there is a path in $I(\mathbb{G}_q \circ \mathbb{G}_p)$ from $u + n$ to $v + n$.

Suppose that $v \neq u$. Since $Q(\mathbb{G}_q)$ is rooted at u there must be a path in $Q(\mathbb{G}_q)$ from u to v . Therefore there must be a positive integer s and distinct vertices $k_0, k_1, k_2, \dots, k_s$ in \mathcal{V} starting at $k_0 = u$ and ending at $k_s = v$, for which $(k_0, k_1), (k_1, k_2), \dots, (k_{s-1}, k_s)$ are arcs in $Q(\mathbb{G}_q)$. Thus for each $i \in \{1, 2, \dots, s\}$ there must be an arc in \mathbb{G}_q from at least (1) k_{i-1} to k_i or (2) k_{i-1} to $k_i + n$, or (3) $k_{i-1} + n$ to k_i or (4) $k_{i-1} + n$ to $k_i + n$. In view of property p3 of graphs in \mathcal{G} , cases (1) and (3) respectively imply the existence of arcs in \mathbb{G}_q from k_{i-1} or $k_{i-1} + n$ to $k_i + n$. Thus under all four conditions there must be an arc from at least one vertex in the set $[k_{i-1}]$ to $k_i + n$.

By assumption \mathbb{G}_p is attached and is in \mathcal{G} which means that for each $i \in \{1, 2, \dots, s\}$, there must be arcs in \mathbb{G}_p from $k_{i-1} + n$ to k_{i-1} and from $k_{i-1} + n$ to $k_{i-1} + n$. Therefore for each $i \in \{1, 2, \dots, s\}$, there must be an arc in $\mathbb{G}_q \circ \mathbb{G}_p$ from $k_{i-1} + n$ to $k_i + n$. This means that for $i \in \{1, 2, \dots, s\}$ there must be an arc in $I(\mathbb{G}_q \circ \mathbb{G}_p)$ from $k_{i-1} + n$ to $k_i + n$. Thus there must be a path in $I(\mathbb{G}_q \circ \mathbb{G}_p)$ from $u + n = k_0 + n$ to $v = k_s + n$. Since this is clearly true for all $v + n$ in $\mathcal{V} + \{n\}$, $I(\mathbb{G}_q \circ \mathbb{G}_p)$ must be rooted at $u + n$. ■

Proof of Proposition 4: For simplicity we write \mathbb{G}_i for \mathbb{G}_{p_i} throughout this proof. By assumption, for $i \in \{1, 2, \dots, m\}$ the graphs \mathbb{G}_{2i} and \mathbb{G}_{2i-1} are quotient rooted and attached respectively. Thus for $i \in \{1, 2, \dots, m\}$ the graphs $I(\mathbb{G}_{2i} \circ \mathbb{G}_{2i-1})$ are rooted because of Lemma 2. Since all graphs in $\bar{\mathcal{G}}_{sa}$ have self-arcs, and $m \geq (n-1)^2$ Proposition 3 applies and it can thus be concluded that $I(\mathbb{G}_{2m} \circ \mathbb{G}_{2m-1}) \circ \dots \circ I(\mathbb{G}_2 \circ \mathbb{G}_1)$ is strongly rooted. But for each $i \in \{1, 2, \dots, m\}$ every arc in $I(\mathbb{G}_{2i} \circ \mathbb{G}_{2i-1})$ is an arc in $\mathbb{G}_{2i} \circ \mathbb{G}_{2i-1}$. This implies that every arc in $I(\mathbb{G}_{2m} \circ \mathbb{G}_{2m-1}) \circ \dots \circ I(\mathbb{G}_2 \circ \mathbb{G}_1)$ is an arc in $\mathbb{G}_{2m} \circ \dots \circ \mathbb{G}_1$ between vertices in $\mathcal{V} + \{n\}$. Therefore every arc in $I(\mathbb{G}_{2m} \circ \mathbb{G}_{2m-1}) \circ \dots \circ I(\mathbb{G}_2 \circ \mathbb{G}_1)$ is an arc in $I(\mathbb{G}_{2m} \circ \dots \circ \mathbb{G}_1)$. Since $I(\mathbb{G}_{2m} \circ \mathbb{G}_{2m-1}) \circ \dots \circ I(\mathbb{G}_2 \circ \mathbb{G}_1)$ is strongly rooted, $I(\mathbb{G}_{2m} \circ \dots \circ \mathbb{G}_1)$ must be strongly rooted as well.

Let $v + n$ be a root of $I(\mathbb{G}_{2m} \circ \dots \circ \mathbb{G}_1)$ from which every other vertex is reachable along a path of length one. We claim that in $\mathbb{G}_{2m+1} \circ \mathbb{G}_{2m} \circ \dots \circ \mathbb{G}_1$ there is an arc from $v + n$ to every vertex in $\{1, 2, \dots, 2n\}$. To prove that this is so, and thus that $\mathbb{G}_{2m+1} \circ \mathbb{G}_{2m} \circ \dots \circ \mathbb{G}_1$ is strongly rooted, first suppose that $u + n$ is any vertex in $\mathcal{V} + \{n\}$. Then there is an arc from $v + n$ to $u + n$ in $I(\mathbb{G}_{2m} \circ \dots \circ \mathbb{G}_1)$ and consequently in $\mathbb{G}_{2m} \circ \dots \circ \mathbb{G}_1$ because $I(\mathbb{G}_{2m} \circ \dots \circ \mathbb{G}_1)$ is strongly rooted at $v + n$; but this arc must also be in $\mathbb{G}_{2m+1} \circ \mathbb{G}_{2m} \circ \dots \circ \mathbb{G}_1$ because \mathbb{G}_{2m+1} has a self arc about vertex $u + n$.

Now suppose that u is any vertex in \mathcal{V} . As before, and for the same reasons, there is an arc in $\mathbb{G}_{2m} \circ \dots \circ \mathbb{G}_1$ from $v + n$ to $u + n$. But there is an arc in \mathbb{G}_{2m+1} from $u + n$ to u because \mathbb{G}_{2m+1} is attached. Therefore there must be an arc in $\mathbb{G}_{2m+1} \circ \mathbb{G}_{2m} \circ \dots \circ \mathbb{G}_1$ from $v + n$ to u . ■

Proposition 5 is a direct consequence of the following lemma.

Lemma 3: If $\mathbb{G} \in \mathcal{G}$ is attached at $i \in \mathcal{V}$, then for any graph $\mathbb{G}_p \in \mathcal{G}$, $\mathbb{G}_p \circ \mathbb{G}$ and $\mathbb{G} \circ \mathbb{G}_p$ are also attached at i .

Proof of Lemma 3: Since \mathbb{G} is attached at i , $(i + n, i)$ is one of its arcs. In view of property p1 of graphs in \mathcal{G} , there is a self arc at $i + n$ in both \mathbb{G} and \mathbb{G}_p . Since $\mathbb{G}_p \in \mathcal{G}$, either (i, i) or $(i + n, i)$ must be one of its arcs because of property p2. In either case $(i + n, i)$ must be one of the arcs of both $\mathbb{G}_p \circ \mathbb{G}$ and $\mathbb{G} \circ \mathbb{G}_p$ because of the definition of graph composition. ■

Proposition 6 depends on the following lemmas.

Lemma 4: Let \mathbb{G}_p and \mathbb{G}_q be graphs in \mathcal{G} . If (i, j) is an arc in $Q(\mathbb{G}_q) \circ Q(\mathbb{G}_p)$, then there is a path in $Q(\mathbb{G}_q \circ \mathbb{G}_p)$ from i to j .

Proof of Lemma 4: Let $(i, j) \in \mathcal{A}(Q(\mathbb{G}_q) \circ Q(\mathbb{G}_p))$ be fixed. If $i = j$, then there is a path in $Q(\mathbb{G}_q \circ \mathbb{G}_p)$ from i to j because all vertices in $Q(\mathbb{G}_q \circ \mathbb{G}_p)$ have self arcs.

Suppose $i \neq j$. Then for some $k \in \mathcal{V}$, $(i, k) \in \mathcal{A}(Q(\mathbb{G}_p))$ and $(k, j) \in \mathcal{A}(Q(\mathbb{G}_q))$. Therefore $([i], [k]) \in \mathcal{A}(\mathbb{G}_p)$ and $([k], [j]) \in \mathcal{A}(\mathbb{G}_q)$. Thus either $([i], k)$ or $([i], k + n)$ is in $\mathcal{A}(\mathbb{G}_p)$ and either $(k, [j])$ or $(k + n, [j])$ is in $\mathcal{A}(\mathbb{G}_q)$. If $([i], k) \in \mathcal{A}(\mathbb{G}_p)$ and $(k, [j]) \in \mathcal{A}(\mathbb{G}_q)$, then $([i], [j]) \in \mathcal{A}(\mathbb{G}_q \circ \mathbb{G}_p)$ which implies that $(i, j) \in \mathcal{A}(Q(\mathbb{G}_q \circ \mathbb{G}_p))$. By the same reasoning, $(i, j) \in \mathcal{A}(Q(\mathbb{G}_q \circ \mathbb{G}_p))$ if $([i], k + n) \in \mathcal{A}(\mathbb{G}_p)$ and $(k + n, [j]) \in \mathcal{A}(\mathbb{G}_q)$. To complete the proof, we need to show that $(i, j) \in \mathcal{A}(Q(\mathbb{G}_q \circ \mathbb{G}_p))$ if either (i) $([i], k) \in \mathcal{A}(\mathbb{G}_p)$ and $(k + n, [j]) \in \mathcal{A}(\mathbb{G}_q)$ is true or (ii) $([i], k + n) \in \mathcal{A}(\mathbb{G}_p)$ and $(k, [j]) \in \mathcal{A}(\mathbb{G}_q)$ is true.

Consider case (i) and suppose that $i = k$. Then $(i + n, [j]) \in \mathcal{A}(\mathbb{G}_q)$. $(i + n, i + n) \in \mathcal{A}(\mathbb{G}_p)$. Therefore $(i + n, [j]) \in \mathcal{A}(\mathbb{G}_q \circ \mathbb{G}_p)$. This implies that $(i, j) \in \mathcal{A}(Q(\mathbb{G}_q \circ \mathbb{G}_p))$.

Now consider case (i) assuming that $i \neq k$. If $(i, k) \in \mathcal{A}(\mathbb{G}_p)$, then $(i, k + n) \in \mathcal{A}(\mathbb{G}_p)$ because of property p3. Similarly if $(i + n, k) \in \mathcal{A}(\mathbb{G}_p)$, then $(i + n, k + n) \in \mathcal{A}(\mathbb{G}_p)$ because of property p3. Therefore $([i], k + n) \in \mathcal{A}(\mathbb{G}_p)$. But $(k + n, [j]) \in \mathcal{A}(\mathbb{G}_q)$ so $([i], [j]) \in \mathcal{A}(\mathbb{G}_q \circ \mathbb{G}_p)$. This implies that $(i, j) \in \mathcal{A}(Q(\mathbb{G}_q \circ \mathbb{G}_p))$.

Now consider case (ii). If $k = i$ then $([i], [j]) \in \mathcal{A}(\mathbb{G}_q \circ \mathbb{G}_p)$ and consequently $(i, j) \in \mathcal{A}(Q(\mathbb{G}_q \circ \mathbb{G}_p))$ because $(i, [j]) \in \mathcal{A}(\mathbb{G}_q)$ and $([i], i) \in \mathcal{A}(\mathbb{G}_p)$ via property p2. If $k \neq i$ then $(k + n, k) \in \mathcal{A}(\mathbb{G}_p)$ because of property p4. Thus $(k + n, [j]) \in \mathcal{A}(\mathbb{G}_q \circ \mathbb{G}_p)$. Moreover $([i], k + n) \in \mathcal{A}(\mathbb{G}_q \circ \mathbb{G}_p)$ because $(k + n, k + n) \in \mathcal{A}(\mathbb{G}_q)$. Thus there is a path from $[i]$ to $[j]$ in $\mathbb{G}_q \circ \mathbb{G}_p$. ■

Proposition 6 is an immediate consequence of the following lemma.

Lemma 5: Let $\mathbb{G}_{p_1}, \dots, \mathbb{G}_{p_m}$ be a sequence of $m > 1$ graphs from \mathcal{G} . If for some $i, j \in \mathcal{V}$, $Q(\mathbb{G}_{p_m}) \circ \dots \circ Q(\mathbb{G}_{p_1})$ contains a path from i to j , then $Q(\mathbb{G}_{p_m} \circ \dots \circ \mathbb{G}_{p_1})$ also contains a path from i to j .

Proof of Lemma 5: We claim first that if $\mathbb{G}_p, \mathbb{G}_q$ are graphs in \mathcal{G} for which $Q(\mathbb{G}_q) \circ Q(\mathbb{G}_p)$ contains a path from u to v , for

some $u, v \in \mathcal{V}$, then $Q(\mathbb{G}_q \circ \mathbb{G}_p)$ also contains a path from u to v . To prove that this is so, fix $u, v \in \mathcal{V}$ and $\mathbb{G}_p, \mathbb{G}_q \in \mathcal{G}$ and suppose that $Q(\mathbb{G}_q) \circ Q(\mathbb{G}_p)$ contains a path from u to v . Then there must be a positive integer s and vertices k_1, k_2, \dots, k_s ending at $k_s = v$, for which $(u, k_1), (k_1, k_2), \dots, (k_{s-1}, k_s)$ are arcs in $Q(\mathbb{G}_q) \circ Q(\mathbb{G}_p)$. In view of Lemma 4, there must be paths in $Q(\mathbb{G}_q \circ \mathbb{G}_p)$ from u to k_1, k_1 to k_2, \dots , and k_{s-1} to k_s . It follows that there must be a path in $Q(\mathbb{G}_q \circ \mathbb{G}_p)$ from u to v . Thus the claim is established.

It will now be shown by induction for each $s \in \{2, \dots, m\}$ that if $Q(\mathbb{G}_{p_s}) \circ \dots \circ Q(\mathbb{G}_{p_1})$ contains a path from i to some $j_s \in \mathcal{V}$, then $Q(\mathbb{G}_{p_m} \circ \dots \circ \mathbb{G}_{p_1})$ also contains a path from i to j_s . In view of the claim just proved above, the assertion is true if $s = 2$. Suppose the assertion is true for all $s \in \{2, 3, \dots, t\}$ where t is some integer in $\{2, \dots, m-1\}$. Suppose that $Q(\mathbb{G}_{p_{t+1}}) \circ \dots \circ Q(\mathbb{G}_{p_1})$ contains a path from i to j_{t+1} . Then there must be an integer k such that $Q(\mathbb{G}_{p_t}) \circ \dots \circ Q(\mathbb{G}_{p_1})$ contains a path from i to k and $Q(\mathbb{G}_{p_{t+1}})$ contains a path from k to j_{t+1} . In view of the inductive hypothesis, $Q(\mathbb{G}_{p_t} \circ \dots \circ \mathbb{G}_{p_1})$ contains a path from i to k . Therefore $Q(\mathbb{G}_{p_{t+1}}) \circ Q(\mathbb{G}_{p_t} \circ \dots \circ \mathbb{G}_{p_1})$ has a path from i to j_{t+1} . Hence the claim established at the beginning of this proof applies and it can be concluded that $Q(\mathbb{G}_{p_{t+1}} \circ \mathbb{G}_{p_t} \circ \dots \circ \mathbb{G}_{p_1})$ has a path from i to j_{t+1} . Therefore by induction the aforementioned assertion is true. ■

Proof of Proposition 7: Let $S \in \text{image } \mathbf{F}^p$ be fixed. Then for some $\{\mathbb{N}_1, \mathbb{N}_2, \dots, \mathbb{N}_p\} \in \mathcal{G}_{sa}^p, \{\mu_1, \mu_2, \dots, \mu_p\} \in \mathcal{R}^p$ and $\{b_1, b_2, \dots, b_p\} \in \mathcal{B}^p, S = F_p \dots F_1$ where $F_i = \mathbf{F}(\mathbb{N}_i, \mu_i, b_i), i \in \{1, \dots, p\}$. By assumption, $\mathbb{N}_1, \mathbb{N}_2, \dots, \mathbb{N}_p$ is a jointly rooted sequence. Since $Q(\gamma(F_i)) = \mathbb{N}_i, i \in \{1, \dots, p\}$, the sequence $Q(\gamma(F_1)), \dots, Q(\gamma(F_p))$ is also jointly rooted. Thus $Q(\gamma(F_p)) \circ \dots \circ Q(\gamma(F_1))$ is rooted. In view of Proposition 6, $Q(\gamma(F_p) \circ \dots \circ \gamma(F_1))$ is rooted.

By hypothesis, for each $i \in \{1, \dots, n\}$, at least one of the vectors in the sequence b_1, \dots, b_p has a 1 in its i th row. Therefore for each $i \in \{1, \dots, n\}$, at least one of the graphs in the sequence $\gamma(F_1), \dots, \gamma(F_p)$ must be attached at i . Thus by Proposition 5, $\gamma(F_p) \circ \dots \circ \gamma(F_1)$ is attached.

In view of (24) and the definition of $S, \gamma(F_p) \circ \dots \circ \gamma(F_1) = \gamma(S)$. Therefore $Q(\gamma(S))$ is rooted and $\gamma(S)$ is attached. Therefore the graph of every matrix in image \mathbf{F}^p is attached and has a rooted quotient graph.

Let \bar{S} be any matrix in $\mathcal{F}^p(q)$. Then there must be matrices $S_i \in \text{image } \mathbf{F}^p, i \in \{1, \dots, q\}$ such that $\bar{S} = S_q \dots S_1$. Then each graph $\gamma(S_i), i \in \{1, \dots, q\}$, must be attached and must have a rooted quotient graph $Q(\gamma(S))$. Therefore by Proposition 4, $\gamma(S_q) \circ \dots \circ \gamma(S_1)$ must be strongly rooted. From this, (24) and the fact that $\bar{S} = S_q \dots S_1$, it follows that \bar{S} has a strongly rooted graph. Therefore every matrix in $\mathcal{F}^p(q)$ is strongly rooted. ■

VII. COMPARISONS WITH PRIOR RESEARCH

As already noted, various versions of asynchronous consensus have been studied before [10], [13]–[15], [18]. Not surprisingly, there are similarities and differences between the

problems addressed in these papers and the problem treated here. As for the similarities, heading updating in [10], [13]–[15], [18] is assumed to go on forever; the same is true here as (1) clearly implies. In addition, the convergence results derived here are to some extent similar to the corresponding results [10], [13], [14]. We will expand on this point in a moment. While the results of this paper are perhaps not surprising to some, proving them is far from obvious. There are several reasons for this. First, in order to carry out a proof, one needs first to go through analytic synchronization; although some of the steps in this process are implicit in the work of [13]–[15], one can hardly call the process straightforward. For example, on first pass one might find it a bit surprising to learn that the definition of the state of \mathcal{S} should include not only all of the agents' headings, but all of their way-points as well. Second, at the technical level there is a very sharp difference between what's encountered here and what was encountered in earlier work on synchronous consensus. In particular, the stochastic matrices which arise here do not have all positive diagonal elements {equivalently the graphs of these matrices do not have self-arcs at all vertices} and because of this, convergence tools used in [6]–[10] are not sufficient to deal with the problem addressed here. Indeed at least half of this paper, namely Section VI, is focused exactly on developing convergence tools appropriate to the type of stochastic matrices which are involved.

A. Comparison of Models

It is possible to derive from the equations which model \mathcal{S} , namely (13)–(16), an asynchronous model similar to which the findings of [13]–[15] depend. Unlike \mathcal{S} , neither the model used in [13]–[15] nor the model we are about to derive, are state space systems; instead they are what we will call “delay-operator” models.

For each $i \in \{1, 2, \dots, n\}$, let $\pi_{ij} : \mathcal{T}_i \rightarrow \mathcal{T}_j$ denote that function for which $t_{\pi_{ij}(\tau)}$ is the largest event time in \mathcal{T}_j which does not exceed $t_{\tau-1}$. From (13) and (14) it follows:

$$\bar{\theta}_j(\pi_{ij}(\tau)) = \bar{w}_j(\pi_{ij}(\tau) - 1) \quad (30)$$

$$\bar{\theta}_j(\pi'_{ij}(\tau)) = \bar{w}_j(\pi'_{ij}(\tau) - 1) \quad (31)$$

$$\bar{\theta}_j(s) = \bar{\theta}_j(s - 1), \quad \pi'_{ij}(\tau) < s < \pi_{ij}(\tau) \quad (32)$$

where $t_{\pi'_{ij}(\tau)}$ is the event time in \mathcal{T}_j just before $t_{\pi_{ij}(\tau)}$. Equations (31) and (32) imply that

$$\bar{\theta}_j(s - 1) = \bar{w}_j(\pi'_{ij}(\tau) - 1), \quad \pi'_{ij}(\tau) < s < \pi_{ij}(\tau). \quad (33)$$

Note that if $\pi_{ij}(\tau) = \tau - 1$, then $\bar{\theta}_j(\tau - 1) = \bar{w}_j(\pi_{ij}(\tau) - 1)$ because of (30); on the other hand, if $\pi_{ij}(\tau) < \tau - 1$, then $\bar{\theta}_j(\tau - 1) = \bar{w}_j(\pi'_{ij}(\tau) - 1)$ because of (33). In either case $\bar{\theta}_j(\tau - 1) = \bar{w}_j(\phi_{ij}(\tau) - 1)$ where $\phi_{ij}(\tau) = \pi_{ij}(\tau)$ if $\pi_{ij}(\tau) = \tau - 1$ and $\phi_{ij}(\tau) = \pi'_{ij}(\tau)$ if $\pi_{ij}(\tau) < \tau - 1$. This enables us to rewrite the subsystem (15) and (16) in the form

$$\bar{w}_i(\tau) = \frac{1}{\bar{n}_i(\tau)} \sum_{j \in \bar{\mathcal{N}}_i(\tau)} \{(1 - \bar{\mu}_j(\tau))\bar{w}_j(\phi_{ij}(\tau)) + \bar{\mu}_j(\tau)\bar{w}_j(\tau - 1)\}, \quad t_\tau \in \mathcal{T}_i \quad (34)$$

$$\bar{w}_i(\tau) = \bar{w}_i(\tau - 1), \quad t_\tau \notin \mathcal{T}_i. \quad (35)$$

Therefore, if we define $w = [w_1, w_2, \dots, w_n]'$, then from the preceding it is clear that

$$w(\tau) = A(\tau, \delta)w(\tau - 1) \quad (36)$$

where δ is the one-unit delay operator and $A(\tau, \delta)$ is an $n \times n$ matrix whose elements $a_{ij}(\tau, \delta)$ are real polynomials in δ .

Let us note that like any equation of this form, (36) can be realized as a state space system using standard lifting techniques. One would expect such a realization to have a dimension of roughly the same size as the largest degree among the polynomial entries in A . It is thus surprising that the underlying asynchronous process which leads to this equation can be described by \mathbb{S} , because \mathbb{S} is only a $2n$ -dimensional state-space system.

The model described by (36) is similar to the model used in [13]–[15] to study asynchronous consensus. Both models are linear time-varying, delay-operator equations evolving on the sequence which results when the event time sequences of the n agents are merged into one sequence. There are however, some important differences between the two models in addition to the obvious and not so important fact that one is written in terms of way points and the other is written in terms of headings {or something equivalent}. Two main features distinguish the models. First, the $a_{ij}(\tau, \delta)$ in $A(\tau, \delta)$ above can be linear combinations of two distinct powers of δ whereas the $a_{ij}(\tau, \delta)$ which appear in the models in [13]–[15] are {in effect} explicitly assumed to be scalar multiples of powers of δ . Second, the nonzero coefficients of $a_{ij}(\tau, \delta)$ in the models in [13]–[15] are assumed to be bounded below uniformly by a positive constant α ; this assumption is not satisfied by the model in (36) because the $\bar{\mu}_i(\tau)$ can take on values arbitrarily close to zero. This difference is especially important because the existence of a positive underbound α is key to the convergence analysis upon which the results in [13]–[15] depend. In particular, the bounding parameter α appears explicitly in the rate of change of the function $V = \max\{\theta_1, \theta_2, \dots, \theta_n\} - \min\{\theta_1, \theta_2, \dots, \theta_n\}$ used in [13]–[15] and were it 0, one could not conclude from the analysis as it stands that V 's limit is zero. Observe that it is precisely when V tends to zero that a consensus is reached. It would be quite interesting and useful to see if the analysis in [13]–[15] could be generalized to handle the delay-operator model defined by (36).

B. Comparison of Results

The version of the asynchronous consensus problem considered here significantly generalizes our earlier work [17]. In particular, the present version of the problem can deal with continuous heading changes whereas the version of the problem solved in [17] cannot. Because the problem considered in [17] is a special case of the problem in this paper, Theorem 2 applies; the theorem provides a slightly more general condition for reaching a consensus than does the main result of [17].

The consensus problem considered in [17] proves to be essentially the same as the delay-free asynchronous consensus problem considered in [13], [14], and so a meaningful comparison of results is possible. To make the comparison, we will refer to [10] rather than [13], [14] since [10] provides a clear and concise summary of the relevant results from [13], [14].

It is possible to compare the hypotheses of Theorem 2 in this paper with the corresponding hypotheses for exponential convergence stated in [10], namely assumptions 2 and 3 of that paper. To do this, let us agree to say that the *union* of a set of graphs $\mathbb{G}_{r_1}, \mathbb{G}_{r_2}, \dots, \mathbb{G}_{r_k}$ with vertex set \mathcal{V} is that graph with vertex set \mathcal{V} and arc set consisting of the union of the arcs of all of the graphs $\mathbb{G}_{r_1}, \mathbb{G}_{r_2}, \dots, \mathbb{G}_{r_k}$. Taken together, assumptions 2 and 3 of [10] are more or less equivalent to assuming that there are finite positive integers q and s such that the *union*

$$\mathbb{G}(k) \triangleq \bar{\mathbb{N}}((k+1)q-1) \cup \bar{\mathbb{N}}((k+1)q-2) \cup \dots \cup \bar{\mathbb{N}}(kq)$$

is strongly connected and independent of k for $k \geq s$. By way of comparison, the hypothesis of Theorem 2 is equivalent to assuming that there is a finite positive integer q such that the *composition*

$$\bar{\mathbb{G}}(k) \triangleq \bar{\mathbb{N}}((k+1)q-1) \circ \bar{\mathbb{N}}((k+1)q-2) \circ \dots \circ \bar{\mathbb{N}}(kq)$$

is rooted for $k \geq 0$. The latter assumption is weaker than the former for several reasons. First, the arc set of $\mathbb{G}(k)$ is always a subset of the arc set of $\bar{\mathbb{G}}(k)$ and in some cases the containment may be strict. Second, $\bar{\mathbb{G}}(k)$ is not assumed to be independent of k , even for k sufficiently large, whereas $\mathbb{G}(k)$ is; in other words, $\bar{\mathbb{G}}(k)$ is not assumed to converge whereas $\mathbb{G}(k)$ is. Third, each $\mathbb{G}(k)$ is assumed to be strongly connected whereas each $\bar{\mathbb{G}}(k)$ need only be rooted; note that a strongly connected graph is a special type of rooted graph in which every vertex is a root. From these comparisons it is clear that the hypotheses of Theorem 2 are non-trivially less restrictive than those made in [10]. Finally, as we've already noted, the hypotheses of Theorem 2 are essentially the same as those which apply to previously derived results for the synchronous version of the problem [7], [8] and so what this paper does is bring our understanding of convergence of asynchronous consensus up to the same level of understanding as we've already had for the synchronous version of the problem.

VIII. CONCLUSION

The asynchronous consensus problem we've considered serves as an example of the type of problem to which the idea of analytic synchronization can be applied. The asynchronous version of the multi-agent rendezvous problem considered in [19] and [20] provides another. Despite these examples, there are several unsettled issues concerning the analytic synchronization idea. First, it is not clear what the general process is for choosing a state vector. Second, it is also not clear what the exact conditions are on an asynchronously interacting set of dynamical systems for analytic synchronization to be possible. The examples provided by this paper and by [19] and [20] may help to more precisely formulate these issues and to lead to their resolution.

It is possible to formulate and solve a "continuous" version of Vicsek's problem in which each agent's heading is adjusted by controlling its differential rate. Because of changing neighbor sets this can lead to a differential equation model with a discontinuous vector field in which chattering may conceivably occur. To avoid this one can introduce "dwell times" as was done in [6] for the leader-follower version of the problem. As a result, the question of synchronization again arises, in this case with

event times being the times at which each agent's dwell time periods begin. Thus, although one might think that the question of synchronization is irrelevant in some continuous versions of the problem, this appears to only be true if one is willing to accept generalized solutions to differential equations and the possibility of chattering. Of course one could redefine what is meant by a neighbor and by a sensing range to avoid switching dynamics altogether. An intriguing version of the consensus problem along these lines, which avoids both the asynchronous issue and chattering, is considered in [24].

REFERENCES

- [1] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, "Novel type of phase transition in a system of self-driven particles," *Phys. Rev. Lett.*, vol. 75, pp. 1226–1229, 1995.
- [2] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Stable flocking of mobile agents," in *Proc. 42nd IEEE CDC*, 2003, pp. 2010–2015.
- [3] M. H. De Groot, "Reaching a consensus," *J. Amer. Stat. Assoc.*, vol. 69, pp. 118–121, 1974.
- [4] M. Pease, R. Shostak, and L. Lamport, "Reaching agreement in the presence of faults," *J. ACM*, vol. 27, pp. 228–234, 1980.
- [5] M. J. Fischer, N. A. Lynch, and M. S. Paterson, "Impossibility of distributed consensus with one faulty process," *J. ACM*, vol. 32, pp. 347–382, 1985.
- [6] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Automat. Control*, vol. 48, no. 6, pp. 988–1001, Jun. 2003.
- [7] L. Moreau, "Stability of multi-agent systems with time-dependent communication links," *IEEE Trans. Automat. Control*, vol. 50, no. 2, pp. 169–182, Feb. 2005.
- [8] W. Ren and R. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Automat. Control*, vol. 50, no. 5, pp. 655–661, May 2005.
- [9] D. Angeli and P. A. Bliman, "Extension of a result by Moreau on stability of leaderless multi-agent systems," in *Proc. 44th IEEE CDC*, 2005, pp. 759–764.
- [10] V. D. Blondel, J. M. Hendrichx, A. Olshevsky, and J. N. Tsitsiklis, "Convergence in multiagent coordination, consensus, and flocking," in *Proc. 44th IEEE CDC*, 2005, pp. 2996–3000.
- [11] R. Olfati-Saber and R. M. Murray, "Consensus seeking in networks of agents with switching topology and time-delays," *IEEE Trans. Automat. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.
- [12] S. Chatterjee and E. Seneta, "Towards consensus: Some convergence theorems on repeated averaging," *J. Appl. Prob.*, vol. 14, pp. 89–97, 1977.
- [13] J. N. Tsitsiklis, "Problems in Decentralized Decision Making and Computation," Ph.D. dissertation, MIT, Cambridge, MA, 1984.
- [14] J. N. Tsitsiklis, D. P. Bertsekas, and M. Athans, "Distributed asynchronous deterministic and stochastic gradient optimization algorithms," *IEEE Trans. Automat. Control*, vol. AC-31, no. 9, pp. 803–812, Sep. 1986.
- [15] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and Distributed Computation*. Englewood Cliffs, NJ: Prentice Hall, 1989.
- [16] J. A. Fax, "Optimal and Cooperative Control of Vehicle Formations," Ph.D. dissertation, California Inst. of Technol., Pasadena, 2001.
- [17] M. Cao, A. S. Morse, and B. D. O. Anderson, "Coordination of an asynchronous multi-agent system via averaging," in *Proc. IFAC Congress*, 2005, [CD ROM].
- [18] L. Fang, P. J. Antsaklis, and A. Tzimas, "Asynchronous consensus protocols: Preliminary results, simulations and open questions," in *Proc. 44th IEEE CDC*, 2005, pp. 2194–2199.
- [19] J. Lin, A. S. Morse, and B. D. O. Anderson, "The multi-agent rendezvous problem—The asynchronous case," in *Proc. 43rd IEEE CDC*, 2004, pp. 1926–1931.
- [20] J. Lin, A. S. Morse, and B. D. O. Anderson, "The multi-agent rendezvous problem—Part 2: The asynchronous case," *SIAM J. Control Optim.*, vol. 46, pp. 2120–2147, 2007.
- [21] A. S. Morse, "Logically switched dynamical systems," in *Nonlinear and Optimal Control Theory*. New York: Springer-Verlag, 2008, pp. 1–84.
- [22] M. Cao, A. S. Morse, and B. D. O. Anderson, "Reaching a consensus in a dynamically changing environment—A graphical approach," *SIAM J. Control Optim.*, vol. 47, no. 2, pp. 575–600, Mar. 2008.
- [23] R. Diestel, *Graph Theory*. New York: Springer-Verlag, 1997.
- [24] F. Cucker and S. Smale, "On the mathematics of emergence," *Jpn. J. Math.*, vol. 2, pp. 197–227, 2007.



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