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THE MULTI-AGENT RENDEZVOUS PROBLEM. PART 1: THE SYNCHRONOUS CASE*

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Abstract. This paper is concerned with the collective behavior of a group of $n > 1$ mobile autonomous agents, labelled 1 through n, which can all move in the plane. Each agent is able to continuously track the positions of all other agents currently within its "sensing region," where by an agent's *sensing region* we mean a closed disk of positive radius r centered at the agent's current position. The *multi-agent rendezvous problem* is to devise "local" control strategies, one for each agent, which without any active communication between agents cause all members of the group to eventually rendezvous at a single unspecified location. This paper describes a solution to this problem consisting of individual agent strategies which are mutually synchronized in the sense that all depend on a common clock.

Key words. cooperative control, distributed control, multi-agent systems

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1. Introduction.

Current interest in cooperative control has led to the development of a number of distributed control algorithms capable of causing large groups of mobile autonomous agents to perform useful tasks [11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Of particular interest here are provably correct algorithms which solve what we shall refer to as the "multi-agent rendezvous problem." This problem, which was considered previously in [19, 1], is concerned with the collective behavior of a group of $n > 1$ mobile autonomous agents, labelled 1 through n, which can all move in the plane. Each agent is able to continuously track the positions of all other agents currently within its "sensing region," where by an agent's *sensing region* we mean a closed disk of positive radius r centered at the agent's current position. The *multi-agent rendezvous problem* is to devise "local" control strategies, one for each agent, which without any active communication between agents cause all members of the group to eventually rendezvous at a single unspecified location.

In this paper, as in [1], we consider distributed strategies which guide each agent toward rendezvous by performing a sequence of "stop-and-go" maneuvers. A *stop-and-go maneuver* takes place within a time interval consisting of two consecutive subintervals. The first, called a *sensing period*, is an interval of fixed length during which the agent is stationary. The second, called a *maneuvering period*, is an interval of variable length during which the agent moves from its current position to its next

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"way-point" and again comes to rest. Successive way-points for each agent are chosen to be within r_M units of each other, where r_M is a prespecified positive distance no larger than r. It is assumed that there has been chosen for each agent i a positive number τ_{M_i}, called a *maneuver time*, which is large enough so that the required maneuver for agent i from any one way-point to the next can be accomplished in at most τ_{M_i} seconds. Since our interest here is exclusively with devising *high level* strategies which dictate when and where agents are to move, we will use point models for agents and shall not deal with how maneuvers are actually carried out or with how vehicle collisions are to be avoided.

In this paper we describe a family of stop-and-go strategies which solves the problem. The family includes the specific strategies proposed in [1] and consists of agent strategies which are mutually synchronized in the sense that all depend on a common clock. In a sequel to this paper [12] we propose and analyze families of strategies which also solve the problem, but without the need for synchronization.

In the synchronous case treated here, the kth maneuvering periods of all n agents begin at the same time $\bar{t}_k$. The kth way-point of each agent is a function of the positions of its "registered neighbors" at time $\bar{t}_k$. Agent i's registered neighbors at time $\bar{t}_k$ are all those other agents positioned within its sensing region at time $\bar{t}_k$. This notion of a neighbor induces a *symmetric* relation on the agent group since agent j is a registered neighbor of agent i at time $\bar{t}_k$ just in case agent i is a registered neighbor of agent j at the same time. Because of this it is possible to characterize neighbor relationships at time $\bar{t}_k$ with a simple graph whose vertices represent agents and whose edges represent existing neighbor relationships (see section 2.2). Although the neighbor relation is symmetric, it is clearly not transitive. On the other hand, if agent i is at the same position as neighbor j at time $\bar{t}_k$, then any registered neighbor of agent j at time $\bar{t}_k$ must certainly be a registered neighbor of agent i at the same time. It is precisely because of this *weak transitivity* property that one can infer a *global* condition of the entire agent group from a *local* condition of one agent and its neighbors. In particular, if the graph characterizing neighbor relationships at time $\bar{t}_k$ is connected, and any one agent is at the same position as all of its neighbors, then the weak transitivity property guarantees at once that all n agents have rendezvoused at time $\bar{t}_k$.

One way to ensure that a neighbor graph is connected at time $\bar{t}_k$, assuming it is connected when the rendezvousing process begins, is to constrain each agent's way-points to be positioned in such a way so that no agent can lose any of its registered neighbors when it moves from one way-point to the next. This can be accomplished using a clever idea taken from [1]. An immediate consequence is that each agent's set of registered neighbors is nondecreasing and, because of this, ultimately converges to a fixed neighbor set for $\bar{t}_k$ sufficiently large.

A second local constraint is to require the way-point of each agent i at the beginning of its kth maneuvering period to lie in the "local" convex hull $\mathcal{H}_i(k)$ of agent i's own position at time $\bar{t}_k$ and the sensed positions of its registered neighbors at the same time. It is quite easy to prove that doing this causes the global convex hull $\mathcal{H}(k+1)$ of all n agent positions at time $\bar{t}_{k+1}$ to be contained in the corresponding global convex hull $\mathcal{H}(k)$ at time $\bar{t}_k$.

A third constraint is to stipulate that for each i, the only condition under which agent i's kth way-point can be positioned at a corner of $\mathcal{H}_i(k)$ is when $\mathcal{H}_i(k)$ is a single point. The global implication of doing this is that the diameter of $\mathcal{H}(k+1)$ must either be strictly smaller than the diameter of $\mathcal{H}(k)$ or every agent must be at the same position as all of its registered neighbors at time $\bar{t}_k$—and this is true whether or not the graph characterizing neighbor relationships at time $\bar{t}_k$ is connected.

In section 4, a more or less standard Lyapunov-based argument is used to prove that if the preceding constraints are adopted by all agents and if the graph characterizing initial neighbor positions is connected, then all n agents will eventually rendezvous at a single point. Not surprisingly, the Lyapunov function used for this purpose is the diameter of the global convex hull. However, although connectivity of the graph characterizing initial neighbor positions is sufficient for rendezvousing, it is not necessary. An example illustrating this is given in section 3.2. The example deals with the situation when the initial neighbor graph consists of two connected components, with one "encircling" the other in a suitably defined sense.

2. The synchronous agent system. In the synchronous case treated in this paper, the maneuvering times for all agents are all the same positive value τ_M. Along any trajectory of the system to be considered, the real time axis can be partitioned into a sequence of consecutive time intervals $[0, t_1), [t_1, t_2), \ldots [t_{k-1}, t_k), \ldots$, each of length at least τ_M. Each interval consists of a sensing period followed by a maneuvering period of fixed length τ_M. All agents function in synchronization in the sense that all are at rest during sensing periods and all can maneuver only during maneuvering periods. In particular, all agents actions are synchronized to the time sequence $\bar{t}_1, \bar{t}_2, \ldots \bar{t}_k \ldots$, where $\bar{t}_k$ denotes the real time $t_k - \tau_M$ at which the kth maneuvering period begins. Agent i's *registered neighbors* at the beginning of its kth maneuvering period, $[\bar{t}_k, t_k)$, are those agents, except for agent i, which are within agent i's sensing region at time $\bar{t}_k$. Note that this definition is a symmetric relation on the set of all agents; i.e., if agent i is a registered neighbor of agent j at the beginning of maneuvering period k, then agent j is a registered neighbor of agent i at the beginning of the same maneuvering period.

2.1. Pairwise motion constraint. A pair of agents which are registered neighbors at the beginning of maneuvering period k are said to satisfy the *pairwise motion constraint* during the period if the positions to which they move at time t_k are both within a closed disk of diameter r centered at the mean of their registered positions at time $\bar{t}_k$. The definition implies that any two agents which are registered neighbors at the beginning of maneuvering period k will be registered neighbors at the beginning of maneuvering period $k+1$ if they satisfy the pairwise motion constraint during the kth maneuvering period. We are interested in strategies possessing this property and accordingly make the following assumption.

Cooperation assumption. During each maneuvering period k, each pair of agents which are registered neighbors at the beginning of the period restrict their motions to satisfy the pairwise motion constraint.

Agent i's *kth way-point* is the point to which agent i is to move at time t_k. Thus if $x_i(t)$ denotes the position of agent i at time t represented in a world coordinate system, then $x_i(t_k)$ and agent i's kth way-point are one and the same. The rule which determines each such way-point is a function depending only on the number and relative positions of agent i's registered neighbors. In particular, if agent i has m_i registered neighbors at time $\bar{t}_k$, positioned relative to agent i at points

$$z_j \triangleq x_{i_j}(\bar{t}_k) - x_i(\bar{t}_k), \quad j \in \{1, 2, \ldots, m_i\}, \tag{1}$$

then agent i's kth way-point is

$$x_i(t_{k-1}) + u_{m_i}(z_1, z_2, \ldots, z_{m_i}), \tag{2}$$

where $u_0 = 0$, $u_m : \mathbb{D}^m \to \mathbb{D}_M$, $m \in \{1, \dots, n-1\}$, and $\mathbb{D}$ and $\mathbb{D}_M$ are the closed disks of radii r and r_M, respectively, centered at the origin in $\mathbb{R}^2$. In other words, if agent i has no registered neighbors at time $\bar{t}_k$ (i.e., $m_i = 0$), it does not move during the kth maneuvering period. On the other hand, if agent i has $m_i > 0$ neighbors at time $\bar{t}_k$ with relative positions $z_1, z_2, \dots, z_{m_i}$, then agent i moves to the position $x_i(t_{k-1}) + u_{m_i}(z_1, z_2, \dots, z_{m_i})$ at time t_k. Thus

$$x_i(t_k) = x_i(t_{k-1}) + u_{m_i(\bar{t}_k)}(x_{i_1}(\bar{t}_k) - x_i(\bar{t}_k),\ x_{i_2}(\bar{t}_k) - x_i(\bar{t}_k), \dots,\ x_{i_{m_i(\bar{t}_k)}}(\bar{t}_k) - x_i(\bar{t}_k)). \tag{3}$$

In what follows we will explain how the u_m are defined. At the very least we will require each to be a continuous function.

2.2. Definition of u_m. We have already defined $u_0 = 0$. To define u_m for $m > 0$ it is necessary to take into account the pairwise motion constraint. Toward this end, for each $z \in \mathbb{D}$, let $\mathcal{C}(z)$ denote the closed disk of diameter r centered at the point $\frac{1}{2}z$. More generally, for each $\{z_1, z_2, \dots, z_m\} \in \mathbb{D}^m$, let

$$\mathcal{C}(z_1, z_2, \dots, z_m) = \bigcap_{j=1}^{m} \mathcal{C}(z_j). \tag{4}$$

Note that 0 is in each $\mathcal{C}(z_i)$ and, moreover, that each such $\mathcal{C}(z_i)$ is closed and strictly convex. Consequently $\mathcal{C}(z_1, z_2, \dots, z_m)$ is either the singleton $\{0\}$ or a strictly convex, closed set containing 0. We can now define u_m to be any continuous function on $\mathbb{D}^m$ satisfying

$$u_m(z_1, z_2, \dots, z_m) \in \mathbb{D}_M \cap \mathcal{C}(z_1, z_2, \dots, z_m) \cap \langle 0, z_1, z_2, \dots, z_m \rangle \ \forall \{z_1, z_2, \dots, z_m\} \in \mathbb{D}^m, \tag{5}$$

where $\langle 0, z_1, z_2, \dots, z_m \rangle$ is the convex hull of the points $0, z_1, z_2, \dots, z_m$. The u_m are further required to have the property that

$$u_m(z_1, z_2, \dots, z_m) \neq \text{a corner}^1 \text{ of } \langle 0, z_1, z_2, \dots, z_m \rangle \tag{6}$$

unless $z_1 = z_2 = \cdots = z_m = 0$. In other words, u_m is required to be (i) a continuous function on $\mathbb{D}^m$ which maps each $\{z_1, z_2, \dots, z_m\} \in \mathbb{D}^m$ into $\mathbb{D}_M \cap \mathcal{C}(z_1, z_2, \dots, z_m) \cap \langle 0, z_1, z_2, \dots, z_m \rangle$ and (ii) a function with the property that $u_m(z_1, z_2, \dots, z_m)$ is not a corner of $\langle 0, z_1, z_2, \dots, z_m \rangle$ unless $z_1 = z_2 = \cdots = z_m = 0$. Examples of functions satisfying these conditions will be given in what follows.

2.3. Target points. One way to go about defining specific u_m which are continuous and which satisfy (5) and (6) is by first defining what we shall refer to as a "target point." By a *target point* we mean a continuous function $\tau : \mathbb{D}^m \to \langle 0, z_1, z_2, \dots, z_m \rangle$ defined in such a way that for each $\{z_1, z_2, \dots, z_m\} \in \mathbb{D}^m$ for which 0 is a corner of $\langle 0, z_1, z_2, \dots, z_m \rangle$, the segment of the line from 0 to $\tau(z_1, z_2, \dots, z_m)$ which lies within $\mathcal{C}(z_1, z_2, \dots, z_m)$ has positive length. For should it be possible to define such a τ, one could satisfy (5) and (6) as well as the continuity requirement with a control of the form

$$u_m = g(z_1, z_2, \dots, z_m)\tau(z_1, z_2, \dots, z_m),$$

[1]Recall that a point x in a polytope $\mathbb{P}$ in $\mathbb{R}^m$ is a *corner* if the only points y and z in $\mathbb{P}$ for which x is a convex combination are $y = z = x$.

where $g : \mathbb{D}^m \to \mathbb{R}$ is any continuous, positive definite function satisfying

$$g < \max_{[0,1]} \left\{ \mu : \mu\tau \in \mathbb{D}_M \bigcap \mathcal{C}(z_1, z_2, \ldots, z_m) \right\}.$$

Note that $g\tau \in \langle 0, z_1, z_2, \ldots, z_m \rangle$ for all $g \in [0, 1]$ because $0 \in \langle 0, z_1, z_2, \ldots, z_m \rangle$. The role of g is therefore to scale down the magnitude of τ enough to ensure that $g\tau$ is in the constraint set $\mathbb{D}_M \bigcap \mathcal{C}(z_1, z_2, \ldots, z_m)$.

It might be thought that one could choose for τ the centroid of $\langle 0, z_1, z_2, \ldots, z_m \rangle$ or perhaps the average of the z_i and 0, namely

$$\tau \triangleq \frac{1}{m+1} \sum_{i=1}^{m} z_i.$$

Both candidate definitions satisfy the requirement that $\tau(z_1, z_2, \ldots, z_m)$ must be a point in $\langle 0, z_1, z_2, \ldots, z_m \rangle$. Unfortunately, simple examples show that the centroid definition does not necessarily yield a function which satisfies the continuity requirement, while the averaging definition may lead to a function which fails to satisfy the requirement that when 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$, the segment of the line from 0 to $\tau(z_1, z_2, \ldots, z_m)$ which lies within $\mathcal{C}(z_1, z_2, \ldots, z_m)$ has positive length. For example, the centroid of the convex hull of the points $(0, 0)$, $z_1 = (0, 1)$, and $z_2 = (p, 1)$ is at $(\frac{p}{3}, \frac{2}{3})$ for $p > 0$ and at $(0, \frac{1}{2})$ for $p = 0$ so the centroid is discontinuous at $p = 0$. As a counterexample to the use of coordinate averaging to define a target point, note that the average of the four points located at $(0, 0)$, $z_1 = (-r, 0)$, $z_2 = (\frac{2r}{3}, \frac{r}{2})$, and $z_3 = (\frac{r}{3}, \frac{r}{2})$ is at $(0, \frac{r}{4})$, while the constraint set $\mathcal{C}(z_1, z_2, z_3)$ determined by these points must be contained in the constraint disk $\mathcal{C}(z_1)$. Since the line $\mathcal{L}$ from $(0, 0)$ to $(0, \frac{r}{4})$ is tangent to this disk at the origin, the intersection of $\mathcal{L}$ with $\mathcal{C}(z_1, z_2, z_3)$ is just the point $(0, 0)$ and consequently not a line segment of positive length.

In what follows we shall approach the problem of defining τ in a slightly different way. We begin by stating the following proposition which provides a simple condition on $\tau(\cdot)$, which, if satisfied, automatically implies satisfaction of the requirement that when 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$, the segment of the line from 0 to $\tau(z_1, z_2, \ldots, z_m)$ which lies within $\mathcal{C}(z_1, z_2, \ldots, z_m)$ has positive length.

PROPOSITION 1. *Let $z_1, z_2, \ldots, z_m$ be a set of $m > 0$ points in $\mathbb{D}$ which are not all 0. If 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$ and z is any nonzero point in $\mathbb{D}$ within r units of each point in $\{z_1, z_2, \ldots, z_m\}$, then the segment of the line from 0 to z which lies in $\mathcal{C}(z_1, z_2, \ldots, z_m)$ has positive length.*

The proofs of this and subsequent propositions and lemmas are in section 6.

Proposition 1 suggests the following approach for defining a target point. First, for each $z \in \mathbb{D}$, let $\mathcal{D}(z)$ denote a closed disk of radius r centered at z. More generally, for any set of $m > 0$ points $z_1, z_2, \ldots, z_m$ in $\mathbb{D}$, write

$$\mathcal{D}(z_1, z_2, \ldots, z_m) = \bigcap_{i=1}^{m} \mathcal{D}(z_i).$$

By construction, each point in $\mathcal{D}(z_1, z_2, \ldots, z_m)$ is within r units of each point in $\{z_1, z_2, \ldots, z_m\}$ and conversely. Thus $0 \in \mathcal{D}(z_1, z_2, \ldots, z_m)$ because $z_i \in \mathbb{D}$, $i \in \{1, 2, \ldots, m\}$.

Second, note that if $z_1, z_2, \ldots, z_m$ is any set of $m > 0$ points in $\mathbb{D}$ which are not all zero and for which 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$, then by Proposition 1 the segment of the line from 0 to any nonzero point $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ which lies in

$\mathcal{C}(z_1, z_2, \ldots, z_m)$ must have positive length. It follows that any continuous function $\tau : \mathbb{D}^m \rightarrow \langle 0, z_1, z_2, \ldots, z_m \rangle$ which satisfies

$$\tau(z_1, z_2, \ldots, z_m) \in \mathbb{D} \bigcap \mathcal{D}(z_1, z_2, \ldots, z_m) \bigcap \langle 0, z_1, z_2, \ldots, z_m \rangle$$

and which is nonzero whenever 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$ and $z_1, z_2, \ldots, z_m$ are not all zero fulfills all the conditions required to be a target point. In what follows we will show that there are at least two different ways to so define τ.

2.3.1. The centroid of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$. In order for the centroid of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ to be a target point, it must depend continuously on the z_i and, in addition, must have the property that it is nonzero for any set of m points in $\mathbb{D}$ which are not all zero and for which 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$. These properties are guaranteed by the following two propositions.

PROPOSITION 2. *Let $z_1, z_2, \ldots, z_m$ be a set of $m > 0$ points in $\mathbb{D}$ which are not all 0. Then the centroid of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ is in $\langle 0, z_1, z_2, \ldots, z_m \rangle$. If, in addition, 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$, then $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ has a nonempty interior, and the centroid of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ cannot be at 0.*

PROPOSITION 3. *The function which assigns to each set of $m > 0$ points $z_1, z_2, \ldots, z_m$ in $\mathbb{D}$ the centroid of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ is continuous.*

Examination of the proof of Proposition 3 given in section 6 reveals that the continuity of the centroid of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ depends crucially on the fact that the centroid is at 0 whenever the area of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ is zero. This property is not shared by the centroid of $\langle 0, z_1, z_2, \ldots, z_m \rangle$, and it is for this reason that the centroid of $\langle 0, z_1, z_2, \ldots, z_m \rangle$ is not a continuous function of the z_i.

It turns out that Propositions 2 and 3 both hold if the set $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ is replaced throughout by the constraint set $\mathbb{D} \cap \mathcal{C}(z_1, z_2, \ldots, z_m)$. This can be shown using essentially the same proofs of the propositions as those given in the appendix. What this means then is that the centroid of $\mathbb{D} \cap \mathcal{C}(z_1, z_2, \ldots, z_m)$ is also a valid target point.

2.3.2. The center of the smallest circle containing $\langle 0, z_1, z_2, \ldots, z_m \rangle$. It is also possible to define τ to be the center of the smallest circle containing $\langle 0, z_1, z_2, \ldots, z_m \rangle$. To understand why this is so, let us note first that for any set of points $z_i \in \mathbb{D}$, $i \in \{1, 2, \ldots, m\}$, the set of points $\mathcal{Q} \stackrel{\Delta}{=} \{0, z_1, \ldots, z_m\}$ is contained in a circle of radius r centered at 0. It follows that the center of this circle is at most r units from every point in $\mathcal{Q}$. This suggests that one might choose for $\tau(z_1, z_2, \ldots, z_m)$ the center $\tau_C(z_1, z_2, \ldots z_m)$ of the smallest circle containing $\mathcal{Q}$ or, equivalently, $\langle 0, z_1, z_2, \ldots, z_m \rangle$, since $\tau_C(z_1, z_2, \ldots, z_m)$ would have to be within r units of every point in $\mathcal{Q}$. It is known that there is such a smallest circle [17] and that if the z_i are not all zero, $\tau_C(z_1, z_2, \ldots z_m)$ is either the midpoint between two of the points in $\mathcal{Q}$ or a point within the interior of a triangle formed from at least one set of three points in $\mathcal{Q}$ [1]. In either case it is clear that $\tau_C(z_1, z_2, \ldots z_m) \in \langle 0, z_1, z_2, \ldots, z_m \rangle$ and, if the z_i are not all zero and 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$, that $\tau_C(z_1, z_2, \ldots z_m)$ is nonzero as well. Furthermore it can be shown that $\tau_C(z_1, z_2, \ldots z_m)$ depends continuously on the z_i [18]. In other words, $\tau_C(z_1, z_2, \ldots z_m)$ satisfies all the conditions required to be a target point. This elegant choice for τ is the one proposed in [1].

3. Main results. Define $t_0 = 0$. Note that because agents do not move during sensing periods, for $k \geq 1$ the position of each agent at time t_{k-1} is the same as its position at time $\bar{t}_k$. Thus (3) can be rewritten as

$$x_i(t_k) = x_i(t_{k-1}) + u_{m_i(t_{k-1})}(x_{i_1}(t_{k-1}) - x_i(t_{k-1}),\ x_{i_2}(t_{k-1}) - x_i(t_{k-1}), \dots,\ x_{i_{m_i(t_{k-1})}}(t_{k-1}) - x_i(t_{k-1})), \tag{7}$$

where $m_i(t_{k-1}) \triangleq m_i(\bar{t}_k)$. Because of this, the system just defined admits the model of a nonlinear discrete-time system with state $x(t_k) = \text{column}\ \{x_1(t_k), x_2(t_k), \dots x_n(t_k)\}$ evolving on the time set $t_0, t_1, \dots t_k, \dots$. Analysis of this system depends on the relationships between neighbors and how they evolve with time. These relationships can be conveniently described by a simple, undirected graph with vertex set $\{1, 2, \dots, n\}$ which is defined so that (i, j) is one of the graph's edges just in case agents i and j are registered neighbors at the beginning of maneuvering period k. Since these relationships can change from one maneuvering period to the next, so can the graph which describes them. In what follows we use the symbol $\mathcal{P}$ to denote a suitably defined set, indexing the class of all simple graphs $\mathbb{G}_p$ on n vertices. Let us partially order the set $\{\mathbb{G}_p : p \in \mathcal{P}\}$ by agreeing to say that $\mathbb{G}_p$ is contained in $\mathbb{G}_q$ if the edge set of $\mathbb{G}_p$ is a subset on the edge set of $\mathbb{G}_q$. It is natural then to define the *union* of a collection of such graphs, $\{\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \dots, \mathbb{G}_{p_m}\}$, to be the simple graph $\mathbb{G}$ with vertex set $\{1, 2, \dots, n\}$ and edge set equaling the union of the edge sets of all of the graphs in the collection.

Let $\sigma(k)$ denote the index of the graph in $\{\mathbb{G}_p : p \in \mathcal{P}\}$ which describes the relationship between registered neighbors at the beginning of maneuvering period k. Because of the cooperation assumption, we know that each agent keeps all of its registered neighbors as the system evolves. What this means is the sequence of graphs $\mathbb{G}_{\sigma(1)}, \mathbb{G}_{\sigma(2)}, \dots, \mathbb{G}_{\sigma(k)}, \dots$ forms the ascending chain

$$\mathbb{G}_{\sigma(1)} \subset \mathbb{G}_{\sigma(2)} \subset \cdots \subset \mathbb{G}_{\sigma(k)} \subset \cdots. \tag{8}$$

Because $\{G_p : p \in \mathcal{P}\}$ is a finite set, the chain must converge to the graph

$$\mathbb{G} \triangleq \bigcup_{k=1}^{\infty} \mathbb{G}_{\sigma(k)} \tag{9}$$

in a finite number of steps. Since the sequence of graphs stops changing in a finite number of steps, rendezvousing at a single point can only occur if $\mathbb{G}$ is a complete graph. There is, however, no a priori guarantee that, along a particular trajectory, $\mathbb{G}$ will turn out to be complete. On the other hand, it is clear that $\mathbb{G}$ will always be at least connected if the initial graph $\mathbb{G}_{\sigma(1)}$ in the ascending chain is. It turns out that connectivity of $\mathbb{G}_{\sigma(1)}$ implies not only that $\mathbb{G}$ is connected but also that the types of distributed control strategies just described actually cause all agents to rendezvous at a single point.

3.1. Rendezvousing.

THEOREM 1. *Let $u_0 = 0 \in \mathbb{D}_M$ and for each $m \in \{1, 2, \dots, n-1\}$, let $u_m : \mathbb{D}^m \to \mathbb{D}_M$ be any continuous function satisfying* (5) *and* (6). *For each set of initial agent positions $x_1(0), x_2(0), \dots, x_n(0)$, each agent's position $x_i(t)$ converges to a unique point $p_i \in \mathbb{R}^2$ such that for each $i, j \in \{1, 2, \dots, n\}$, either $p_i = p_j$ or $||p_i - p_j|| > r$. Moreover, if agents i and j are registered neighbors at any time t, then $p_i = p_j$.*

The proof of this theorem is given in section 4.

Theorem 1 states that the strategies under consideration cause all agents' positions to converge to points in the plane with the property that each two such points are either equal to each other or separated by a distance greater than r units. The theorem further states that if two agents are ever registered neighbors of each other, then their positions converge to the same point. We are led to the following corollary.

COROLLARY 1. *If the graph characterizing registered neighbors at the beginning of period* 1 *is connected, then the positions of all* n *agents converge to a common point in the plane.*

It is quite straightforward to extend these results to the leader-follower case when the rendezvous point is specified at the outset. This can be accomplished by simply fixing one additional agent (i.e., a virtual agent) at the desired rendezvous point and letting the remaining n agents maneuver just as before. With initial graph connectivity of all $n+1$ agent positions, convergence to the position of the virtual agent is then assured.

A more interesting case occurs when two virtual agents are fixed at distinct points in the plane. In this case it can be shown that with initial connectivity of the $(n+2)$-agent graph, all n agents will eventually move to positions on the line connecting the two virtual agents and will distribute themselves in a predictable manner depending on only the number of agents, r, and the distance between the two fixed, virtual agents. This behavior will be explored in greater depth in another paper dealing with forming formations using distributed control.

3.2. Trapping. While the graph connectivity hypothesis of Corollary 1 is sufficient for rendezvousing, it is not necessary. For example, suppose that the $\mathbb{G}_{\sigma(1)}$ has a connected component $\mathbb{G}_C$ which contains a simple closed cycle whose vertices are $i_1, i_2, \dots, i_m$. Then in the plane, the geometric form obtained by connecting by a straight line the initial position of each agent $i_j \in \{i_1, i_2, \dots, i_m\}$ with its registered neighbors with labels in $\{i_1, i_2, \dots, i_m\}$ will be a simple, closed, polygon $\mathbb{P}$. It turns out that if the initial positions of all agents whose labels are not in the vertex set of $\mathbb{G}_C$ are within $\mathbb{P}$, then rendezvous will necessarily occur. While this conclusion might appear to be an obvious consequence of the established property that agents $i_j \in \{i_1, i_2, \dots, i_m\}$ eventually rendezvous at a point, actually proving that this is true is not so straightforward. There are two reasons for this. First, there is no guarantee that the polygon $\mathbb{P}(k)$ formed by the positions at time t_k of agents $i_j \in \{i_1, i_2, \dots, i_m\}$ will remain simple as the system evolves, even if it is initially; thus just what it means for an agent to be "inside" of $\mathbb{P}(k)$ requires a more sophisticated notion of interior than the obvious one for a simple closed curve in the plane, and this in turn complicates the analysis. Second, it is quite possible that an agent initially positioned inside of $\mathbb{P}(0)$ will be outside of $\mathbb{P}(k)$ for some $k > 0$. In what follows we explain how to overcome both of these difficulties and in so doing we establish a rendezvousing result along the lines just described.

We begin by reviewing the concept of a "winding number" and what it means for a point to be inside of a closed curve in $\mathbb{R}^2$. Let $\kappa : [0, 1] \to \mathbb{R}^2$ be any continuous closed curve and let y be any point in $\mathbb{R}^2$ which does not lie on κ. The *winding number* of y with respect to κ, written $\mathrm{wn}(\kappa, y)$, is the number of times a point p traversing κ encircles y in a counterclockwise direction as p makes a full circuit of κ. Points not on κ with nonzero winding numbers are inside of κ, while those with a winding number of zero are outside of κ. There is a well-known formula for $\mathrm{wn}(\kappa, y)$, involving the integral around a closed contour $\tilde{\kappa} : [0, 1] \to \mathbb{C}$ in the complex plane [15]. $\tilde{\kappa}$ is a representation of κ resulting from the assignment to each vector $x = [a \quad b]'$ in $\mathbb{R}^2$ the associated complex number $\tilde{x} \triangleq a + jb$. In this setting, $\text{wn}(\kappa, y)$ is given by the contour integral

$$\text{wn}(\kappa, y) = \frac{1}{2\pi j} \oint_{\tilde{\kappa}} \frac{dz}{z - \tilde{y}}.$$

We will use this formula in what follows to prove Lemma 8.

The closed curves of interest here are of a specific type determined by finite point sets in $\mathbb{R}^2$. In particular, let us note that any ordered set of $m > 0$ points $\{y_1, y_2, \ldots, y_m\}$ in $\mathbb{R}^2$ uniquely determines a continuous, piecewise linear, closed curve $c : [0, m] \to \mathbb{R}^2$ defined so that

$$c(t) = (t + 1 - i)y_{i+1} + (i - t)y_i, \qquad i - 1 \leq t \leq i, \qquad i \in \{1, 2, \ldots, m\},$$

where $y_{m+1} = y_1$. An ordered set $\{y_1, y_2, \ldots, y_m\}$ of three or more such points is called a *cycle* if $||y_{i+1} - y_i|| \leq r,\ i \in \{1, 2, \ldots, m\}$; in what follows we denote such a cycle by $\lceil y_1, y_2, \ldots, y_m \rceil$. A point $z \in \mathbb{R}^2$ is called an *interior point* of $\lceil y_1, y_2, \ldots, y_m \rceil$ if it is an interior point of the closed, piecewise linear curve c determined by $\{y_1, y_2, \ldots, y_m\}$.

A point $z \in \mathbb{R}^2$ is said to be *linked* to a nonempty set of vectors $\{y_1, y_2, \ldots, y_m\}$ in $\mathbb{R}^2$ if for some $i \in \{1, 2, \ldots, m\}$, $||z - y_i|| \leq r$. More generally, z is *connected* to $\{y_1, y_2, \ldots, y_m\}$ through a set of vectors $\{x_1, x_2, \ldots, x_n\}$ in $\mathbb{R}^2$ if there exists a subset $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ with $x_{i_k} \in \{y_1, y_2, \ldots, y_m\}$ such that $||z - x_{i_1}|| \leq r$ and $||x_{i_{s-1}} - x_{i_s}|| \leq r,\ i \in \{2, 3, \ldots k\}$. The following corollary to Theorem 1 will be proved later in this section.

COROLLARY 2. *Suppose that the set of initial positions* $\{x_1(0), x_2(0), \ldots, x_n(0)\}$ *of the* n *agents contains a cycle* $\lceil x_{i_1}(0), x_{i_2}(0), \ldots, x_{i_m}(0) \rceil$. *Then all agents initially positioned inside the cycle eventually rendezvous at one point with all agents with positions initially connected to the cycle through* $\{x_1(0),\ x_2(0), \ldots,\ x_n(0)\}$.

In what follows we use the abbreviated notation $\mathbf{C}(k) \triangleq \lceil x_{i_1}(t_k), x_{i_2}(t_k), \ldots, x_{i_m}(t_k) \rceil, k \geq 0$, and say that a vector x is connected to $\mathbf{C}(k)$ whenever x is connected to $\mathbf{C}(k)$ through $\{x_1(t_k),\ x_2(t_k), \ldots,\ x_n(t_k)\}$. Note that Corollary 2 does not require agents initially positioned inside of $\mathbf{C}(0)$ to be connected to $\mathbf{C}(0)$. It is natural to say that such "disconnected" agents are ultimately *trapped* by those agents whose initial positions comprise $\mathbf{C}(0)$. This particular group behavior is accordingly referred to as "trapping."

Consider the situation hypothesized in Corollary 2. We already know from Theorem 1 that all agents with positions initially connected to $\mathbf{C}(0)$ eventually rendezvous at a single point. So what remains to be shown is that all agents at initial positions interior to $\mathbf{C}(0)$ but not connected to it also rendezvous at the same point. To do this it is enough to show that each such initially disconnected internal agent eventually moves at some finite time t_K to a position which is connected to $\mathbf{C}(K)$—for once this happens, Theorem 1 can be applied with a start time of t_K, thereby enabling one to conclude that the agent under consideration will eventually rendezvous at the same point as the agents with positions initially connected to $\mathbf{C}(0)$. Carrying out this program relies on three key propositions which follow and which are proved in section 6.

PROPOSITION 4. *The interior of any cycle* $\lceil y_1, y_2, \ldots, y_m \rceil$ *in* $\mathbb{R}^2$ *is contained in its convex hull* $\langle y_1, y_2, \ldots, y_m \rangle$.

This proposition is used as follows. Note that because all agents initially positioned at points comprising $\mathbf{C}(0)$ eventually rendezvous at a single point, the diameter of the convex hull $\langle x_{i_1}(t_k), x_{i_2}(t_k), \ldots, x_{i_m}(t_k)\rangle$ must eventually become smaller than r and remain so for all future time. What this and Proposition 4 therefore imply is that any agent whose position remains inside of $\mathbf{C}(k)$ for all time must at some finite time $t_{\bar{k}}$ reach a position connected to $\mathbf{C}(\bar{k})$. Unfortunately not every agent initially positioned at a point inside of and disconnected from $\mathbf{C}(0)$ can be counted on to be so accommodating. We will deal with this situation by proving that when such an agent first leaves $\mathbf{C}(k)$—say at time $t_{\bar{k}}$—it automatically moves to a position connected to $\mathbf{C}(\bar{k})$. Let A be the label of such an agent and let $x_A(t_{\bar{k}})$ denote its position at time $t_{\bar{k}}$. Below we shall argue using the following proposition that all of agent A's registered neighbors at the beginning of maneuvering period $\bar{k}-1$ are inside of $\mathbf{C}(\bar{k}-1)$ at time $t_{\bar{k}-1}$.

Proposition 5. *Let $\lceil y_1, y_2, \ldots, y_m\rceil$ be a cycle in $\mathbb{R}^2$ which contains a point z which is not linked to $\lceil y_1, y_2, \ldots, y_m\rceil$. Then any point within r units of z is either inside of $\lceil y_1, y_2, \ldots, y_m\rceil$ or is linked to $\lceil y_1, y_2, \ldots, y_m\rceil$.*

We've assumed that $x_A(t_{\bar{k}})$ is not inside of $\mathbf{C}(\bar{k})$, and that $x_A(t_{\bar{k}-1})$ is inside of $\mathbf{C}(\bar{k}-1)$ and not connected to $\mathbf{C}(\bar{k}-1)$. Clearly $x_A(t_{\bar{k}-1})$ is not linked to $\mathbf{C}(\bar{k}-1)$. From Proposition 5 it follows that all of agent $A's$ registered neighbors at the beginning of maneuvering period $k-1$ are at positions at time $\bar{t}_{\bar{k}-1}$ {and consequently time $t_{\bar{k}-1}$} which are either inside of $\mathbf{C}(\bar{k}-1)$ or linked to $\mathbf{C}(\bar{k}-1)$. If any registered neighbor's position were connected to $\mathbf{C}(\bar{k}-1)$, then $x_A(t_{\bar{k}-1})$ would be connected to $\mathbf{C}(\bar{k}-1)$, which we have explicitly assumed is not the case. Therefore none of A's registered neighbors is connected (or therefore linked) to $\mathbf{C}(\bar{k}-1)$ at time $t_{\bar{k}-1}$; moreover, all must be inside of $\mathbf{C}(\bar{k}-1)$ because of Proposition 5.

To show that under these conditions, $x_A(t_{\bar{k}})$ is necessarily connected to $\mathbf{C}(\bar{k})$, we will make use of the following concept. Let us agree to call a cycle $\lceil \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m\rceil$ a *successor* of a given cycle $\lceil y_1, y_2, \ldots, y_m\rceil$ if, in addition to the cycle requirement that $||\bar{y}_{i+1}-\bar{y}_i|| \leq r,\ \ i \in \{1, 2, \ldots, n\}$, the inequalities $||\bar{y}_i - y_i|| \leq r$, $||\bar{y}_{i+1} - y_i|| \leq r$, and $||\bar{y}_i - y_{i+1}|| \leq r$ all hold for $i \in \{1, 2, \ldots, m\}$. Observe that each cycle in the sequence $\lceil y_1, y_2, \ldots, y_m\rceil$, $\lceil \bar{y}_1, y_2, \ldots, y_m\rceil$, $\lceil \bar{y}_1, \bar{y}_2, \ldots, y_m\rceil, \ldots, \lceil \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m\rceil$ is a successor of the cycle which precedes it. It is easy to verify that for each $k \geq 0$, $\mathbf{C}(k+1)$ is a successor of $\mathbf{C}(k)$.

Proposition 6. *Let $\lceil \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m\rceil$ be a successor of a given cycle $\lceil y_1, y_2, \ldots, y_m\rceil$ in $\mathbb{R}^2$. Suppose that $z_1, z_2, \ldots, z_k$ are $k > 0$ interior points of $\lceil y_1, y_2, \ldots, y_m\rceil$ which are not linked to $\lceil y_1, y_2, \ldots, y_m\rceil$ and which satisfy $||z_1 - z_i|| \leq r$, $i \in \{2, 3, \ldots, k\}$. Then each point in the convex hull $\langle z_1, z_2, \ldots, z_k\rangle$ is either an interior point of $\lceil \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m\rceil$ or is linked to $\lceil \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m\rceil$.*

Recall that the strategy under consideration puts $x_A(t_{\bar{k}})$ at a point in the convex hull of the set consisting of $x_A(t_{\bar{k}-1})$ and the positions at time $t_{\bar{k}-1}$ of agent A's registered neighbors. Proposition 6 therefore implies that $x_A(t_{\bar{k}})$ must be either inside of $\mathbf{C}(\bar{k})$ or linked to it. Since we have ruled out the former by assumption, $x_A(t_{\bar{k}})$ is linked and therefore connected to $\mathbf{C}(\bar{k})$ as claimed. This completes the proof of Corollary 2. ∎

4. Analysis. The aim of this section is to establish the correctness of Theorem 1. Towards this end, let $\{\{x_1(t_k), x_2(t_k), \ldots, x_n(t_k)\} : k \geq 1\}$ be a system trajectory determined by (7) and any initial set of agent positions. Let k^* denote the value of k for which the ascending chain shown in (8) converges to the limit graph $\mathbb{G}$ in (9). Thus for $t_k \geq t_{k^*}$, the neighbors of each agent do not change. For each $i \in \{1, 2, \ldots, n\}$, let $\{i_1, i_2, \ldots, i_{m_i}\}$ denote the set of indices labelling the neighbors of agent i. For simplicity, we will deal only with the case when each agent has at least one neighbor.

This means that all m_i are positive integers. These assumptions imply that for $k \geq k^*$, the system under consideration will have a state $\{x_1(t_k), x_2(t_k), \ldots, x_n(t_k)\}$ taking values in the space

$$\mathcal{X} = \{\{x_1, x_2, \ldots x_n\} : ||x_j - x_i|| \leq r, \ \ j \in \{i_1, i_2, \ldots, i_{m_i}\}, i \in \{1, 2, \ldots, n\}\}. \tag{10}$$

4.1. Error system. To analyze system behavior it is convenient to introduce a suitably defined "error system." For $\{x_1, x_2, \ldots, x_n\} \in \mathcal{X}$, define

$$e_i = x_i - x_n, \quad i \in \{1, 2, \ldots, n\}, \tag{11}$$

and note that $e_n = 0$. Let $e \triangleq \{e_1, e_2, \ldots, e_{n-1}\}$. In view of (10) and the fact that $e_j - e_i = x_j - x_i$ for all $i, j \in \{1, 2, \ldots, n\}$, we see that e takes values in the closed space

$$\mathcal{E} = \{\{e_1, e_2, \ldots e_{n-1}\} : e_n = 0, ||e_j - e_i|| \leq r, \quad j \in \{i_1, i_2, \ldots, i_{m_i}\},\ i \in \{1, 2, \ldots, n\}\}. \tag{12}$$

Note that

$$x_{i_j}(t_{k-1}) - x_i(t_{k-1}) = e_{i_j}(t_{k-1}) - e_i(t_{k-1}), \quad j \in \{1, 2, \ldots, m_i\}, \quad i \in \{1, 2, \ldots, n\}.$$

It follows that the update equation (7) for x_i can be written as

$$x_i(t_k) = x_i(t_{k-1}) + f_i(e(k-1)), \quad k \geq k^*, \tag{13}$$

where $f_i : \mathcal{E} \to \mathbb{D}$ is the continuous function

$$\{e_1, e_2, \ldots, e_{n-1}\} \longmapsto u_{m_i}(e_{i_1} - e_i,\ e_{i_2} - e_i, \ldots,\ e_{i_{m_i}} - e_i)|_{e_n = 0}.$$

In view of (13) and the definition of the e_i,

$$e_i(t_k) = e_i(t_{k-1}) + f_i(e(t_{k-1})) - f_n(e(t_{k-1})), \quad k > k^*,\ i \in \{1, 2, \ldots, n-1\}. \tag{14}$$

This enables us to define the *error system*

$$e(t_k) = e(t_{k-1}) + f(e(t_{k-1})), \quad k > k^*, \tag{15}$$

where $f(e) = \{f_1(e) - f_n(e),\ f_2(e) - f_n(e), \ldots,\ f_{n-1}(e) - f_n(e)\}$.

4.2. Proving convergence in the style of Lyapunov. In what follows, we will prove that under certain conditions $e(t_k) \to 0$ as $k \to \infty$. We will do this using the positive definite function $V : \mathcal{E} \to \mathbb{R}$ defined by

$$V(e) = \text{dia}\{e_1, e_2, \ldots, e_{n-1}, 0\}, \tag{16}$$

where for any set of vectors $y_1, y_2, \ldots, y_m$ in $\mathbb{R}^2$, $\text{dia}\{y_1, y_2, \ldots, y_m\}$ denotes the diameter[2] of $\langle y_1, y_2, \ldots, y_m \rangle$. The following proposition is central to the proof of Theorem 1.

PROPOSITION 7. *The difference function $\Delta : \mathcal{E} \to \mathbb{R}$ defined by*

$$\Delta(e) = V(e + f(e)) - V(e) \tag{17}$$

is negative semidefinite. Moreover, if $\mathbb{G}$ is connected, then Δ is negative definite.

[2] Recall that the *diameter* of a closed set $\mathcal{S} \subset \mathbb{R}^2$ is the maximum of $||s_1 - s_2||$ over all $s_1, s_2 \in \mathcal{S}$.

Proof of Theorem 1. In general the graph $\mathbb{G}$ to which the ascending chain (8) converges for some finite $k = k^*$ consists of a finite set of connected components. Suppose that $\mathbb{G}_c$ is any one of these. To prove Theorem 1 it is enough to show that the positions of those agents whose indices are the vertices of $\mathbb{G}_c$ converge to a common point. For simplicity we will do this only for the case when $\mathbb{G}_c = \mathbb{G}$, since, except for notation, the proof is essentially the same even if $\mathbb{G}_c \neq \mathbb{G}$.

By hypothesis $n > 1$. Note that if $e(t_k) = 0$ for some $k = \bar{k}$, then all agents are in the same position at time $t_{\bar{k}}$; moreover, any such position will remain fixed for all $t \geq t_{\bar{k}}$ because $f(0) = 0$. Therefore to complete the proof it is enough to show that $e(t_k)$ tends to 0 as $k \to \infty$.

Let $V : \mathcal{E} \to \mathbb{R}$ be defined as in (16). Note that

$$V(e(t_k)) = \text{dia}\{x_1(t_k), x_2(t_k), \dots, x_n(t_k)\} \tag{18}$$

because the diameter of a convex set in $\mathbb{R}^2$ is invariant under translation of the set. From this and Proposition 7, it follows that the difference function

$$\Delta(e(t_k)) = V(e(t_k) + f(e(t_k))) - V(e(t_k))$$

is nonpositive for $k \geq k^*$. Thus $V(e(t_k))$ is a monotone nonincreasing function of k for $k \geq k^*$. Since for $k \geq k^*$, $V(e(t_k))$ is bounded above by $V(e(t_{k^*}))$ and below by 0, there must exist a finite limit

$$V^* \triangleq \lim_{k \to \infty} V(e(t_k)).$$

We claim that $V^* = 0$. To prove this claim, suppose that it is false. Then $V^* > 0$. Let $\mathcal{B}$ denote the set of all points $e \in \mathcal{E}$ such that $V^* \leq V(e) \leq V(e(t_{k^*}))$. Note that $\mathcal{B}$ is closed and bounded because $V(\cdot)$ is continuous and $\mathcal{E}$ is closed. Moreover, $0 \notin \mathcal{B}$ because $V(\cdot)$ is positive definite and bounded away from zero on $\mathcal{B}$. By Proposition 7, $\Delta(\cdot)$ is negative definite. Therefore for all $e \in \mathcal{B}$, $\Delta(e) < 0$. From this, the compactness of $\mathcal{B}$, and the continuity of $\Delta(\cdot)$, it follows that

$$\mu \triangleq \max_{e \in \mathcal{B}} \Delta(e)$$

is a finite negative number. Since $e(t_k) \in \mathcal{B}$ for $k \geq k^*$, it must therefore be true that

$$V(e(t_{k+1})) - V(e(t_k)) = \Delta(e(t_k)) \leq \mu, \quad k \geq k^*.$$

Thus by summing,

$$V(e(t_k)) \leq V(e(t_{k^*})) + (k - k^*)\mu, \quad k \geq k^*.$$

Therefore, for k sufficiently large, $V(e(t_k))$ must be negative because $\mu < 0$. But this is impossible because $V(\cdot)$ is positive definite. Hence V^* cannot be positive. □

The proof just given is basically a standard Lyapunov argument[3] applied to the system (17). It is worth pointing out here that the continuity of $\Delta(\cdot)$ is crucial to the proof as is the fact that $\mathcal{E}$ is closed. If $\mathcal{E}$ were not a closed set, the preceding proof would break down because one could not conclude that $\mathcal{B}$ is closed. The closure of $\mathcal{E}$ is

[3]It is worth noting that a similar proof could also be crafted using recent results by Moreau which appeared in [16] after this paper was submitted in December, 2004.

a direct consequence of the fact that sensing regions are defined to be closed sets. The continuity of $\Delta(\cdot)$ is a consequence of the requirement that the $u_m(\cdot)$ be continuous functions. In summary, for the present analysis to go through, it is essential that sensing regions be closed sets and that the $u_m(\cdot)$ be continuous functions. Whether or not these requirements can be relaxed by approaching convergence differently remains to be seen.

5. Concluding remarks. In this paper we have reconsidered the multi-agent rendezvous problem originally posed in [1] and have described several alternate synchronous solutions. We have provided an example which shows that rendezvousing can in some cases be guaranteed to occur even if the graph characterizing initial relations is not initially connected. In a sequel to this paper [12] we will explain how rendezvousing can be achieved asynchronously, without assuming that the agents share a common clock.

Since this paper and its sequel [12] were written, a number of papers on rendezvous have appeared. We refer the reader to [3] for additional references and for interesting new results on the rendezvous problem posed in higher dimensional spaces and with more general assumptions about sensing and communications.

6. Appendix. The proof of Proposition 1 depends on the following lemma.

LEMMA 1. *Let* $z_1, z_2, \dots, z_m$ *be a set of* $m > 0$ *points in* $\mathbb{D}$ *which are not all* 0. *If* 0 *is a corner of* $\langle 0, z_1, z_2, \dots, z_m\rangle$, *then the constraint set* $\mathcal{C}(z_1, z_2, \dots, z_m)$ *has a nonempty interior.*

Proof of Lemma 1. Suppose that $\mathcal{C}(z_1, z_2, \dots, z_m)$ has an empty interior in which case $\mathcal{C}(z_1, z_2, \dots, z_m)$ is the singleton $\{0\}$. It will be enough to prove that 0 is not a corner of $\langle 0, z_1, z_2, \dots, z_m\rangle$. In what follows we shall assume, without loss of generality, that 0 is on the boundary of each disk in the intersection; for if there were any disks in the intersection which contained the origin in their interiors, all such disks could be removed from the intersection without changing what the intersection equals.

To proceed, let us note first that $m > 1$ because each $\mathcal{C}(z_i)$ has a nonempty interior and, by hypothesis, the intersection $\mathcal{C}(z_1, z_2, \dots, z_m)$ does not. Next observe that since $m > 1$ and $\mathcal{C}(z_1)$ has a nonempty interior, there must be a least integer $j \in \{2, 3, \dots, m\}$ such that $\mathcal{I} \stackrel{\Delta}{=} \bigcap_{i=1}^{j-1} \mathcal{C}(z_j)$ has a nonempty interior and $\mathcal{I} \cap \mathcal{C}(z_j)$ contains just the origin. The intersection of any positive number of disks from $\{\mathcal{C}(z_1), \mathcal{C}(z_2), \dots, \mathcal{C}(z_m)\}$ is either the origin or a convex set with a nonempty interior; moreover, the latter will always be a strictly convex set whose edges are arcs from circles bounding disks in the intersection and whose corners are intersections of such arcs. It follows that $\mathcal{C}(z_j)$ must either be tangent at the origin to an arc which is from a circle bounding some disk $\mathcal{C}(z_k) \in \{\mathcal{C}(z_1), \mathcal{C}(z_2), \dots, \mathcal{C}(z_{j-1})\}$ or $\mathcal{C}(z_j)$'s boundary must pass through a corner of $\mathcal{I}$ at the origin. If the former is true, then z_k must equal $-z_j$. Since $z_j \neq 0$, this means that the origin is halfway between z_k and z_j on the line connecting these two points. Hence the origin cannot be a corner of the polytope $\langle 0, z_1, z_2, \dots, z_m\rangle$.

Now suppose that the boundary of $\mathcal{C}(z_j)$ passes through a corner of $\mathcal{I}$ at the origin. Let $\mathcal{C}(z_k)$ and $\mathcal{C}(z_l)$ denote two disks in $\{\mathcal{C}(z_1), \mathcal{C}(z_2), \dots, \mathcal{C}(z_{j-1})\}$ whose intersection at the origin determines this corner. Under these conditions, $z_k + z_l \neq 0$—for if $z_k + z_l = 0$, then $\mathcal{C}(z_k)$ and $\mathcal{C}(z_l)$ would be tangent, and $\mathcal{I}$ would consequently contain just the origin. Moreover, the intersection $\mathcal{C}(z_j) \cap \mathcal{C}(z_k) \cap \mathcal{C}(z_l)$ must consist of just the origin—for if this were not so, then $\mathcal{I} \cap \mathcal{C}(z_j)$ would have a nonempty interior since $\mathcal{I}$ coincides locally, in an open neighborhood of 0, with $\mathcal{C}(z_k) \cap \mathcal{C}(z_l)$.

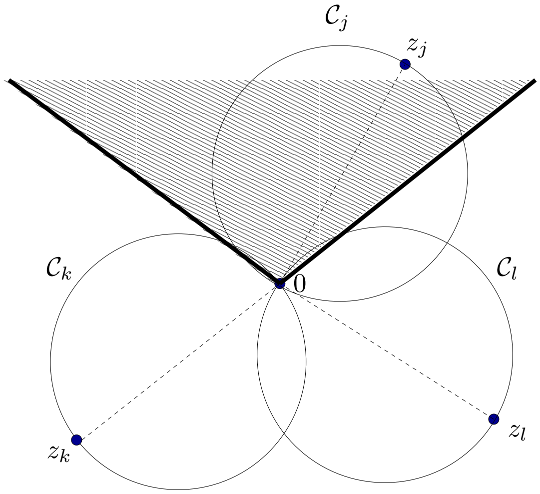

FIG. 1. *Three constraint disks whose intersection is the origin.*

As illustrated in Figure 1, the requirement that $\mathcal{C}(z_j) \cap \mathcal{C}(z_k) \cap \mathcal{C}(z_l)$ consist of just the origin implies that $\mathcal{C}(z_j)$ must be positioned in such a way so that it intersects only at the origin with a cone of points determined by tangents to $\mathcal{C}(z_k)$ and $\mathcal{C}(z_l)$ at the origin. This means that z_j must lie within the opposing cone shown in grey in Figure 1. Hence the origin is within the interior of the convex hull of z_j, z_k, and z_l. Therefore the origin cannot be a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$. $\square$

Proof of Proposition 1. Lemma 1 and the hypothesis that 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$ imply that $\mathcal{C}(z_1, z_2, \ldots, z_m)$ has a nonempty interior. From this and the hypothesis that $z \neq 0$, it follows that if 0 is an interior point of $\mathcal{C}(z_1, z_2, \ldots, z_m)$, then a line segment with the required property must exist.

Suppose next that 0 is on the boundary of $\mathcal{C}(z_1, z_2, \ldots, z_m)$. To complete the proof it is clearly enough to show that the line from 0 to z passes through the interior of each disk $\mathcal{C}(z_j)$ for which 0 is a boundary point. Let $\mathcal{C}(z_j)$ be such a disk in which case $||z_j|| = r$. Suppose that the line from 0 to z does not pass through the interior $\mathcal{C}(z_j)$. This means that $z_j'z \leq 0$ and thus that $||z||^2 - 2z_j'z + ||z_j||^2 \geq ||z||^2 + ||z_j||^2$. Since $||z - z_j||^2 = ||z||^2 - 2z_j'z + ||z_j||^2$ and $||z_j|| = r$, it follows that

$$||z - z_j||^2 \geq ||z||^2 + r^2.$$

But $||z||^2 > 0$ because $z \neq 0$, so

$$||z - z_j|| > r.$$

This contradicts the hypothesis that z is within r units of each point in $\{z_1, z_2, \ldots, z_m\}$. Therefore the line from 0 to z must pass through the interior $\mathcal{C}(z_j)$. $\square$

The proof of Proposition 2 depends on the following two lemmas.

LEMMA 2. *Let $z_1, z_2, \ldots, z_m$ be a set of $m > 0$ points in $\mathbb{D}$ which are not all zero. Let $\mathcal{E}(x, y)$ be an edge of $\langle 0, z_1, z_2, \ldots, z_m \rangle$ with distinct corners x and y. Write $\mathcal{L}(x, y)$ for the line passing through x and y, and let $\mathcal{S}(x, y)$ denote the closed half-plane bounded by this line whose intersection with $\langle 0, z_1, z_2, \ldots, z_m \rangle$ is $\mathcal{E}(x, y)$. If z is any point in $\mathcal{S}(x, y)$ which is also in $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$, then the reflection of z about $\mathcal{L}(x, y)$ is also in $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$.*

Proof of Lemma 2. Note that $\langle 0, z_1, z_2, \ldots, z_m\rangle$ is contained in the half-plane obtained by reflecting $\mathcal{S}(x, y)$ about $\mathcal{L}(x, y)$. Because of this, for each $w \in \mathcal{S}$

$$||\bar{w} - q|| \le ||w - q|| \quad \forall q \in \langle 0, z_1, z_2, \ldots, z_m\rangle,$$

where $\bar{w}$ is the reflection of w about $\mathcal{L}(x, y)$. In particular, this implies that

$$||\bar{z} - z_i| \le ||z - z_i||, \quad i \in \{0, 1, 2, \ldots, m\}, \tag{19}$$

where $z_0 = 0$ and $\bar{z}$ is the reflection of z about $\mathcal{L}(x, y)$. But

$$||z - z_i|| \le r, \quad i \in \{0, 1, 2, \ldots, m\}, \tag{20}$$

because $z \in \mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$. From (19) and (20) it follows that $\bar{z} \in \mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$. ∎

Lemma 3. *Let $\mathcal{L}$ be a line in $\mathbb{R}^2$ which divides a given closed set $\mathcal{D}$ into closed subsets $\mathcal{P}$ and $\mathcal{Q}$ with $\mathcal{Q}$ convex. If the reflection of $\mathcal{P}$ about $\mathcal{L}$ is a subset of $\mathcal{Q}$, then the centroid of $\mathcal{D}$ is in $\mathcal{Q}$.*

Proof. Let $\bar{\mathcal{P}}$ denote the reflection of $\mathcal{P}$ about $\mathcal{L}$. By hypothesis, $\bar{\mathcal{P}} \subset \mathcal{Q}$. Then write $\mathcal{Q} - \bar{\mathcal{P}}$ for the complement of $\bar{\mathcal{P}}$ in $\mathcal{Q}$. By symmetry, the centroid of $\mathcal{P} \cup \bar{\mathcal{P}}$ is in $\mathcal{L} \subset \mathcal{Q}$. Meanwhile, the centroid of $\mathcal{Q} - \bar{\mathcal{P}}$ must also be in $\mathcal{Q}$ because $\mathcal{Q}$ is convex and $\mathcal{Q} - \bar{\mathcal{P}} \subset \mathcal{Q}$. Thus the centroid of $\mathcal{D}$ must be in $\mathcal{Q}$ because it is the average of the centroids of $\mathcal{P} \cup \bar{\mathcal{P}}$ and $\mathcal{Q} - \bar{\mathcal{P}}$ weighted by the areas of $\mathcal{P} \cup \bar{\mathcal{P}}$ and $\mathcal{Q} - \bar{\mathcal{P}}$, respectively. ∎

Proof of Proposition 2. Write $\mathcal{D}$ for $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ and let $\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_k\}$ denote the set of edges of $\langle 0, z_1, z_2, \ldots, z_m\rangle$. For each such edge $\mathcal{E}_i$, let $\mathcal{L}_i$ denote the line in $\mathbb{R}^2$ containing $\mathcal{E}_i$ and write $\mathcal{S}_i$ for the closed half-plane bounded by this line whose intersection with $\langle 0, z_1, z_2, \ldots, z_m\rangle$ is $\mathcal{E}_i$. Let $\bar{\mathcal{S}}_i$ denote the reflection of $\mathcal{S}_i$ about $\mathcal{L}_i$. In view of Lemma 2,

$$\overline{\mathcal{S}_i \cap \mathcal{D}} \subset \mathcal{D}, \quad i \in \{1, 2, \ldots, k\},$$

where $\overline{\mathcal{S}_i \cap \mathcal{D}}$ is the reflection of $\mathcal{S}_i \cap \mathcal{D}$ about $\mathcal{L}_i$. Since $\overline{\mathcal{S}_i \cap \mathcal{D}}$ is also a subset of $\bar{\mathcal{S}}_i$,

$$\overline{\mathcal{S}_i \cap \mathcal{D}} \subset \bar{\mathcal{S}}_i \cap \mathcal{D}, \quad i \in \{1, 2, \ldots, k\}. \tag{21}$$

Moreover, by de Morgan's rule

$$\{\mathcal{S}_i \cap \mathcal{D}\} \cup \{\bar{\mathcal{S}}_i \cap \mathcal{D}\} = \mathcal{D}, \quad i \in \{1, 2, \ldots, k\},$$

because $\mathcal{S}_i \cup \bar{\mathcal{S}}_i = \mathbb{R}^2$, $i \in \{1, 2, \ldots, k\}$. Thus for each $i \in \{1, 2, \ldots\}$, $\mathcal{L}_i$ divides $\mathcal{D}$ into two closed convex regions, namely $\mathcal{S}_i \cap \mathcal{D}$ and $\bar{\mathcal{S}}_i \cap \mathcal{D}$. From this, (21), and Lemma 3 it follows that

$$\text{centroid}\{\mathcal{D}\} \in \bar{\mathcal{S}}_i \cap \mathcal{D}, \quad i \in \{1, 2, \ldots, k\}.$$

Therefore

$$\text{centroid}\{\mathcal{D}\} \in \bigcap_{i=1}^{k} \{\bar{\mathcal{S}}_i \cap \mathcal{D}\}. \tag{22}$$

But

$$\bigcap_{i=1}^{k} \{\bar{\mathcal{S}}_i \cap \mathcal{D}\} = \langle 0, z_1, z_2, \ldots, z_m\rangle \bigcap \mathcal{D} \tag{23}$$

because

$$\langle 0, z_1, z_2, \dots, z_m \rangle = \bigcap_{i=1}^{k} \bar{\mathcal{S}}_i.$$

From (22) and (23) it follows that centroid$\{\mathcal{D}\} \in \langle 0, z_1, z_2, \dots, z_m \rangle$.

Now suppose that 0 is a corner of $\langle 0, z_1, z_2, \dots, z_m \rangle$. Then in view of Lemma 1, $\mathbb{D} \cap \mathcal{C}(z_1, z_2, \dots, z_m)$ has a nonempty interior. To prove that $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \dots, z_m)$ also has a nonempty interior, it is therefore enough to show that

$$\mathcal{C}(z_1, z_2, \dots, z_m) \subset \mathcal{D}(z_1, z_2, \dots, z_m). \tag{24}$$

Recall that for $z \in \mathbb{D}$, $\mathcal{D}(z) = \{x : ||z - x|| \le r\}$ and $\mathcal{C}(z) = \{x : ||\frac{1}{2}z - x|| \le \frac{r}{2}\}$. Thus for $x \in \mathcal{C}(z)$

$$||z - x|| = \left\| \frac{1}{2}z - x + \frac{1}{2}z \right\| \le \left\| \frac{1}{2}z - x \right\| + \left\| \frac{1}{2}z \right\| \le \frac{r}{2} + \frac{1}{2}||z|| \le r;$$

thus $x \in \mathcal{D}(z)$. Hence $\mathcal{C}(z) \subset \mathcal{D}(z),\ z \in \mathbb{D}$, from which (24) follows.

To prove that the centroid of $\mathcal{D}$ is not at 0, it is enough to show that it is not at 0 whenever it lies on an edge of $\langle 0, z_1, z_2, \dots, z_m \rangle$ which contains 0. Accordingly, let $\mathcal{E}_j$ be an edge of $\langle 0, z_1, z_2, \dots, z_m \rangle$ which contains both 0 and the centroid of $\mathcal{D}$. Since the centroid of $\mathcal{D}$ lies in $\mathcal{L}_j$, and both $\mathcal{S}_j \cap \mathcal{D}$ and $\bar{\mathcal{S}}_j \cap \mathcal{D}$ have nonempty interiors, it must be true that

$$\text{area}\{\mathcal{S}_j \cap \mathcal{D}\}d = \text{area}\{\bar{\mathcal{S}}_j \cap \mathcal{D}\}\bar{d},$$

where d is the distance from the centroid of $\mathcal{S}_j \cap \mathcal{D}$ to the point closest on $\mathcal{L}_j$ and $\bar{d}$ is correspondingly the distance from the centroid of $\bar{\mathcal{S}}_j \cap \mathcal{D}$ to the point closest on $\mathcal{L}_j$. But area$\{\overline{\mathcal{S}_j \cap \mathcal{D}}\} = $ area$\{\mathcal{S}_j \cap \mathcal{D}\}$, so

$$\text{area}\{\overline{\mathcal{S}_j \cap \mathcal{D}}\}d = \text{area}\{\bar{\mathcal{S}}_j \cap \mathcal{D}\}\bar{d}. \tag{25}$$

We claim that

$$\overline{\mathcal{S}_j \cap \mathcal{D}} = \bar{\mathcal{S}}_j \cap \mathcal{D}. \tag{26}$$

To establish this claim, we first note that $\overline{\mathcal{S}_j \cap \mathcal{D}} \subset \bar{\mathcal{S}}_j \cap \mathcal{D}$ because (21) holds for all $i \in \{1, 2, \dots, k\}$. Thus to prove (26) it is enough to show that the complement of $\overline{\mathcal{S}_j \cap \mathcal{D}}$ in $\bar{\mathcal{S}}_j \cap \mathcal{D}$, denoted by $\mathcal{W}$, is empty. Towards this end, suppose that $\mathcal{W}$ is nonempty and has a nonempty interior. Since $\bar{\mathcal{S}}_j \cap \mathcal{D} = \{\overline{\mathcal{S}_j \cap \mathcal{D}}\} \cup \mathcal{W}$ and $\{\overline{\mathcal{S}_j \cap \mathcal{D}}\} \cap \mathcal{W}$ is empty, it must be true that

$$\text{area } \{\bar{\mathcal{S}}_j \cap \mathcal{D}\}\bar{d} = \text{area } \{\overline{\mathcal{S}_j \cap \mathcal{D}}\}d_1 + \text{area } \{\mathcal{W}\}d_2,$$

where d_1 is the distance from the centroid of $\{\overline{\mathcal{S}_j \cap \mathcal{D}}\}$ to the point closest on $\mathcal{L}_j$ and d_2 is correspondingly the distance from the centroid of $\mathcal{W}$ to the point closest on $\mathcal{L}_j$. But $\overline{\mathcal{S}_j \cap \mathcal{D}}$ is the reflection of $\mathcal{S}_j \cap \mathcal{D}$ about $\mathcal{L}_j$, and thus $d_1 = d$. Therefore

$$\text{area } \{\bar{\mathcal{S}}_j \cap \mathcal{D}\}\bar{d} = \text{area } \{\overline{\mathcal{S}_j \cap \mathcal{D}}\}d + \text{area } \{\mathcal{W}\}d_2.$$

This and (25) imply that area $\{\mathcal{W}\}d_2 = 0$. But $d_2 \neq 0$ because we have assumed that $\mathcal{W}$ has a nonempty interior. This implies that area $\{\mathcal{W}\} = 0$, which contradicts the hypothesis that $\mathcal{W}$ has a nonempty interior. Therefore $\mathcal{W}$ has an empty interior.

To show that $\mathcal{W}$ is actually empty or, equivalently, that (26) holds, it is enough to prove that the interior of $\bar{\mathcal{S}}_j \cap \mathcal{D}$ is contained in $\overline{\mathcal{S}_j \cap \mathcal{D}}$. For if this is true, then (26) holds, because both sets are closed and convex with nonempty interiors and $\overline{\mathcal{S}_j \cap \mathcal{D}} \subset \bar{\mathcal{S}}_j \cap \mathcal{D}$.

Suppose that the interior of $\bar{\mathcal{S}}_j \cap \mathcal{D}$ is not contained in $\overline{\mathcal{S}_j \cap \mathcal{D}}$. Then there must be a point p in the interior of $\bar{\mathcal{S}}_j \cap \mathcal{D}$ which is not in $\overline{\mathcal{S}_j \cap \mathcal{D}}$. Since $\{p\}$ and $\overline{\mathcal{S}_j \cap \mathcal{D}}$ are disjoint and each is a closed, convex set, there must be a line $\bar{\mathcal{L}}$ which separates the two and intersects neither. From this it is clear that there is an open set $\mathcal{N}_p \subset \bar{\mathcal{S}}_j \cap \mathcal{D}$ which contains p and which does not intersect $\bar{\mathcal{L}}$. It follows that $\mathcal{N}_p$ and $\overline{\mathcal{S}_j \cap \mathcal{D}}$ are disjoint and thus that $\mathcal{N}_p \subset \mathcal{W}$. But this is impossible because $\mathcal{W}$ has no interior. Therefore $\mathcal{W}$ is empty. We have therefore proved that $\mathcal{D}$ is *symmetric* about $\mathcal{L}_j$ in the sense that (26) holds.

Since $\mathcal{D}$ has a nonempty interior, its boundary consists of circular arcs resulting from the intersection of $m+1$ disks of radius r. Let $\mathcal{A}$ denote a circular arc of positive length which lies in $\mathcal{S}_j$ and which comprises part of the boundary of $\mathcal{D}$. In view of $\mathcal{D}$'s symmetry about $\mathcal{L}_j$ as defined by (26), the reflection of $\mathcal{A}$ about $\mathcal{L}_j$, namely $\bar{\mathcal{A}}$, must be a circular arc of positive length which lies in $\bar{\mathcal{S}}_j$ and which comprises part of the boundary of $\mathcal{D}$. Let x and y be points in $\{0, z_1, z_2, \dots, z_m\}$ which define disks $\mathcal{D}(x)$ and $\mathcal{D}(y)$ whose boundaries contain $\mathcal{A}$ and $\bar{\mathcal{A}}$, respectively. Clearly the reflection of $\mathcal{D}(x)$ about $\mathcal{L}_j$ must equal $\mathcal{D}(y)$, which implies that $\bar{x} = y$. Thus $\bar{x} \in \langle 0, z_1, z_2, \dots, z_m\rangle$. Since either $\bar{x}$ or x must be in $\mathcal{S}_j$, at least one of these two points must be in $\mathcal{S}_j \cap \langle 0, z_1, z_2, \dots, z_m\rangle$, which is equal to $\mathcal{E}_j$. This can only occur if $\bar{x} = x$. In summary we have shown that if $\mathcal{A}$ is any circular arc of positive length comprising part of the boundary of $\mathcal{D}$, and if x is any point in $\{0, z_1, z_2, \dots, z_m\}$ which defines a disk $\mathcal{D}(x)$ whose boundary contains $\mathcal{A}$, then x must be in $\mathcal{E}_j$.

Now let y be the nonzero endpoint of the edge $\mathcal{E}_j$, let $\mathcal{A}$ be any circular arc of positive length comprising part of the boundary of $\mathcal{D}$, and let $x_{\mathcal{A}}$ be any point in $\{0, z_1, z_2, \dots, z_m\}$ which defines a disk $\mathcal{D}(x_{\mathcal{A}})$ whose boundary contains $\mathcal{A}$. As we have just shown, $x_{\mathcal{A}} \in \mathcal{E}_j$. This means there must be a number $\lambda \in [0, 1]$ such that $x_{\mathcal{A}} = \lambda y$. Let z be any point in $\mathbb{D} \cap \mathcal{D}(y)$. Then by definition $||z|| \le r$ and $||y - z|| \le r$. Therefore

$$\begin{aligned} ||x_{\mathcal{A}} - z|| &= ||\lambda y - z|| = ||\lambda(y - z) - (1-\lambda)z|| \\ &\le ||\lambda(y - z)|| + ||(1-\lambda)z|| \le \lambda||y - z|| + (1-\lambda)||z|| \le r, \end{aligned}$$

so $z \in \mathcal{D}(x_{\mathcal{A}})$. Since z was chosen arbitrarily,

$$\mathbb{D} \cap \mathcal{D}(y) \subset \mathcal{D}(x_{\mathcal{A}}).$$

This containment holds for each disk $\mathcal{D}(x_{\mathcal{A}})$ whose boundary contains a circular arc $\mathcal{A}$ of positive length comprising part of the boundary of $\mathcal{D}$. Since the intersection of the $\mathcal{D}(x_{\mathcal{A}})$ over all such $\mathcal{A}$ is $\mathcal{D}$, it must therefore be true that

$$\mathbb{D} \cap \mathcal{D}(y) \subset \mathcal{D}. \tag{27}$$

On the other hand, $\mathcal{D} \subset \mathcal{D}(y)$ since $y \in \langle 0, z_1, z_2, \dots, z_m\rangle$. Thus $\mathcal{D} \subset \mathbb{D} \cap \mathcal{D}(y)$. This and (27) thus imply that

$$\mathbb{D} \cap \mathcal{D}(y) = \mathcal{D}.$$

It follows that the centroid of $\mathcal{D}$ must be the centroid of $\mathbb{D} \cap \mathcal{D}(y)$. But the centroid of two intersection disks with the same radius must be at the midpoint between their centers. Therefore the centroid of $\mathcal{D}$ is at $\frac{1}{2}y$ which is not 0. ∎

Proof of Proposition 3. In what follows we write z for the n-tuple $\{z_1, z_2, \dots, z_m\} \in \mathbb{D}^m$, and $\mathcal{S}(z)$ for the intersection $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \dots, z_m)$. Thus for $x, y \in \mathbb{D}^m$, $\mathcal{S}(x) \cap \mathcal{S}(y) = \mathbb{D} \cap \mathcal{D}(x_1, x_2, \dots, x_m) \cap \mathcal{D}(y_1, y_2, \dots, y_m)$. For $x \in \mathbb{D}^m$, let $\alpha(\mathcal{S}(x))$ and $\sigma(\mathcal{S}(x))$ denote, respectively, the area and centroid of $\mathcal{S}(x)$. Note that $\sigma(\mathcal{S}(x)) = 0$ whenever $\alpha(\mathcal{S}(x)) = 0$. This crucial property (which is not true for polygons) is a consequence of the fact that $\mathcal{S}(x)$ is either strictly convex with nonempty interior or the singleton 0.

It will first be shown that $z \longmapsto \sigma(\mathcal{S}(z))$ is continuous at each point $x \in \mathbb{D}^m$ at which $\alpha(\mathcal{S}(x)) = 0$. Let x be any such point. Clearly $\sigma(\mathcal{S}(x)) = 0$. Let $\epsilon > 0$ be fixed. Since $z \longmapsto \text{diameter}(\mathcal{S}(z))$ is continuous on $\mathbb{D}^m$, there must be a number $\delta > 0$ such that $\text{diameter}(\mathcal{S}(z)) \leq \epsilon$ whenever $||z - x|| \leq \delta$. But both 0 and $\sigma(\mathcal{S}(z))$ are points in $\mathcal{S}(z)$ for all $z \in \mathbb{D}^m$. Hence $||\sigma(\mathcal{S}(z))|| \leq \text{diameter}(\mathcal{S}(z)),\ z \in \mathbb{D}^m$; thus $|\sigma(\mathcal{S}(z))| \leq \epsilon$ whenever $||z - x|| \leq \delta$. Therefore $z \longmapsto \sigma(\mathcal{S}(z))$ is continuous at each point $x \in \mathbb{D}^m$ at which $\alpha(\mathcal{S}(x)) = 0$.

It will now be shown that $z \longmapsto \sigma(\mathcal{S}(z))$ is continuous at each point $x \in \mathbb{D}^m$ at which $\alpha(\mathcal{S}(x)) > 0$. Let x be such a point. Pick $\epsilon > 0$ and define

$$\bar{\epsilon} = \frac{\epsilon}{\epsilon + 4r} \alpha(\mathcal{S}(x)). \tag{28}$$

Since $z \longmapsto \alpha(\mathcal{S}(z))$ and $z \longmapsto \alpha(\mathcal{S}(x) \cap \mathcal{S}(z))$ are continuous on $\mathbb{D}^m$ and $\alpha(\mathcal{S}(x) \cap \mathcal{S}(x)) = \alpha(\mathcal{S}(x))$, there must be a number $\delta > 0$ such that

$$|\alpha(\mathcal{S}(x)) - \alpha(\mathcal{S}(z))| \leq \bar{\epsilon} \quad \text{and} \quad |\alpha(\mathcal{S}(x)) - \alpha(\mathcal{S}(x) \cap \mathcal{S}(z))| \leq \bar{\epsilon} \tag{29}$$

whenever $||z - x|| \leq \delta$. Fix z at any such value. To complete the proof it is enough to show that

$$||\sigma(\mathcal{S}(x)) - \sigma(\mathcal{S}(z))|| \leq \epsilon. \tag{30}$$

From the first inequality in (29), $\alpha(\mathcal{S}(z)) \geq \alpha(\mathcal{S}(x)) - \bar{\epsilon}$. But from (28), $\alpha(\mathcal{S}(x)) - \bar{\epsilon} = \frac{4r\bar{\epsilon}}{\epsilon}$ and thus

$$\alpha(\mathcal{S}(z)) \geq \frac{4r\bar{\epsilon}}{\epsilon}. \tag{31}$$

In general

$$\mathcal{S}(x) = (\mathcal{S}(x) \cap \mathcal{S}(z)) \cup \mathcal{X} \quad \text{and} \quad \mathcal{S}(z) = (\mathcal{S}(x) \cap \mathcal{S}(z)) \cup \mathcal{Z}, \tag{32}$$

where $\mathcal{X}$ and $\mathcal{Z}$ are the complements of $\mathcal{S}(x) \cap \mathcal{S}(z)$ in $\mathcal{S}(x)$ and $\mathcal{S}(z)$, respectively. If $\mathcal{S}(x) \cap \mathcal{S}(z)$ is a strictly proper subset of $\mathcal{S}(x)$ (respectively, $\mathcal{S}(z)$), then $\mathcal{X}$ (respectively, $\mathcal{Z}$) is a subset with nonempty interior; in this case $\alpha(\mathcal{X})$ and $\sigma(\mathcal{X})$ (respectively, $\alpha(\mathcal{Z})$ and $\sigma(\mathcal{Z})$) are well defined. If, on the other hand, $\mathcal{S}(x) \cap \mathcal{S}(z)$ equals $\mathcal{S}(x)$ (respectively, $\mathcal{S}(z)$), then $\mathcal{X}$ (respectively, $\mathcal{Z}$) is the empty set; in this case $\alpha(\mathcal{X})$ (respectively, $\alpha(\mathcal{Z})$) is zero, and $\sigma(\mathcal{X})$ (respectively, $\sigma(\mathcal{Z})$) is taken to be the 0 vector in $\mathbb{R}^2$.

In view of (32)

$$\alpha(\mathcal{S}(x)) = \alpha(\mathcal{S}(x) \cap \mathcal{S}(z)) + \alpha(\mathcal{X}), \tag{33}$$

$$\alpha(\mathcal{S}(z)) = \alpha(\mathcal{S}(x) \cap \mathcal{S}(z)) + \alpha(\mathcal{Z}), \tag{34}$$

$$\alpha(\mathcal{S}(x))\sigma(\mathcal{S}(x)) = \alpha(\mathcal{S}(x) \cap \mathcal{S}(z))\sigma(\mathcal{S}(x) \cap \mathcal{S}(z)) + \alpha(\mathcal{X})\sigma(\mathcal{X}), \tag{35}$$

$$\alpha(\mathcal{S}(z))\sigma(\mathcal{S}(z)) = \alpha(\mathcal{S}(x) \cap \mathcal{S}(z))\sigma(\mathcal{S}(x) \cap \mathcal{S}(z)) + \alpha(\mathcal{Z})\sigma(\mathcal{Z}). \tag{36}$$

Subtracting (33) from (34) and (35) from (36), one obtains

$$\alpha(\mathcal{S}(z)) - \alpha(\mathcal{S}(x)) = \alpha(\mathcal{Z}) - \alpha(\mathcal{X}) \tag{37}$$

and

$$\alpha(\mathcal{S}(z))\sigma(\mathcal{S}(z)) - \alpha(\mathcal{S}(x))\sigma(\mathcal{S}(x)) = \alpha(\mathcal{Z})\sigma(\mathcal{Z}) - \alpha(\mathcal{X})\sigma(\mathcal{X}), \tag{38}$$

respectively. Using (37) to eliminate $\alpha(\mathcal{Z})$ from (38), there results

$$\alpha(\mathcal{S}(z))\sigma(\mathcal{S}(z)) - \alpha(\mathcal{S}(x))\sigma(\mathcal{S}(x)) = \alpha(\mathcal{X})\{\sigma(\mathcal{Z}) - \sigma(\mathcal{X})\} + \{\alpha(\mathcal{S}(z)) - \alpha(\mathcal{S}(x))\}\sigma(\mathcal{Z}),$$

which can be rewritten as

$$\sigma(\mathcal{S}(z)) - \sigma(\mathcal{S}(x)) = \frac{1}{\alpha(\mathcal{S}(z))} \times \{\alpha(\mathcal{X})\{\sigma(\mathcal{Z}) - \sigma(\mathcal{X})\} + \{\alpha(\mathcal{S}(z)) - \alpha(\mathcal{S}(x))\}\{\sigma(\mathcal{Z}) - \sigma(\mathcal{S}(x))\}\}. \tag{39}$$

Since the centroids of $\mathcal{Z}, \mathcal{X}, \mathcal{S}(z)$, and $\mathcal{S}(x)$ are all in $\mathbb{D}$, it must be true that the norm of each is bounded above by r. This and (40) imply that

$$||\sigma(\mathcal{S}(z)) - \sigma(\mathcal{S}(x))|| \leq \left\|\frac{1}{\alpha(\mathcal{S}(z))}\right\| \{2r||\alpha(\mathcal{X})|| + 2r||\alpha(\mathcal{S}(z)) - \alpha(\mathcal{S}(x))||\}. \tag{40}$$

But $\|\frac{1}{\alpha(\mathcal{S}(z))}\| \leq \frac{\epsilon}{4r\bar{\epsilon}}$ because of (31); moreover, $||\alpha(\mathcal{X})|| \leq \bar{\epsilon}$ because of (33) and the second inequality in (29). From these inequalities, the first inequality in (29), and (40), it follows that (30) is true. $\square$

Proof of Proposition 4. Note first that each point on the piecewise linear curve c determined by the points $y_1, y_2, \ldots, y_m$ is on a line connecting two of these points. It follows that each point on c is contained in $\langle y_1, y_2, \ldots, y_m\rangle$. Let y be an interior point of $\lceil y_1, y_2, \ldots, y_m\rceil$; in other words, $\text{wn}(y, c) \neq 0$. Because of this, c must encircle y at least once. Since y is an interior point, any line of sufficient length which passes through y must intersect c at a minimum of two distinct points. Since points on c are in $\langle y_1, y_2, \ldots, y_m\rangle$, y must therefore be in $\langle y_1, y_2, \ldots, y_m\rangle$ as well. $\square$

The proof of Proposition 5 depends on the following fact.

LEMMA 4. *Let a, b, c, d be four points in the plane positioned so that the line from a to b intersects the line from c to d, and so that $||a - b|| \leq r$ and $||c - d|| \leq r$. Then*

$$\min\{||a - d||, ||b - c||\} \leq r \quad \text{and} \quad \min\{||a - c||, ||b - d||\} \leq r. \tag{41}$$

Proof of Lemma 4. Let e denote any point at which the line from a to b intersects the line from c to d. Since $a-d = (a-e)+(e-d)$ and $c-b = (c-e)+(e-b)$, we can use the triangle inequality to get $||a-d|| \leq ||a-e||+||e-d||$ and $||c-b|| = ||c-e||+||e-d||$, respectively. Adding these inequalities yields

$$||a - d|| + ||c - b|| \leq ||a - e|| + ||e - d|| + ||c - e|| + ||e - b||.$$

But because a, b, and e are colinear and c, d, and e are colinear, $||a - e|| + ||e - b|| = ||a - b||$ and $||c - e|| + ||e - d|| = ||c - d||$, respectively. Therefore

$$||a - d|| + ||c - b|| \leq ||a - b|| + ||c - d|| \leq 2r.$$

It follows that either $||a-d|| \le r$ or $||c-b|| \le r$. By the same reasoning, either $||a-c|| \le r$ or $||d-b|| \le r$. Therefore (41) is true. □

Proof of Proposition 5. Suppose w is within r units of z and is not interior to $\lceil y_1, y_2, \dots, y_m \rceil$. Then the line connecting z and w must intersect the line from y_j to y_{j+1} for some $j \in \{1, 2, \dots, m\}$. Since $||y_j - y_{j+1}|| \le r$, Lemma 4 can be applied with $a = z$, $b = w$, $c = y_j$, and $d = y_{j+1}$. It follows from (41) that either z or w is linked to $\lceil y_1, y_2, \dots, y_m \rceil$. Therefore w is so linked. □

The proof of Proposition 6 depends on the following lemmas.

LEMMA 5. *Let a, b, and c be three points in the plane such that $||a-b|| \le r$ and $||a-c|| \le r$. Any point in the convex hull of a, b, and c is within at most r units of both a and either b or c.*

Proof of Lemma 5. Let $d = \frac{1}{2}(b+c)$. Since $d-a = \frac{1}{2}\{(a-b)+(a-c)\}$, it must be true that $||d-a|| \le \frac{1}{2}\{||a-b|| + ||a-c||\}$. From this and the hypotheses $||a-b|| \le r$ and $||a-c|| \le r$, it follows that $||d-a|| \le r$. Moreover, from the triangle inequality, $||b-c|| \le ||b-a|| + ||a-c||$. Therefore $||b-c|| \le 2r$. Since d is the midpoint between b and c, $||b-d|| \le r$ and $||c-d|| \le r$. Thus the sets $\langle a, b, d \rangle$ and $\langle a, c, d \rangle$ each have diameter no greater than r. Since $\langle a, b, c \rangle = \langle a, b, d \rangle \cup \langle a, c, d \rangle$, it follows that any point in $\langle a, b, c \rangle$ must be in $\langle a, b, d \rangle$ or $\langle a, c, d \rangle$ and consequently within r units of a and either b or c. □

LEMMA 6. *Suppose that $z_1, z_2, \dots, z_k$ are $k > 0$ interior points of a given cycle $\lceil y_1, y_2, \dots, y_m \rceil$ which are not linked to $\lceil y_1, y_2, \dots, y_m \rceil$ and which satisfy $||z_1 - z_i|| \le r$, $i \in \{2, 3, \dots, k\}$. Then each point in the convex hull $\langle z_1, z_2, \dots, z_k \rangle$ is an interior point of $\lceil y_1, y_2, \dots, y_m \rceil$.*

Proof of Lemma 6. Note first that if there is any point $z \in \langle z_1, z_2, \dots, z_k \rangle$ which is not an interior point of $[y_1, y_2, \dots, y_m]$, then z would have to be either on or outside of the piecewise linear curve c determined by $y_1, y_2, \dots, y_m$; in either case this would mean that the line connecting z to any point in $\{z_1, z_2, \dots, z_k\}$ would have to intersect c since, by assumption, $z_1, z_2, \dots, z_k$ are interior points of c. Since any such line is contained in $\langle z_1, z_2, \dots, z_k \rangle$, the convex hull itself would have to intersect c. Thus to prove the lemma it is enough to show that $\langle z_1, z_2, \dots, z_k \rangle$ does not intersect c. To do this it is sufficient to show that for each pair of points $z_i, z_j \in \{z_1, z_2, \dots, z_k\}$, the line ℓ_{ij} from z_i to z_j does not intersect c. To do this we suppose the contrary, namely that there is a pair of points $z_i, z_j \in \{z_1, z_2, \dots, z_k\}$ such that ℓ_{ij} intersects c. Suppose this intersection occurs on the line ℓ between y_q and y_{q+1}.

First consider the case when either z_i or z_j equals z_1 in which case $||z_i - z_j|| \le r$. To prove that ℓ_{ij} does not intersect ℓ, it is sufficient to prove that for any $s \in \{2, 3, \dots, k\}$, ℓ_{1s} and ℓ do not intersect. Suppose that for some such s such an intersection exists. Since $||y_q - y_{q+1}|| \le r$ and $||z_1 - z_s|| \le r$, Lemma 4 applies with $a = z_1$, $b = z_s$, $c = y_q$, and $d = y_{q+1}$. It follows from (41) that either z_1 or z_s is within r units of either y_q or y_{q+1}. This means that either z_1 or z_s is linked to $\lceil y_1, y_2, \dots, y_m \rceil$, which is a contradiction. Therefore for any $s \in \{2, 3, \dots, k\}$, ℓ_{1s} and ℓ do not intersect. In particular, ℓ_{ij} does not intersect ℓ if either z_i or z_j equals z_1.

Now suppose that neither z_i nor z_j equals z_1. From what has just been shown we can conclude that ℓ does not intersect either ℓ_{1i} or ℓ_{1j}. Since ℓ is assumed to intersect ℓ_{ij}, either y_q or y_{q+1} must be in the convex hull $\langle z_1, z_i, z_j \rangle$. But $||z_1 - z_i|| \le r$ and $||z_1 - z_j|| \le r$; thus from Lemma 5 we can conclude that either y_q or y_{q+1} must be within r units of z_1. But this is a contradiction of the hypothesis that z_1 is not linked to $\lceil y_1, y_2, \dots, y_m \rceil$. Hence ℓ_{ij} and ℓ do not intersect. □

LEMMA 7. *For any four points a, b, c, d in $\mathbb{R}^2$, the set $\langle a, b, d \rangle \cup \langle a, c, d \rangle \cup \langle b, c, d \rangle$ is convex.*

Proof of Lemma 7. For the case when $d \in \langle a,b,c\rangle$, the union $\langle a,b,d\rangle \cup \langle a,c,d\rangle \cup \langle b,c,d\rangle$ equals $\langle a,b,c\rangle$ which is convex. Suppose therefore that $d \notin \langle a,b,c\rangle$. Then the line through at least one of the bounding edges of $\langle a,b,c\rangle$—say the edge from b to c—must separate d and $\langle a,b,c\rangle$. We claim that the four- (or less) corner polygon $\mathbb{P} = \langle a,b,d\rangle \cup \langle a,c,d\rangle$ is convex. This certainly must be true if either $c \in \langle a,b,d\rangle$ or $b \in \langle a,c,d\rangle$, since in either case $\mathbb{P}$ would be a polygon with at most three corners. On the other hand, if neither of these cases holds, then the line segment from b to c must lie totally within $\mathbb{P}$. Thus in this case the line segment between any pair of corners of $\mathbb{P}$ must lie completely within $\mathbb{P}$. Since any four- (or less) corner polygon in the plane with this property is necessarily convex, $\mathbb{P}$ is convex. Finally we note that $\langle b,c,d\rangle \subset \mathbb{P}$ because b, c, and d are in $\mathbb{P}$. Thus $\langle a,b,d\rangle \cup \langle a,c,d\rangle \cup \langle b,c,d\rangle = \mathbb{P}$ so $\langle a,b,d\rangle \cup \langle a,c,d\rangle \cup \langle b,c,d\rangle$ is convex as claimed. □

LEMMA 8. *Let $\kappa : [0,1] \to \mathbb{R}^2$ be any continuous closed curve, and let a and b be any two distinct points on κ. Let κ_1 be the closed curve consisting of the segment of κ from a to b together with the straight line segment from b to a. Let κ_2 be the closed curve consisting of the segment of κ from b to a together with the straight line segment from a to b. Then for any point $y \in \mathbb{R}^2$ which is not on κ or on the line from a to b,*

$$\mathrm{wn}(y,\kappa) = \mathrm{wn}(y,\kappa_1) + \mathrm{wn}(y,\kappa_2). \tag{42}$$

Proof. Write ϕ_1 and ϕ_2 for the segments of κ from a to b and b to a, respectively, and let ℓ_1 and ℓ_2 denote the line segments from a to b and b to a, respectively. Then

$$\begin{aligned}\mathrm{wn}(y,\kappa_1) + \mathrm{wn}(y,\kappa_2) &= \frac{1}{2\pi j}\left\{\oint_{\tilde{\kappa_1}} \frac{dz}{z-\tilde{y}} + \oint_{\tilde{\kappa_2}} \frac{dz}{z-\tilde{y}}\right\}\\ &= \frac{1}{2\pi j}\left\{\int_{\tilde{\phi_1}} \frac{dz}{z-\tilde{y}} + \int_{\tilde{\ell_1}} \frac{dz}{z-\tilde{y}} + \int_{\tilde{\phi_2}} \frac{dz}{z-\tilde{y}} + \int_{\tilde{\ell_2}} \frac{dz}{z-\tilde{y}} + \right\}.\end{aligned}$$

But

$$\int_{\tilde{\ell_1}} \frac{dz}{z-\tilde{y}} + \int_{\tilde{\ell_2}} \frac{dz}{z-\tilde{y}} = 0,$$

so

$$\mathrm{wn}(y,\kappa_1) + \mathrm{wn}(y,\kappa_2) = \frac{1}{2\pi j}\left\{\int_{\tilde{\phi_1}} \frac{dz}{z-\tilde{y}} + \int_{\tilde{\phi_2}} \frac{dz}{z-\tilde{y}}\right\} = \frac{1}{2\pi j}\oint_{\tilde{\kappa}} \frac{dz}{z-\tilde{y}}$$

from which (42) follows. □

LEMMA 9. *Let $\lceil y_1, y_2, \dots, y_m \rceil$ and $\lceil \bar{y}_1, y_2, \dots, y_m \rceil$ be cycles such that $||y_1 - \bar{y}_1|| \le r$. If z is an interior point of $\lceil y_1, y_2, \dots, y_m \rceil$, then either $||z - \bar{y}_1|| \le r$ or z is an interior point of $\lceil \bar{y}_1, y_2, \dots, y_m \rceil$ or both.*

Proof of Lemma 9. Suppose z is not an interior point of $\lceil \bar{y}_1, y_2, \dots, y_m \rceil$. It is enough to prove that $||z - \bar{y}_1|| \le r$. Towards this end let c, $\bar{c}$, c_1, c_2, and $\bar{c}_2$ denote the piecewise linear closed curves determined by the ordered point sets $\{y_1, y_2, \dots, y_m\}$, $\{\bar{y}_1, y_2, \dots, y_m\}$, $\{y_2, y_3, \dots, y_m\}$, $\{y_1, y_2, y_m\}$, and $\{\bar{y}_1, y_2, y_m\}$, respectively.

Suppose first that z is inside or on c_2; that is, $z \in \langle y_1, y_2, y_m\rangle$. By Lemma 7, $\langle y_1, y_2, \bar{y}_1\rangle \cup \langle y_1, y_m, \bar{y}_1\rangle \cup \langle y_2, y_m, \bar{y}_1\rangle$ is a convex set. Thus $\langle y_1, y_2, y_m\rangle \subset \langle y_1, y_2, \bar{y}_1\rangle \cup \langle y_1, y_m, \bar{y}_1\rangle \cup \langle y_2, y_m, \bar{y}_1\rangle$ because y_1, y_2, and y_m are all in the union. Therefore $z \in \langle y_1, y_2, \bar{y}_1\rangle \cup \langle y_1, y_m, \bar{y}_1\rangle \cup \langle y_2, y_m, \bar{y}_1\rangle$. We have assumed $||y_1 - \bar{y}_1|| \le r$. Moreover, $||\bar{y}_1 - y_2|| \le r$ and $||\bar{y}_1 - y_m|| \le r$ because $\lceil \bar{y}_1, y_2, \dots, y_m \rceil$ is assumed to be a cycle.

Thus no matter whether z is in $\langle y_1, y_2, \bar{y}_1\rangle$, $\langle y_1, y_m, \bar{y}_1\rangle$, or $\langle y_2, y_m, \bar{y}_1\rangle$, Lemma 5 applies, and it can be concluded that $||z - \bar{y}_1|| \leq r$ as claimed.

Consider next the case when z is outside of c_2; in other words, $\text{wn}(z, c_2) = 0$. Since z is not on c_2, it is clearly not on the line segment from y_2 to y_m. Therefore Lemma 8 can be applied to c, c_1, and c_2 providing that $\text{wn}(z, c) = \text{wn}(z, c_1) + \text{wn}(z, c_2)$. But by assumption z is an interior point of $\lceil y_1, y_2, \dots, y_m\rceil$, so $\text{wn}(z, c) \neq 0$. Therefore

$$\text{wn}(z, c_1) \neq 0. \tag{43}$$

By assumption, z is not an interior point of $\lceil \bar{y}_1, y_2, \dots, y_m\rceil$. Thus z must be either on $\bar{c}$ or outside of $\bar{c}$. If z is on $\bar{c}$, then it is linked to $\{\bar{y}_1, y_2, \dots, y_m\}$. On the other hand, if z is outside of $\bar{c}$, then $\text{wn}(z, \bar{c}) = 0$. Moreover, in this case $\text{wn}(z, \bar{c}) = \text{wn}(z, c_1) + \text{wn}(z, \bar{c}_2)$ because of Lemma 8. From this and (43), it follows that $\text{wn}(z, \bar{c}_2) \neq 0$. Thus z is inside of $\bar{c}_2$. But $\lceil \bar{y}_1, y_2, \dots, y_m\rceil$ is a cycle, so $||\bar{y}_1 - y_2|| \leq r$ and $||y_m - \bar{y}_1|| \leq r$. From this and Lemma 5, it follows that $||z - \bar{y}_1|| \leq r$ as claimed. □

Proof of Proposition 6. Consider the sequence of cycles $\lceil y_1, y_2, \dots, y_m\rceil$, $\lceil \bar{y}_1, y_2, \dots, y_m\rceil$, $\lceil \bar{y}_1, \bar{y}_2, \dots, y_m\rceil, \dots, \lceil \bar{y}_1, \bar{y}_2, \dots, \bar{y}_m\rceil$, each being a successor of the one before it. Let z be any point in $\langle z_1, z_2, \dots, z_k\rangle$. By Lemma 6, z is an interior point of $\lceil y_1, y_2, \dots, y_m\rceil$. Therefore by Lemma 9, either $||z - \bar{y}_1|| \leq r$ or z is an interior point of $\lceil \bar{y}_1, y_2, \dots, y_m\rceil$. If the former is true, then z is clearly linked to $\lceil \bar{y}_1, \bar{y}_2, \dots, \bar{y}_m\rceil$. On the other hand, if the latter is true, Lemma 9 can again be used, this time to reach the conclusion that either $||z - \bar{y}_2|| \leq r$ or z is an interior point of $\lceil \bar{y}_1, \bar{y}_2, \dots, y_m\rceil$ which is not linked to $\lceil \bar{y}_1, \bar{y}_2, \dots, y_m\rceil$. Continuing this process a finite number of times completes the proof. □

The proof of Proposition 7 is a simple consequence of the following lemmas.

LEMMA 10. *Let $\mathcal{S}$ be a closed, bounded convex set in $\mathbb{R}^m$. If x and y are vectors in $\mathcal{S}$ for which*

$$||x - y|| = \text{diameter}(\mathcal{S}), \tag{44}$$

then x and y are corners of $\mathcal{S}$.

Proof of Lemma 10. Suppose (44) holds. It is enough to show that y is a corner of $\mathcal{S}$. Suppose that it is not. Then there must be *distinct* vectors x_1 and x_2 in $\mathcal{S}$ and a number $\alpha \in (0, 1)$ for which $y = \alpha x_1 + (1 - \alpha)x_2$. In view of (44) and the definition of $\text{dia}(\mathcal{S})$, the function $f(\lambda) \triangleq ||x - \lambda x_1 - (1 - \lambda)x_2||^2$ must attain its maximum on $[0, 1]$ at the interior point $\lambda = \alpha$. But this is impossible because $f(\lambda)$ is a nonconstant, convex function of λ. Therefore, by contradiction y must be a corner of $\mathcal{S}$. □

LEMMA 11. *Let $\{x_1, x_2, \dots, x_n\} \in \mathcal{X}$ be fixed. Then*

$$\text{dia}\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\} \leq \text{dia}\{x_1, x_2, \dots, x_n\}, \tag{45}$$

where for $i \in \{1, 2, \dots, n\}$,

$$\bar{x}_i = x_i + u_{m_i}(x_{i_1} - x_i,\ x_{i_2} - x_i, \dots,\ x_{i_{m_i}} - x_i). \tag{46}$$

Moreover, if $\mathbb{G}$ is connected, then either the inequality in (45) *is strict or $x_1 = x_2 = \dots = x_n$.*

Proof of Lemma 11. By definition, for $m > 0$, $u_m(\cdot)$ maps the vectors $z_i \in \mathbb{D}$, $i \in \{1, 2, \dots, m\}$, into a point $\bar{z}$ in the convex hull $\langle 0, z_1, z_2, \dots z_m\rangle$; moreover, $\bar{z}$ is not a corner of $\langle 0, z_1, z_2, \dots, z_m\rangle$ unless $z_1 = z_2 = \dots = z_m = 0$. In the present context this means that for $i \in \{1, 2, \dots, n\}$, $x_i + u_{m_i}(\cdot)$ maps the vectors $x_{i_j} - x_i \in \mathbb{D}, j \in$

$\{1, 2, \dots, m_i\}$, into the point $\bar{x}_i$ in the convex hull $\langle x_i, x_{i_1}, x_{i_2}, \dots, x_{i_{m_i}} \rangle$; moreover, $\bar{x}_i$ is not a corner of $\langle x_i, x_{i_1}, x_{i_2}, \dots, x_{i_{m_i}} \rangle$ unless $x_i = x_{i_1} = x_{i_2} = \cdots = x_{i_{m_i}}$. Since each $\langle x_i, x_{i_1}, x_{i_2}, \dots, x_{i_{m_i}} \rangle$ is a subset of $\langle x_1, x_2, \dots, x_n \rangle$, it must be true that

$$\langle \bar{x}_1, \bar{x}_2, \dots \bar{x}_n \rangle \subset \langle x_1, x_2, \dots x_n \rangle. \tag{47}$$

Moreover, $\bar{x}_i$ is not a corner of $\langle x_1, x_2, \dots, x_n \rangle$ unless $x_i = x_{i_1} = x_{i_2} = \cdots = x_{i_{m_i}}$. Inequality (45) is a direct consequence of (47).

Now suppose that $\mathbb{G}$ is connected and that the x_i are not all equal. Then for each $i \in \{1, 2, \dots, n\}$, there is at least one $i_j \in \{i_1, i_2, \dots, i_{m_i}\}$ for which $x_{i_j} \neq x_i$. This means that it cannot be true that $x_i = x_{i_1} = x_{i_2} = \cdots = x_{i_{m_i}}$ for any value of $i \in \{1, 2, \dots, n\}$. Therefore $\bar{x}_i$, $i \in \{1, 2, \dots, n\}$, is not a corner of $\langle x_1, x_2, \dots, x_n \rangle$. From this and Lemma 10 it follows that the inequality in (45) is strict. ∎

It is worth noting that (47) establishes that the sequence of convex hulls of agent positions generated on successive steps must form a descending chain of convex sets. As a consequence, one can conclude at once that the sequence has a limit set $\mathcal{H}$ into which all agents must eventually move and remain. While this fact does not depend upon the $u_m(\cdot)$ being continuous, the fact that $\mathcal{H}$ is actually a single point does.

Proof of Proposition 7. Note that (11) implies that

$$\text{dia}\{x_1, x_2, \dots, x_n\} = \text{dia}\{e_1, e_2, \dots, e_{n-1}, 0\}$$

because the diameter of a convex set is invariant under translation. Therefore

$$V(e) = \text{dia}\{x_1, x_2, \dots, x_n\}. \tag{48}$$

Next observe that Lemma 11 says that

$$\text{dia}\{x_1 + f_1(e), x_2 + f_2(e), \dots, x_n + f_n(e)\} \leq \text{dia}\{x_1, x_2, \dots, x_n\} \tag{49}$$

with the inequality being strict if $\mathbb{G}$ is connected. But

$$\begin{aligned} &\text{dia}\{x_1 + f_1(e), x_2 + f_2(e), \dots, x_n + f_n(e)\} \\ &\quad = \text{dia}\{e_1 + f_1(e) - f_n(e), e_2 + f_2(e) - f_n(e), \dots, x_{n-1} + f_{n-1}(e) - f_n(e), 0\} \\ &\quad = V(e + f(e)). \end{aligned}$$

From this, (49), and (48) it is clear that

$$V(e + f(e)) - V(e)$$

is a negative semidefinite function and actually a negative definite function if $\mathbb{G}$ is connected. ∎

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