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Properties of Zero-Free Transfer Function Matrices

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Abstract: Transfer functions of linear, time-invariant finite-dimensional systems with more outputs than inputs, as arise in factor analysis (for example in econometrics), have, for state-variable descriptions with generic entries in the relevant matrices, no finite zeros. This paper gives a number of characterizations of such systems (and indeed square discrete-time systems with no zeros), using state-variable, impulse response, and matrix-fraction descriptions. Key properties include the ability to recover the input values at any time from a bounded interval of output values, without any knowledge of an initial state, and an ability to verify the no-zero property in terms of a property of the impulse response coefficient matrices. Results are particularized to cases where the transfer function matrix in question may or may not have a zero at infinity or a zero at zero.

Key Words: linear system, zeros, system inversion.

1. The Problem of Interest

Our motivation for studying problems of tall transfer function matrices and in particular of zeroless ones comes from dynamic factor models which arise in a number of fields, e.g. econometrics. Our contention is that a number of the properties of such transfer function matrices will be applicable in that context; indeed it was through working in the first instance with an econometric problem that we became conscious of the extensive system theoretic properties of these transfer function matrices. This paper does not describe the application of the idea to factor analysis, but before describing the content of the paper, we outline the context. In linear dynamic factor models, the so called latent (i.e. noisefree) variables are often assumed as stationary with rational *singular* spectral density and thus may be represented by a stable finite-dimensional linear system excited by white noise. Further, in these models, the singularity of the spectral density implies there are more scalar outputs than scalar inputs. There is interest both in modeling, i.e. estimation of the parameters of such a system, for instance in state space form, and in using such models for prediction. In particular, in recent times, in econometrics, so-called generalized dynamic factor models [1],[2] have been developed and the system theory of such transfer function matrices may be useful in this field.

In our work with such models, it became evident that filters and predictors constructed using Wiener or Kalman filtering procedures had unusual properties which in the end could be largely understood in terms of properties of the underlying linear system. The purpose of this paper is to set out these properties without, as noted above, exploring the embedding of the problem into general consideration of factor models. As it turns out, it is also possible to consider these unusual properties using properties of the power spectrum matrix, or covariance sequence, of the underlying system output when that system is excited with white noise. This will be done in a separate paper.

Our starting point is the easily overlooked fact (reviewed in the next section) that when the matrices of a minimal state-variable realization of a *tall* transfer function matrix have entries with generic values, the system has no zeros.

Following a brief review of the concept of zeros for a transfer function matrix, we study in Section 3 an algorithm due to Moylan [3]. This algorithm was originally presented in continuous time, with the core result that when a system has no zeros and has at least as many scalar outputs as inputs, the input trajectory can be recovered (at least formally–differentiations are not excluded) from the output trajectory without knowledge of an initial state. Our presentation, in discrete time, establishes this fact, which is hardly a surprise, but also shows that the input sequence over any interval, $[k, k+L-1]$ say, can be determined by the output values over an interval $[k, k+L-1]$, where L is an integer no greater than the McMillan degree of the underlying transfer function matrix and is necessarily at least as large as the observability index of the underlying system. This idea, applicable to rational transfer function matrices with no finite zeros, is extended in two directions: to transfer function matrices with no finite zeros and no zero at infinity, and to transfer function matrices with no finite zeros, except possibly at the origin. In the first of these extensions, it turns out that additionally the state is recoverable at time $k + L$, from the same output data, and in the second of these extensions and with a modification of the definition of L, the input sequence over an interval $[k + J, k + L - 1]$ for some $J > 0$ can be recovered from output data over $[k, k + L - 1]$ for some modified L.

In Section 4, we present alternative views of this result. We characterize an absence of all finite zeros in terms of the system having no nontrivial *output-nulling invariant subspace*, (the concept is reviewed, and was introduced in [4]); this helps us then to characterize the zero-free property in terms of the unique solvability of a certain linear matrix equation involving

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the state-variable parameters and impulse response coefficients. Further study of this condition allows a characterization just in terms of impulse response coefficients (together with an upper bound on the system dimension, something which in principle can be determined from the impulse response coefficients). Then we review the ideas using polynomial matrix fraction descriptions and the Smith-McMillan form. The same extensions which were made in Section 3 are considered in Section 4.

Section 5 contains some concluding remarks.

2. Zeros of Rational Transfer Function Matrices

Suppose that $W(z)$ is a $p \times m$ rational transfer function matrix with minimal realization $\{A, B, C, D\}$ of dimension n. The associated state variable equations can be thought of as

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned} \tag{1}$$

Definition 1 The finite zeros of the transfer function matrix $W(z)$ with minimal realization $\{A, B, C, D\}$ are defined to be the finite values of z for which the rank of the following matrix falls below its normal rank:

$$M(z) = \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} \tag{2}$$

This and the later comments of this section can all be found in [5],[6]. As is easily checked, in the case of scalar transfer functions, the zeros according to this definition coincide with the conventional notion of transfer function zeros, as the zeros of the numerator polynomial of the transfer function, when it is represented as a ratio of coprime polynomials. If a matrix transfer function is represented as a coprime left or right polynomial fraction, the zeros according to the definition are those values of z for which the numerator matrix has rank less than its normal rank. The zeros are also the zeros of the numerator polynomials in a Smith-McMillan decomposition of $W(z)$ as

$$W(z) = U_1(z)V(z)U_2(z) \tag{3}$$

where $U_1(z), U_2(z)$ are square polynomial matrices in z with constant nonzero determinant and $V(z)$ has zero entries other than those in the same row and column, which are rational, and have a certain property: if the ii entry of $V(z)$ is $n_i(z)/d_i(z)$ where n_i, d_i are coprime and monic, then n_i divides n_{i+1} when the latter is nonzero, and d_{i+1} divides d_i when the former is nonzero.The zero set of the n_i and the zero set of the d_i correspond to the zero and pole set of $W(z)$.

We shall need to consider several extensions of Definition 1. First, the definition of zeros can be extended to encompass infinite values of z also, though not straighforwardly for all the various equivalent characterizations of finite zeros. Observe that for $|z|$ sufficiently large, the rank of M is easily seen to be $n+$ rank D. This, and the fact that $W(\infty)$ evaluates as D, motivate the following definition:

Definition 2 The transfer function matrix $W(z)$ with minimal realization $\{A, B, C, D\}$ of dimension n is said to have an infinite zero precisely when n + rank D is less than the normal rank of M, or equivalently, when rank $D <$ normal rank W.

This definition is also equivalent to the following approach: define $\tilde{W}(q) = W(\frac{aq+b}{q+c})$ for some real a, b, c such that a is not a pole of $W(z)$. Then $W(z)$ has an infinite zero if and only if $\tilde{W}(q)$ has a zero at $q = -c$. Suppose further that $p > m$ or that $W(z)$ is tall and that $[B^T \;\; D^T]^T$ has full column rank (else a trivial transformation of the input space could be made and certain inputs discarded, in order to achieve this last property). If the entries of A, B, C, D are generic, it is not hard to see that there will be no values of z for which the rank of M will drop below the normal rank, which is $n + m$, as we now argue. Therefore there will be no zeros, finite or infinite.

Proposition 1 Consider a transfer function matrix $W(z)$ with minimal realization $\{A, B, C, D\}$ of dimension n in which B, C have m columns and p rows respectively with $p > m$. If the entries of A, B, C, D assume generic values, then $W(z)$ has no finite or infinite zeros.

Proof: Observe first that the normal rank (which is the rank for almost all z) of a generic M is $n + m$: to see this, take $A = C = 0$ and D as any full column rank matrix, to get a particular $M(z)$ which for any nonzero z has rank $n + m$. Since the normal rank cannot exceed $n+m$ and this rank is attained for a particular choice of A etc, so $n + m$ must be the normal rank for generic M. Observe also that D generically has rank m, and hence the normal rank of M equals n + rank D, which shows that generically $W(z)$ has no infinite zero. For the finite zeros, observe that any such zero must be a zero of every minor of dimension $(n+m)\times(n+m)$. Since $M(z)$ has normal rank $n+m$, there must be at least one minor of dimension $(n + m) \times (n + m)$ which is nonzero for almost all values of z. Choose A, B and the first m rows of C, D generically, and consider the associated minor. For each of the finite set of values of z for which the minor is zero, determine the associated kernel. Then a generic $(n + m)$-dimensional vector will not be orthogonal to any single one of these kernels, and since there are a finite number of such kernels, a generic $(n + m)$-dimensional vector will not be orthogonal to any of the kernels considered simultaneously. If the next, i.e $(m + 1)$-th, row of $[C \;\; D]$ is set equal to this vector, then any vector in any of the finite set of kernels of the $(n + m)$-dimensional minors formed using the first m rows of $[C \;\; D]$ will not be orthogonal to the added row of $[C \;\; D]$, which means that the $(m+n+1)$ row matrix obtained by adjoining the new row of $[C \;\; D]$ must have an empty kernel for any value of z, i.e. there is no zero. Given that C, D are actually generic and may have more rows again, the result is now evident. □

A square nonunimodular transfer function on the other hand always has zeros, but need not have finite zeros. Consider for example $W(z) = (z^2 + z + 1)^{-1}$.

3. System Inversion with No Zeros–A State-space Algorithm

System inversion is the task of obtaining the input of a system with knowledge of the output. It is generally recognized that the zeros of the original system may end up as poles of any system inverse. If then there are no zeros of the system being inverted, the question arises as to what is the nature of the dynamics of the inverse system. In this section, we argue that if there is a finite-dimensional linear system with a prescribed state-variable form, and if it has no zeros, then knowledge of the output over a certain finite interval allows (unique) computation of the input at the start of that interval, *and* the state value at the start of that interval. Furthermore, with a full column rank condition on the matrix D of the state variable realization, one can also compute the input and state value at the end of that interval.

The key reference is [3]. We begin with the following definition, taken from the above reference and mildly modified:

Definition 3 For all $k \geq 0$ let u_k^1 and u_k^2 be any two inputs at time k to the system (1) with initial conditions x_0^1 and x_0^2 and let y_k^1 and y_k^2 be the corresponding outputs at time k. Then the system is said to be left invertible with unknown initial state if $y_k^1 = y_k^2$ for all k implies that $u_k^1 = u_k^2$ for all k and that $x_0^1 = x_0^2$.

This definition makes no reference to existence of an algorithm for computing the initial state and input sequence given the output sequence; such an algorithm does exist, and is effectively described in [3] for a continuous time system.

When is a system left invertible with unknown initial state? The answer is as follows:

Theorem 1 Consider the system (1) with input, state and output dimensions m, n and p, with $p \geq m$, and with $[B^T \ D^T]^T$ of full column rank. Then it is left invertible with unknown initial state if and only if rank $M(z) = n + m$ for all finite $z \in C$.

The proof of necessity is short and very similar indeed to that of [3], whose proof is in continuous time. The discrete/continuous difference is almost inconsequential for the necessity proof but of substance for the sufficiency proof.

Proof of Necessity: Suppose that the rank condition on $M(z)$ fails. Then for some possibly complex λ, there exist x_0 and u_0 not both zero for which

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} \lambda x_0 \\ 0 \end{bmatrix} \tag{4}$$

In fact, x_0 must be nonzero, else the full column rank condition on $[B^T \ D^T]^T$ would be violated. The system with initial state $x_0 \neq 0$ and input sequence $u_k = \lambda^k u_0$ delivers, as is easily verified, $x_j = \lambda^j x_0$ and $y_j = \lambda^j [C x_0 + D u_0] = 0$ for all j. Thus zero output can arise from both a zero initial state and input, or an initial state x_0 and input sequence $u_k = \lambda^k u_0$. This shows that the system is not left invertible with unknown initial state, and completes the proof of necessity.□

The proof of sufficiency is effectively constructive. An algorithm using a finite number of rational calculations suffices. It is described in [3] for the case of continuous time systems, and involves a differentiation operation; as stated in the penultimate section of [3], 'the change required (for discrete time) is the substitution of "unit predictors" for differentiators; this makes the inverse noncausal, but with the obvious change to the time scale a causal inverse with delay is produced'.

In actual fact, a strengthened result is possible, for which we shall provide a constructive proof:

Theorem 2 Consider the system (1) with input, state and output dimensions m, n and p, with $p \geq m$, and with $[B^T \ D^T]^T$ of full column rank. Suppose that rank $M(z) = n + m$ for all finite $z \in C$. Then for some integer L with $L \leq n$, the state and input at an arbitrary time k, viz. x_k and u_k, are computable from the L measurements $y_k, y_{k+1}, \ldots, y_{k+L-1}$

Proof: The beginning of this proof is drawn from [3], but is included for completeness and because we include an important step omitted in the original.

The broad strategy is to progressively identify in a series of steps parts of the state using output measurements, with the process ending when all parts of the state and then the input have been identified. Each step includes three separate operations, an output basis change, a state-space basis change, and a reduction of state-space dimension. We describe the first step, which starts with the system (1).

Given the condition of the theorem hypothesis that $p \geq m$, there exists a nonsingular coordinate basis change matrix for the output space such that after the basis change,

$$D = \begin{bmatrix} D_0 \\ 0 \end{bmatrix} \tag{5}$$

with D_0 of full row rank. Thus D_0 will have at most m rows and rank at most m. Partition C conformably with D as $C^T = [C_1^T \ C_2^T]$. Now observe (and we will use this fact below, though it is not verified in the original reference) that C_2 must have a nonzero number of rows and be nonzero–for otherwise there will certainly be values of z for which rank $M(z) < n + m$, as the following argument shows. Let α be an eigenvector of $A - BD_0^T(D_0D_0^T)^{-1}C_1$, with eigenvalue z_0. Define $\beta = -D_0^T(D_0D_0^T)^{-1}C_1\alpha$. Then it is easily verified that

$$\begin{aligned} (z_0 I - A)\alpha - B\beta &= 0 \\ C_1\alpha + D_0\beta &= 0 \end{aligned} \tag{6}$$

which demonstrates that, if C_2 has no rows or its rows are all zero, $M(z_0)$ has a nontrivial kernel, contradicting the theorem hypothesis.

Now apply a second output coordinate basis change that leaves D_0 and C_1 invariant, but achieves

$$C_2 = \begin{bmatrix} \tilde{C}_2 \\ 0 \end{bmatrix} \tag{7}$$

where $\tilde{C}_2$ has full row rank, call it q. The zero block may be absent. Necessarily as we have just proved, $q > 0$. The output equation part of (1) after this output basis change is

$$\begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix} = \begin{bmatrix} C_1 \\ \tilde{C}_2 \\ 0 \end{bmatrix} x + \begin{bmatrix} D_0 \\ 0 \\ 0 \end{bmatrix} u \tag{8}$$

Now a state-space basis change is executed. A nonsingular matrix T is found so that $\tilde{C}_2 T^{-1} = [0 \quad I]$ where the zero matrix might not be present. This matrix is used to define the state-space basis change with the result that (1) after the two coordinate basis changes takes the form

$$\begin{aligned} \begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_k \\ \begin{bmatrix} y_k^1 \\ y_k^2 \\ y_k^3 \end{bmatrix} &= \begin{bmatrix} C_{11} & C_{12} \\ 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} + \begin{bmatrix} D_0 \\ 0 \\ 0 \end{bmatrix} u_k \end{aligned} \tag{9}$$

The process of passing from (1) to (9) ensures that the rank of D_0 in (9) is identical with the rank of D in (1). It is evident that the variable y_k^3 contains no information, and so it can be thrown away. It is also evident that y_k^2 is identical with x_k^2, i.e. y_k immediately gives us the value of some of the entries of the state vector x_k.

We will now introduce a new output vector, which is a combination of parts of y_k and y_{k+1}.

$$\begin{bmatrix} Y^1_{k,k+1} \\ Y^2_{k,k+1} \end{bmatrix} = \begin{bmatrix} y^1_k - C_{12} y^2_k \\ y^2_{k+1} - A_{22} y^2_k \end{bmatrix} \tag{10}$$

Then using (9), we can write:

$$x^1_{k+1} = A_{11} x^1_k + A_{12} y^2_k + B_1 u_k \tag{11}$$

$$Y_{k,k+1} = \begin{bmatrix} Y^1_{k,k+1} \\ Y^2_{k,k+1} \end{bmatrix} = \begin{bmatrix} C_{11} \\ A_{21} \end{bmatrix} x^1_k + \begin{bmatrix} D_0 \\ B_2 \end{bmatrix} u_k$$

This now completes one step of the progressive construction procedure. As noted above, a key outcome of this step is that part of the state vector, x_k^2, has been identified. Furthermore though, we have replaced the quadruple A, B, C, D by the quadruple $\{A_{11}, B_1, [C_{11}^T \;\; A_{21}^T]^T, [D_0^T \;\; B_2^T]^T\}$ with smaller state-space dimension. Admittedly, there is an additional known input to the state equation in the form of $A_{12} y_k^2$, but this is inessential to the argument. Observe further that

$$\begin{aligned} \text{rank } M(z) &= \text{rank} \begin{bmatrix} zI - A_{11} & -A_{12} & -B_1 \\ -A_{21} & zI - A_{22} & -B_2 \\ C_{11} & C_{12} & D_0 \\ 0 & I_q & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= q + \text{rank} \begin{bmatrix} zI - A_{11} & 0 & -B_1 \\ -A_{21} & 0 & -B_2 \\ C_{11} & 0 & D_0 \end{bmatrix} \\ &= q + \text{rank} \begin{bmatrix} zI - A_{11} & -B_1 \\ -A_{21} & -B_2 \\ C_{11} & D_0 \end{bmatrix} \end{aligned} \tag{12}$$

Consequently, because $M(z)$ has full column rank for all z, the same must be true of the matrix

$$\hat{M}(z) = \begin{bmatrix} zI - A_{11} & -B_1 \\ -A_{21} & -B_2 \\ C_{11} & D_0 \end{bmatrix} = \begin{bmatrix} zI - \hat{A} & -\hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \tag{13}$$

Therefore the same procedure as used to obtain $\hat{M}(z)$ from $M(z)$ can be repeated on the quadruple $\{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}$, and indeed further repeated. Parts of the state vector (modulo coordinate basis changes) will be identified in a series of these steps, and no step can fail to yield information, until the entire state vector is identified. Each step reduces the state vector dimension by at least one; so the state vector will be completely learnt with $L \le n$ successive values of the output. The matrix replacing $M(z)$ after each step and therefore after the L steps will still have full column rank, and the underlying equation linking u_k to the outputs at that stage will be of the form

$$Y_{k,k+L-1} = \tilde{D} u_k \tag{14}$$

Thus the matrix replacing $M(z)$ is now simply $\tilde{D}$. Since $\tilde{D}$ is of full column rank, u_k will be computable from $Y_{k,k+L-1}$. □

Theorem 2 can be improved upon, by making an additional assumption on the underlying system that it has no infinite zero, and it is useful for us to do that. In this case, we must assume $p > m$ rather than $p \ge m$, since if $p = m$ is permitted, the system necessarily has zeros.

Theorem 3 Consider the system (1) with input, state and output dimensions m, n and p, with $p > m$, and with $[B^T \;\; D^T]^T$ of full column rank. Suppose that rank $M(z) = n + m$ for all $z \in C \cup \infty$. Then for some integer L with $L \le n$ and arbitrary k, the state at time j is computable for $k \le j \le k + L$ and the input at time j is computable for $k \le j \le k + L - 1$, from $y_k, y_{k+1}, \ldots, y_{k+L-1}$

Proof: By Theorem 2, there exists $L \le n$ such that knowledge of $y_k, y_{k+1}, \ldots, y_{k+L-1}$ allows computation of x_k and u_k. Since $x_{k+1} = Ax_k + Bu_k$, we can immediately compute x_{k+1}. Then because $Du_{k+1} = y_{k+1} - Cx_{k+1}$ and because D has full column rank, this following from the assumption of no infinite zero, we can compute u_{k+1}. The argument clearly repeats, for $k + 2, k + 3, \ldots k + L - 1$. □

There is an important extension of these two theorems to deal with systems which have a zero at zero, and possibly at infinity, but nowhere else. It will find application subsequently. To the extent that finding the state and input of a system from the output is a matter of passing that output through an inverse system, we might imagine that if the original system had a zero at the origin but nowhere else, there could be a finite-time transient associated with the zero-input response of the inverse system. This is because the inverse system would have a pole at the origin and nowhere else. Then the output of the inverse system could only be guaranteed to equal the input of the original system after this finite time transient, (unless the initial state of the inverse system just happened to have been correctly set, which is hardly likely); moreover, there would be a lag in the sense that the value of the input of the original system at a particular time would not become known at the output of the inverse system until some time instants later. This is indeed what we find.

Theorem 4 Consider the system (1) with input, state and output dimensions m, n and p, with $p \ge m$, and with $[B^T \;\; D^T]^T$ of full column rank. Suppose that rank $M(z) = n + m$ for all finite $z \in C$ except at $z = 0$. Then there exists a positive integer $J \le n$ such that for some integer L with $J \le L \le n + Jm$ and arbitrary k, the state at time $k + J$ is computable from $y_k, y_{k+1}, \ldots, y_{k+L-1}$. Moreover, if rank $M(z) = n + m$ for $z = \infty$, the state at time j is computable for $k + J \le j \le k + L$, and the input at time j is computable for $k + J \le j \le k + L - 1$ from the same data.

Proof: It is clear that there exists a positive integer J such that $\bar{W}(z) := z^{-J} W(z)$ is proper and has no zero at $z = 0$; it also has no other finite zero, since W has this property. The McMillan degree of $\bar{W}$ will be overbounded by the McMillan degrees of the transfer functions of which it is a product, and regarding it as $W(z) z^{-J} I_m$, the bound is $n + Jm$. Notice that if $\{x_i, u_i, y_i\}, i = \ldots, k, k+1, k+2, \ldots$ is a state, input, output trajectory triple for the system with transfer function $W(z)$, then $\{\bar{x}_i = x_{i+J}, \bar{u}_i = u_{i+J}, y_i\}, i = \ldots, k, k+1, k+2, \ldots$ is a triple for $\bar{W}(z)$ and conversely. By Theorem 2 applied to the $\bar{W}(z)$ system, there exists $L \le n + Jm$ such that $\bar{x}_k = x_{k+J}, \bar{u}_k = u_{k+J}$ is uniquely computable from $y_k, y_{k+1}, \ldots, y_{k+L-1}$ for all k. The argument used for proving Theorem 3 establishes the remaining claim of this theorem. □

In particular cases, one could imagine that smaller J and a smaller upper bound on L might be possible, but we will not explore this.

4. Alternative Approaches, Impulse Response and Matrix Fraction Condition

In this section, we offer an alternative proof of the main result, Theorem 2, of the previous section, and in the process offer a number of very different insights. The proof draws on the concept of an *output-nulling invariant subspace*, which was introduced in [4]. This will help us give an interpretation of the main result in terms of the solvability properties of a certain linear equation in which appear the constituent matrices of the state variable realization, and then we shall formulate a result involving just the impulse response coefficients. We shall also give an interpretation of the result using the concept of polynomial matrix fraction description of a linear finite-dimensional time-invariant system and the Smith-McMillan form.

4.1 Output-nulling Subspaces

Output-nulling subspaces are defined in relation to the system of equation (1).

Definition 4 A subset $\mathcal{V}$ of the state space is an output-nulling invariant subspace if for arbitrary $v_0 \in \mathcal{V}$, there exists some $u_0 \in \mathcal{U}$ (with $\mathcal{U}$ the input space) such that

$$v_1 := Av_0 + Bu_0 \in \mathcal{V} \quad (15)$$

$$Cv_0 + Du_0 = 0 \quad (16)$$

The intuition is that if one begins in state v_0, one can find an input u_0 such that the output y_0 is zero, and the next state, viz. v_1, has the same property as v_0; i.e., one can find an input u_1 so that the output y_1 is zero, and the next state v_2 again has the same property; equivalently, if one begins in state v_0, one can find an infinite sequence of inputs $u_0, u_1, \dots$ resulting in $y_k = 0$ for all $k \geq 0$.

We record without proof two properties from [4]. In the relevant equations, the notation $K^{-1}\mathcal{M}$ for a matrix K and a subspace $\mathcal{M}$ denotes an inverse image, i.e. the subspace $\{x|Kx \in \mathcal{M}\}$.

Proposition 2 There is a largest output-nulling invariant subspace, call it $\mathcal{V}_{max}$, containing all other output-nulling invariant subspaces, and it can be defined by the following algorithm

$$\mathcal{V}_0 = \mathcal{R}^n$$
$$\mathcal{V}_i = \mathcal{V}_{i-1} \cap \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} \mathcal{V}_{i-1} + \begin{bmatrix} B \\ D \end{bmatrix} \mathcal{U} \right\} \quad (17)$$
$$\mathcal{V}_{max} = \mathcal{V}_K$$

where K is the least integer, bounded by n, for which $\mathcal{V}_K = \mathcal{V}_{K+1}$. Further, $\mathcal{V}_K = \mathcal{V}_{K+j}$ for all positive j. In the event that there is no non-trivial output-nulling invariant subspace, the algorithm delivers $\mathcal{V}_{max} = \{0\}$. In the event that $\mathcal{V}_{max}$ is nontrivial, $K < n$.

Proposition 3 A subspace $\mathcal{V}$ is an output-nulling invariant subspace for the system (1) if and only if there exists an F for which there holds

$$(A + BF)\mathcal{V} \subset \mathcal{V} \quad (18)$$

$$(C + DF)\mathcal{V} = 0 \quad (19)$$

4.2 The No-zero Property as a Unique Solvability Property

Let us now interpret the construction of the first of these propositions in the light of the basic system equation (1). In addition to the definitions of the proposition, make the additional definition

$$\mathcal{W}_i = \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} \mathcal{V}_i + \begin{bmatrix} B \\ D \end{bmatrix} \mathcal{U} \right\} \quad (20)$$

Then it is easy to verify that :

$$\begin{aligned}
&\mathcal{W}_0 = \{x_{k+L-1} | \exists u_{k+L-1} \text{ such that } Cx_{k+L-1} + Du_{k+L-1} = 0\} \\
&\mathcal{V}_1 = \mathcal{W}_0 \\
&\mathcal{W}_1 = \{x_{k+L-2} | \exists u_{k+L-2} \text{ such that } Cx_{k+L-2} + Du_{k+L-2} = 0 \\
&\text{and } x_{k+L-1} \in \mathcal{W}_0\} \\
&= \{x_{k+L-2} | \exists u_{k+L-2}, u_{k+L-1} \text{ such that} \\
&Cx_{k+L-2} + Du_{k+L-2} = 0 \text{ and } Cx_{k+L-1} + D_{k+L-1} = 0\} \\
&\mathcal{V}_2 = \mathcal{W}_1 \\
&\vdots \\
&\mathcal{V}_K = \{x_{k+L-K} | \exists u_{k+L-K}, u_{k+L-K+1}, \dots, u_{k+L-1} \text{ such that} \\
&y_{k+L-K} = y_{k+L-K+1} = \cdots = y_{k+L-1} = 0\}
\end{aligned}$$

Hence by identifying K with L and recalling that $\mathcal{V}_{max} = \mathcal{V}_K = \mathcal{V}_{K+j}$ for all j and in particular j such that $K + j = n - 1$, we see that $\mathcal{V}_{max}$ will be nontrivial just when there exists a nonzero state x_k and a choice of inputs $u_k, u_{k+1}, u_{k+2}, \dots, u_{k+n-1}$ such that $y_k = y_{k+1} = y_{k+2} \dots, y_{k+n-1} = 0$.

We shall work with the following equation, which is an immediate consequence of (1):

$$\begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \\ \vdots \\ y_{k+L-1} \end{bmatrix} =
\begin{bmatrix} C & D & 0 & 0 & 0 & \cdots & 0 \\ CA & CB & D & 0 & 0 & \cdots & 0 \\ CA^2 & CAB & CB & D & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ CA^{L-1} & CA^{L-2}B & CA^{L-3}B & CA^{L-4}B & CA^{L-5}B & \cdots & D \end{bmatrix}
\times \begin{bmatrix} x_k \\ u_k \\ u_{k+1} \\ \vdots \\ u_{k+L-1} \end{bmatrix} \quad (21)$$

The key result is the following:

Theorem 5 Consider the system (1), with input, state and output dimensions m, n and p, with $p \geq m$, and with $[B^T \; D^T]^T$ of full column rank, together with the associated matrix $M(z)$ of (2), and the equation (21). Denote by N_L the matrix appearing on the right side of this equation. Then the following conditions are equivalent.

1. $M(z)$ is singular for some $z \in C$

2. For all $L \geq 1$, the matrix N_L has a vector in its kernel for which $[x_k^T \ \ u_k^T]^T$ is nonzero, and so this vector is not uniquely computable from the L measurements $y_k, y_{k+1}, \ldots, y_{k+L-1}$

3. For all $L \geq 1$, the matrix N_L has a vector in its kernel for which x_k is nonzero.

4. For the system (1), $\mathcal{V}_{max}$ is nontrivial.

Proof: We prove first that condition 1 implies condition 2. Suppose then that $M(z)$ is singular for some z. Then for some complex λ, there exists $x_0 \neq 0$ and u_0, such that (4) holds. Now set $x_k = \lambda^k x_0$ and $u_{k+j} = \lambda^{k+j} u_0$ in equation (21). It is straightforward to see that the right hand side is zero for all $L \geq 1$, i.e. that N_L has a vector in its kernel with $[x_k^T \ u_k^T]$ nonzero.

Now suppose that condition 2 holds; we shall demonstrate condition 3. If $x_k \neq 0$ there is nothing to prove. Suppose then that $x_k = 0$. The first row of (21) then yields $Du_k = 0$, and the full column rank condition on $[B^T \ D^T]^T$ ensures that $x_{k+1} = Bu_k \neq 0$. Further, we now have for all $L > 1$

$$\begin{bmatrix} y_{k+1} \\ y_{k+2} \\ y_{k+3} \\ \vdots \\ y_{k+L-1} \end{bmatrix} = \begin{bmatrix} C & D & 0 & 0 & 0 & \cdots & 0 \\ CA & CB & D & 0 & 0 & \cdots & 0 \\ CA^2 & CAB & CB & D & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ CA^{L-2} & CA^{L-3}B & CA^{L-4}B & CA^{L-5}B & CA^{L-6}B & \cdots & D \end{bmatrix} \times \begin{bmatrix} x_{k+1} \\ u_{k+1} \\ u_{k+2} \\ \vdots \\ u_{k+L-1} \end{bmatrix} \quad (22)$$

The matrix on the right is evidently N_{L-1}, and it has a vector in its kernel with the first n entries guaranteed to not be all zero. This establishes condition 3. On the other hand, condition 2 is an immediate consequence of condition 3. So conditions 2 and 3 are equivalent.

Now we establish that condition 3 is equivalent to condition 4. Condition 3 is equivalent to the property that there exists a nonzero x_k and a choice of inputs $u_k, u_{k+1}, u_{k+2}, \ldots$ such that $y_k = y_{k+1} = y_{k+2} \cdots = 0$. In the discussion of the algorithm for computing $\mathcal{V}_{max}$ we observed that this was equivalent to the existence of a nontrivial $\mathcal{V}_{max}$.

Since we have shown that condition 1 implies condition 2, and that conditions 2, 3 and 4 are equivalent, it remains to show that condition 4 implies condition 1.

Choose a matrix F existing by the second proposition, where $\mathcal{V}$ is identified with $\mathcal{V}_{max}$. Then there necessarily exists a nonzero $v \in \mathcal{V}$ and a constant λ such that $(A + BF)v = \lambda v$ and $(C + DF)v = 0$. [In more detail, think of $\mathcal{R}^n$ as the direct sum of $\mathcal{V}_{max}$ and the quotient subspace $\mathcal{R}^n/\mathcal{V}_{max}$, and choose the coordinate basis so that $\mathcal{R}^n/\mathcal{V}_{max} = \{[0 \ w^T]^T\}$. It is trivial to conclude that in this coordinate basis, $C + DF = [0 \ X]$ for some matrix X, and $A + BF$ is block upper triangular. The $1 - 1$ block necessarily has an eigenvalue, which yields λ and an eigenvector, which with zero entries appended yields v]. It can then immediately be seen that the vector $[v^T \ \ Fv^T]^T$ is in the kernel of $M(\lambda)$.

□

The key consequence we draw from this Theorem involves the negation of some of the conditions. The observability index ν_o, equal to the maximum Kronecker index of an observable pair (C, A), is the least integer for which $[C^T \ (CA)^T \ \ldots (CA^{\nu_o - 1})^T]$ has rank n.

Corollary 1 Adopt the same hypotheses as for Theorem 5. Then the following conditions are equivalent.

1. $W(z)$ has no zero in C.

2. There exists $L \leq n$ and necessarily equal to or greater than the observability index ν_o of (C, A) such that for all k (21) can be uniquely solved for $[x_k^T \ \ u_k^T]^T$, or equivalently, x_k and u_k are computable from the L measurements $y_k, y_{k+1}, \ldots, y_{k+L-1}$.

3. There exists $L \leq n$ and necessarily equal to or greater than the observability index ν_o of (C, A)such that for all k, (21) can be uniquely solved for x_k, or equivalently, x_k is computable from the L measurements $y_k, y_{k+1}, \ldots, y_{k+L-1}$.

Moreover, if $p > m$ the following additional equivalences hold:

1. $W(z)$ has no zero in $C \cup \infty$.

2. There exists $L \leq n$ and necessarily equal to or greater than the observability index of (C, A) such that $x_k, x_{k+1}, \ldots, x_{k+L}$ and $u_k, u_{k+1}, \ldots, u_{k+L-1}$ are computable from the L measurements $y_k, y_{k+1}, \ldots, y_{k+L-1}$.

3. There exists $L \leq n$ and necessarily equal to or greater than the observability index of (C, A) such that the matrix N_L has full column rank.

Proof: By Theorem 5, condition 1 is equivalent to the requirement that $\mathcal{V}_{max}$ is trivial, i.e. $\mathcal{V}_{max} = \{0\}$. Since $\mathcal{V}_{max} = \mathcal{V}_K$ for some $K \leq n$, $\mathcal{V}_K$ is trivial, i.e. there exists no nonzero x_k and input sequence $u_k, u_{k+1}, \ldots, u_{k+K-1}$ such that $y_k, y_{k+1}, \ldots, y_{k+K-1} = 0$. Equivalently, for $L = K$ with $L \leq n$, (21) can be uniquely solved for $[x_k^T \ u_k^T]^T$, i.e. condition 2 holds apart from the lower bound on L. However given the definition of the observability index, it is trivial that if $L < \nu$, then there will exist a nonzero x_k and zero $u_k, u_{k+1}, \ldots, u_{k+L-1}$ in the kernel of N_L. Conditions 2 and 3 are equivalent as negations of the corresponding equivalent conditions in Theorem 5.

For the second part, assume condition 1 holds. As argued in the previous section, $W(z)$ has no zero at ∞ if and only if D has full column rank. From x_k and u_k, x_{k+1} is immediate, from $Du_{k+1} = y_{k+1} - Cx_{k+1}$ the value of u_{k+1} follows, and so on. The argument is trivially reversible to give condition 1 as a consequence of condition 2. Condition 3 is a restatement of condition 2. □

There are extensions of the results of this subsection to cover the case when $W(z)$ has a zero at $z = 0$ and nowhere else. The extensions are obtainable by using exactly the same device as was used at the end of the previous section in establishing Theorem 4. For such an $M(z)$ associated with a transfer function $W(z)$, there necesssarily exists a positive integer J such that $z^{-J}W(z)$, call it $\bar{W}(z)$, has no finite zero, though it may have an infinite zero even if $W(z)$ has no infinite zero. One works with $\bar{M}(z)$ formed in the obvious way, and recognizes that input, state and output sequences for $\bar{W}(z)$, call them $\{\bar{x}_i, \bar{u}_i, y_i\}$ for $i = \ldots, k, k+1, \ldots$ are related to the corresponding sequences for $W(z)$, i.e. $\{x_i, u_i, y_i\}$ for $i = \ldots, k, k+1, \ldots$ by $\bar{x}_i = x_{i+J}, \bar{u}_i = u_{i+J}$. The results of this section can be applied to $\bar{M}(z)$ and $\bar{W}(z)$ and then the time translation of input and state applied to recover results applicable to $M(z)$ and $W(z)$. The key conclusion paralleling Corollary 1 is that if $M(z)$ has no zero in C except at $z = 0$, then there exists a $J > 0$ and an $\bar{L} \leq n + mJ$ such that x_{k+J} and u_{k+J} are uniquely computable from the measurements $y_k, y_{k+1}, \ldots, y_{k+\bar{L}-1}$. Moreover, if additionally $W(z)$ has no zero at infinity, one can compute $x_{k+J+1}, \ldots, x_{K+\bar{L}}$ and $u_{k+J+1}, \ldots, u_{k+\bar{L}-1}$.

4.3 Characterization of the No-zero Property using Impulse Response Data

A slightly unattractive feature of the matrix N_L introduced in the previous subsection is that it requires a state-variable realization of the underlying linear system, and it mixes state-variable realization dependent quantities with impulse response quantities. We shall now give a condition equivalent to those of Corollary 1 that simply involves the impulse response data alone. The corresponding matrix equation also involves output and input sequences but not the state variable.

Corollary 2 Adopt the same hypotheses as for Theorem 5. Let $w_0, w_1, \ldots$ denote the impulse response sequence $D, CB, CAB, \ldots$. Then the first three equivalent conditions of Corollary 1 are equivalent to the following condition: for any $k \geq n$ and some $L \leq n$, the following equation is uniquely solvable for $u_k, u_{k+1}, \ldots, u_{k+L-1}$:

$$\begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \\ \vdots \\ y_{k+L-1} \end{bmatrix} = \begin{bmatrix} w_k & \cdots & w_1 & w_0 & 0 & 0 & \cdots & 0 \\ w_{k+1} & \cdots & w_2 & w_1 & w_0 & 0 & \cdots & 0 \\ w_{k+2} & \cdots & w_3 & w_2 & w_1 & w_0 & \cdots & 0 \\ \vdots & & & & & \vdots & & \\ w_{k+L-1} & \cdots & w_L & w_{L-1} & w_{L-2} & w_{L-3} & \cdots & w_0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_k \\ u_{k+1} \\ \vdots \\ u_{k+L-1} \end{bmatrix} \tag{23}$$

Proof: Assume the conditions of Corollary 1 hold. Fix arbitrary $k \geq n$. Suppose that the condition of this corollary does not hold. Then there will be two input sequences from time 0 to $k + L - 1$ which are guaranteed to be distinct over the time interval $[k, k + L - 1]$ which give rise to the same outputs $y_k, y_{k+1}, \ldots, y_{k+L-1}$. Let the difference of these two sequences be $\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_{k+L-1}$. Let $\tilde{x}_k$ be the state which results when $x_0 = 0$ and the sequence $\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_{k-1}$ is applied. Then it is trivial to see that $\tilde{x}_k$ and the $\tilde{u}_j$ for $k \leq j \leq k + L - 1$ when stacked together define a (nonzero) vector in the kernel of N_L, violating the conditions of Corollary 1. Hence the condition of this corollary holds.

Conversely, suppose the condition of this corollary holds but the conditions of Corollary 1 do not hold. Then N_L has a nontrivial kernel, and we may identify a nonzero vector in the kernel by its components $\tilde{x}_k$ and $\tilde{u}_j$ for $k \leq j \leq k + L - 1$. Let $\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_{k-1}$ be any sequence generating $\tilde{x}_k$ when $x_0 = 0$; such a sequence exists because $k \geq n$. Because $\tilde{x}_k$ and $\tilde{u}_j$ for $k \leq j \leq k + L - 1$ are not all zero, not all $\tilde{u}_j$ for $0 \leq j \leq k + L - 1$ are zero. Then it is trivial that the nonzero sequence $\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_{k+L-1}$ produces $y_k = y_{k+1} = \cdots = y_{k+L-1} = 0$; again we have a contradiction to the condition of this corollary. □

Note that there is a straightforward algorithm for verifying the condition of Corollary 2. One must have an upper bound $\bar{n}$ on the McMillan degree n of the underlying transfer function matrix (and then one may choose $k = \bar{n}$), and one requires at most $2n$ impulse response parameters (assuming $L = n$), though fewer might suffice and one can always use more. In fact, one can choose $k = \nu_c$, the controllability index, as is revealed by the proof of the corollary. Note that both the controllability index and the observablity index are determinable from impulse response data (knowing a bound on the McMillan degree), by studying row and column dependencies in the associated finite Hankel matrix.

Partition the matrix on the right side of (23) into two matrices W_1, W_2 with k block columns and L block columns respectively, and for convenience partition the vector of u_i conformally into U_1 and U_2. There results

$$\begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \\ \vdots \\ y_{k+L-1} \end{bmatrix} = [W_1 \ W_2] \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \tag{24}$$

and it is U_2 that needs to be uniquely determinable. Operate on the rows of this equation through premultiplication by a nonsingular square matrix, S say, to convert W_1 to row echelon form (also known as Hermite normal form,[7]. (Thus all zero rows are at the bottom of the matrix, the first nonzero entry of each nonzero row after the first occurs to the right of the first nonzero entry of the previous row, the first nonzero entry in any nonzero row is 1, and all entries in the column above and below such a first nonzero entry are zero). Because W_1 is a truncated Hankel matrix, it is known to have rank at most n, and indeed the rank equals n for $L \geq \nu_0$ and $k \geq \nu_c$. The equation will then take the following form:

$$S \begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \\ \vdots \\ y_{k+L-1} \end{bmatrix} = \begin{bmatrix} \tilde{W}_{11} & \tilde{W}_{12} \\ 0 & \tilde{W}_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \tag{25}$$

Then the condition of the corollary is fulfilled if and only if $\tilde{W}_{22}$ has full column rank.

To understand the adjustment that needs to be made when there is a zero at zero, observe that if $\bar{W}(z) = z^{-J}W(z)$ then with obvious definition of symbols, $\bar{w}_j = 0$ for $j < J$ and otherwise $\bar{w}_{j+J} = w_j$. Suppose then that $W(z)$ has no zero, except at zero and possibly at infinity. Choose J so that $\bar{W}(z)$ has no finite zero. The McMillan degree of $\bar{W}(z)$ is bounded by $n+mJ$. Notice that if $\{x_i, u_i, y_i\}, i = \ldots, k, k+1, k+2, \ldots$ is a state, input, output trajectory triple for the system with transfer function $W(z)$, then $\{\bar{x}_i = x_{i+J}, \bar{u}_i = u_{i+J}, y_i\}, i = \ldots, k, k+1, k+2, \ldots$ is a triple for $\bar{W}(z)$ and conversely. Observing that the result of Corollary 2 applied to $\bar{W}(z)$ then yields that for some $\bar{L} \le n+mJ$ and any $n \ge n+mJ$, the following equation is uniquely solvable for $u_{k+J}, u_{k+J+1}, \ldots, u_{k+J+\bar{L}-1}$:

$$\begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \\ \vdots \\ y_{k+\bar{L}-1} \end{bmatrix} = \begin{bmatrix} w_{k-J} & \cdots & w_1 & w_0 & 0 & 0 & 0 & \cdots & 0 \\ w_{k+1-J} & \cdots & w_2 & w_1 & w_0 & 0 & 0 & \cdots & 0 \\ w_{k+2-J} & \cdots & w_3 & w_2 & w_1 & w_0 & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots & & \\ w_{k+\bar{L}-1-J} & \cdots & w_{\bar{L}} & w_{\bar{L}-1} & w_{\bar{L}-2} & w_{\bar{L}-3} & w_{\bar{L}-4} & \cdots & w_0 \end{bmatrix} \times \begin{bmatrix} u_J \\ u_{1+J} \\ \vdots \\ u_{k+J} \\ u_{k+1+J} \\ \vdots \\ u_{k+J+L-1} \end{bmatrix} \quad (26)$$

To sum up, there is a modification of the impulse response condition which, if fulfilled, ensures that from a finite interval of output data of suitable length, a finite interval of input data can be obtained, save that, because of the occurrence of the zero, it takes J times units following the first time index of the output data to arrive at the first time index of the computed input data. This is hardly surprising, given that having a zero is rather like having a delay. If the transfer function is $z/(z+1/2)$, and the associated equation relating input and output is $y_k + (1/2)y_{k-1} = u_k$, it is evident that knowing y_k, y_{k-1} we can compute u_k but not u_{k-1}. There is a simple modification of the algorithm based on constructing an echelon form that can of course be used.

4.4 Matrix Fraction Characterization

A further tool beyond those already considered for describing a linear, finite-dimensional time-invariant system is a matrix fraction. Here, we shall restrict attention to a certain sort of matrix fraction, viz., a left matrix fraction defined using a variable z^{-1} rather than z. Such descriptions arise naturally when we consider a system equation written in the form

$$\begin{aligned} y_k + A_1 y_{k-1} + A_2 y_{k-2} + \cdots + A_\alpha y_{k-\alpha} \\ = B_0 u_k + B_1 u_{k-1} + B_2 u_{k-2} + \cdots + B_\beta u_{k-\beta} \end{aligned} \quad (27)$$

where α, β are nonnegative integers. The associated transfer function matrix is

$$W(z) = A^{-1}(z^{-1})B(z^{-1}) \quad (28)$$

with obvious definitions of the polynomials $A(z^{-1})$ and $B(z^{-1})$, which are assumed to be coprime as polynomials in z^{-1} and with $A(0) = I$. When the input dimension is no greater than the output dimension, having no zero at infinity or finite nonzero value in the z description or equivalently no zero at zero or finite nonzero value in the z^{-1} description corresponds to the matrix

$$B(z^{-1}) = B_0 + B_1 z^{-1} + B_2 z^{-2} + \cdots + B_\beta z^{-\beta} \quad (29)$$

having full column rank for all finite z^{-1}. [In particular, taking $z^{-1} = 0$, we have that B_0 has full column rank.] The full column rank property may occur if $B(z^{-1})$ is square and unimodular, or if there are more rows than columns, and the B_i coefficients are generic.

Notice that there is no mention as to whether z^{-1} has a zero at infinity. Equivalently, no assumption is made about the z description having a zero at zero. We expect to be able to recover a finite interval of inputs from a finite interval of outputs, but with a delay in the time index of the first input relative to the time index of the first output. We shall now see this directly, working with the matrix fraction description..

Proposition 4 Consider the linear finite-dimensional system defined by (27), where the dimension of the output vector y_k is greater than or equal to the dimension of the input vector u_k. Suppose further the polynomials $A(z^{-1})$ and $B(z^{-1})$ are left coprime. Then $B(z^{-1})$ has full rank for all finite values of its argument if and only if there exists a polynomial matrix $D(z^{-1})$ such that (27) implies, for any input sequence that

$$u_k = D(z^{-1})y_k \quad (30)$$

Proof: First, assume the full rank property. It is a trivial consequence of the Smith normal form decompostion [7] that a polynomial matrix with full rank at all finite values of its argument and with equal or more rows than columns has a left multiplier , also a polynomial matrix, which is an inverse, i.e. there exists a matrix $C(z^{-1})$ polynomial in z^{-1} such that $C(z^{-1})B(z^{-1}) = I$. Then observe that

$$C(z^{-1})A(z^{-1})y_k = C(z^{-1})B(z^{-1})u_k = u_k \quad (31)$$

and we see that u_k can be obtained as a linear combination of y_k and a finite number of earlier values, with the weights given by the coefficients of the polynomial $D(z^{-1}) = C(z^{-1})A(z^{-1})$. [It is however less straightforward to keep track of the degrees of the various polynomial matrices and how they relate to the McMillan degree of the linear system.]. Conversely, suppose that for some polynomial matrix $D(z^{-1})$, there holds $u_k = D(z^{-1})y_k$. Because of the assumption that $A(z^{-1}), B(z^{-1})$ are coprime, there exist polynomial matrices $E(z^{-1}), F(z^{-1})$ with $I = AE+BF$. Notice also that because $u = Dy = DA^{-1}Bu$, one has $DA^{-1}B = I$ and then

$$\begin{aligned} I &= DA^{-1}B = DA^{-1}[AE + BF]B \\ &= [DE + DA^{-1}BF]B = (DE + F)B \end{aligned} \quad (32)$$

Thus B has a polynomial left inverse $C = DE + F$, and consequently has full rank for all finite values of its argument. □

If the degree in z^{-1} of the polynomial matrix $D(z^{-1})$ is γ say, then an interval of values $y_k, y_{k+1}, \ldots, y_{k+\gamma}$ suffices to yield $u_{k+\gamma}$.

One can also say that the inverse system producing the u_k sequence from the y_k sequence has poles at the origin of the z-plane, and accordingly, for an arbitrary initial condition in this system (including an initial condition of zero), one must expect a transient depending on the initial condition to be added to the output that would otherwise be obtained; the transient will be of finite-time duration, since it is associated with a pole at the origin. After γ time instants in fact, it will be gone. The situation is analogous to that described prior to Theorem 4.

4.5 Smith-McMillan Form Characterization

Using the concept of the Smith-McMillan form, we can understand the inversion results associated both with systems which have no zeros in the finite z-plane (and may or may not have a zero at infinity), and systems which have no zeros, apart from one at zero, in $C \cup \infty$. If a transfer function has no finite zeros, and has more rows than columns, then the decomposition of (3) will have

$$V^T(z) = [\text{diag } d_i^{-1}(z) \;\; 0] \tag{33}$$

where each $d_i(z)$ is a monic polynomial. Accordingly, in mixed-time domain and z-transform notation, there will hold

$$U_2^{-1}(z)[\text{diag } d_i(z) \;\; E(z)]U_1^{-1}(z)y_k = u_k \tag{34}$$

where $E(z)$ is an arbitrary polynomial matrix. Let us take it as the zero matrix. The matrix product on the left side is polynomial in z; assume the highest power of z occurring is δ. This shows that u_k is computable as a linear expression involving $y_k, y_{k+1}, \ldots, y_{k+\delta}$. Whether or not we know that $W(z)$ has no zero at infinity is not of use here; the Smith-McMillan form completely hides that information.

If a transfer function has no finite zero apart from zero, and no zero at infinity, we may switch from z to z^{-1} and perform a Smith-McMillan decomposition with the variable z^{-1}. Similar calculations to those above using this form will express u_k in terms of $y_k, y_{k-1}, \ldots, y_{k-\gamma}$ for some finite γ.

5. Conclusions

In this paper, we have analysed the consequences for a finite-dimensional linear system to have more outputs than inputs. A number of interesting properties flow from the fact that generically, such systems have no finite zeros. The key properties revolve round the invertibility of the system, and in particular, the ability to *exactly* recover an input value (or sequence of values) given only a finite interval of output values (or sliding interval of such values). Different characterizations of the property are possible, using state-variable, matrix fraction, Smith-McMillan form and impulse response data.

In a companion paper, we shall explore the ideas where the systems in question are excited by white noise, and properties are examined using output spectral data.

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