A model reference approach to safe controller changes in iterative identification and control

A. Lecchini [a,1] A. Lanzon [b,c,2] B.D.O. Anderson [b,c,2]

[a] *Department of Engineering - University of Cambridge Trumpington Street, Cambridge, CB2 1PZ, U.K.*

[b] *Department of Systems Engineering, Research School of Information Sciences and Engineering, The Australian National University, Canberra ACT 0200, Australia.*

[c] *National ICT Australia Ltd., Locked Bag 8001, Canberra ACT 2601, Australia.*

Abstract

A controller change from a current controller which stabilises the plant to a new controller, designed on the basis of an approximate model of the plant and with guaranteed bounds on the stability properties of the true closed loop, is called a safe controller change. In this paper, we present a model reference approach to the determination of safe controller changes on the basis of approximate closed loop models of the plant and robust stability results in the ν-gap.

1 Introduction

The identification of an unknown plant in practice always delivers an approximate model. It is a recognised fact that the mismatch between the plant and the identified model is influenced by the experimental conditions under which the identification has been carried out. This fact has been broadly investigated in the last ten years in the context of closed loop identification [6,9,11]; for a recent overview on this area the reader is referred to [6].

Email addresses: `al394@cam.ac.uk` (A. Lecchini), `Alexander.Lanzon@anu.edu.au` (A. Lanzon), `Brian.Anderson@anu.edu.au` (B.D.O. Anderson).

[1] The first author would like to acknowledge the Belgian Programme on Interuniversity Poles of Attraction and the European Research Network on System Identification (ERNSI) funded by the European Union.

[2] The second and third authors would like to acknowledge the support of the Australian Government through the Department of Communications, Information Technology and the Arts and the Australian Research Council via the Centre of Excellence program and an ARC Discovery-Projects grant.

A typical closed loop identification scenario is as follows. Let P be an unknown plant operating in feedback connection with a controller C_0 and let P_0 be a model identified from data collected under such an operating condition. Let $[P, C_0]$ denote the closed loop system formed by the plant P and the controller C_0. Then, the model P_0 is expected to give rise to a closed loop system $[P_0, C_0]$ whose closed loop transfer function is similar to the closed loop transfer function of $[P, C_0]$ and in this sense P_0 approximates P. However, it is not guaranteed that for some different controller C_1 (designed using P_0) the closed loop transfer functions of $[P, C_1]$ and of $[P_0, C_1]$ would be similar. In particular, even if $[P_0, C_1]$ has very good stability properties, in general the designer is not assured that also $[P, C_1]$ will have good stability properties. In certain situations the closed loop system $[P, C_1]$ could even be unstable. Thus, there are practical limitations applying in the redesign of controllers based on identified models.

The observation in effect imposes a need for small controller changes [1–3,8], where "smallness" is a concept which still needs definition. The rationale behind this is intuitive: if the change between C_0 and C_1 is small enough, then also the change between the closed loop transfer functions of $[P, C_0]$ and of $[P, C_1]$ should be small. Thus in principle, by limiting the change in the controller to be sufficiently small, one can limit also the degradation of the stability properties that can occur in the actual closed loop.

The quantification of small controller changes can be obtained by using the framework of [12] where distances between controllers are measured by the ν-gap, or pointwise in frequency by the chordal distance, and stability is measured by the stability margin (see Section 2). In particular, in this paper, we denote by the term "safe controller change" a small controller change from C_0 to C_1 such that the new real closed loop system $[P, C_1]$ has some known guaranteed bound on its stability margin. In Section 3 safe controller changes will be characterised in terms of chordal distance from the current stabilising controller C_0 under the assumption of some known worst-case bound on the error between the closed loop transfer functions of the real system $[P, C_0]$ and of the nominal system $[P_0, C_0]$. The idea of quantifying small controller changes via the ν-gap metric for the purposes of adaptive control was introduced in [3]. In [2,1,8], the idea has been applied to multiple model adaptive control in order to assure safe switchings in a set of candidate controllers.

The main contribution of this paper is a procedure to select safe controller changes motivated by iterative identification and control methods (see e.g. [11,13]). The general iterative identification and control method consists in the following successive steps: (1) identification of a model of the plant from data obtained from the current closed loop system; (2) controller redesign based on the closed loop model; (3) update of the current controller with the redesigned controller and evaluation of achieved performance. If the design criterion assumed in step (2) offers no a priori robustness guarantee it is not guaranteed that the designed controller will stabilise the real plant in step (3) (recall for example the case of an $\mathscr{H}_2$ criterion which can lead to closed loop margins that are arbitrarily small [5]). On the other hand, before implementing the new controller on the plant, the designer would like to be assured for stability. If the new controller is a considerable distance away from the current stabilising controller the above discussion on safety implies that the designer should not implement this new controller directly on the plant. Obviously, there exists the possibility to reduce the size of the controller change to a safe/small change. However, care must exercised, since the designer would like, at the same time, to maintain some portion of the performance improvement that was achieved by the controller calculated at step (2).

In this paper we approach the problem of finding a safe controller change, from C_0 to C_1, which also achieves performance improvement. Different definitions of performance improvement will be considered. Let us denote by C_* the desired controller designed at step (2) using the model P_0. In Section 4, we define performance improvement in terms of nominal closed loop transfer functions as the condition that the mismatch between the closed loop transfer functions of $[P_0, C_1]$ and of $[P_0, C_*]$ is smaller than the mismatch between the closed loop transfer functions of $[P_0, C_0]$ and of $[P_0, C_*]$. In Section 5 we extend performance improvement to the real closed loop $[P, C_1]$. As will be shown, the identification assumptions needed to obtain the improvement of Section 4 are less strict than those for the improvement considered in Section 5.

The safe controller C_1 is then obtained as the solution of a suitable model reference control problem which is stated in Section 4. An important point in the formulation of our model reference problem is that we will explicitly take into account the fact that the method used to calculate C_1 could also not deliver the exact solution to the problem, typically because of a controller order reduction step which replaces a controller of excessive order exactly solving the model reference problem by a controller of acceptable order approximately solving the problem.

Let us remark that, since the procedure proposed in this paper is not a complete design method but is intended to be used in order to introduce safety in more general iterative identification and control design methods, it is not clear at this stage which measure of performance improvement is more convenient to adopt, whether improvement in nominal performance or guaranteed improvement on the real plant with more strict identification requirements. The main point is, in fact, to have the guaranteed stability margin for $[P, C_1]$. The new controller can then be tested on the real plant and iteratively redesigned if necessary.

2 Vinnicombe's tools for robust stability

In this section, we introduce the notation and we recall some robust stability results from [12].

We shall consider MIMO linear time-invariant systems. In the notation, we will not distinguish between the continuous-time and the discrete-time cases. The frequency response of a transfer function T is indicated by $T(\omega)$. If T is

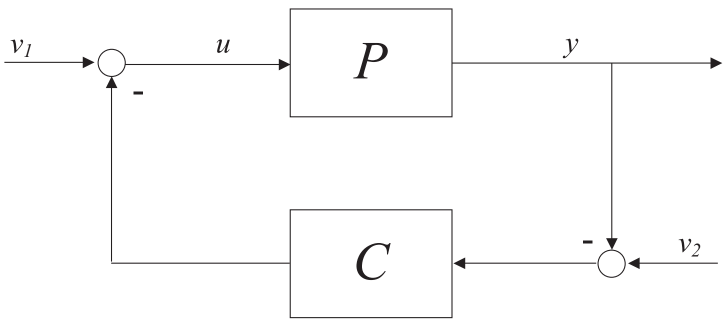


Fig. 1. The feedback configuration.

a continuous-time transfer function it should be read $T(\omega) = T(j\omega)$. If T is a discrete-time transfer function then $T(\omega) = T(e^{j\omega})$. The frequency dependent maximum singular value of a matrix transfer function T is denoted by $\bar{\sigma}(T, \omega)$. The transfer function of the plant is denoted by P. The feedback connection of the plant P and a controller C is depicted in Fig. 1. We denote by $T(P, C)$ the closed loop transfer function from $[v_2 \quad v_1]^T$ to $[y \quad u]^T$. It is given by

$$T(P, C) = \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} [-C \quad I] \,.$$

The results on theorems below provide sufficient conditions on the modification of a current stabilising controller to a new controller with guaranteed preservation of stability.
Firstly, we introduce the following definitions.

Definition 1 (Condition $\mathcal{C}$) *Two continuous-time transfer functions C_0 and C_1 satisfy Condition $\mathcal{C}$ if*

$$\det(I + C_1(\omega)^* C_0(\omega)) \neq 0 \ \forall \omega \text{ and } \mathrm{wno}\left(\det(I + C_1^* C_0)\right) + \eta(C_0) - \bar{\eta}(C_1) = 0,$$

where wno(.) *indicates the winding number of the Nyquist diagram of a scalar transfer function, evaluated on a contour along the imaginary axis and indented to the right around any pure imaginary pole, and $\eta(C)$ ($\bar{\eta}(C)$) is the number of open (closed) right-half-plane poles of C.*

The statement of Condition $\mathcal{C}$ for discrete-time transfer functions is similar but with the obvious modifications in the wording when considering the z-plane instead of the s-plane.

Definition 2 (Chordal distance) *The chordal distance $\kappa(C_0, C_1, \omega)$ is given by*

$$\kappa(C_0, C_1, \omega) = \bar{\sigma}\left((I + C_1 C_1^*)^{-\frac{1}{2}} (C_1 - C_0) (I + C_0^* C_0)^{-\frac{1}{2}}, \omega \right) .$$

Definition 3 (Frequency dependent stability margin) *The frequency dependent stability margin of the stable closed loop system* $[P, C]$ *is given by*

$$\rho(P, C, \omega) = \bar{\sigma}\left(T(P, C), \omega\right)^{-1}.$$

If the closed loop system $[P, C]$ *is unstable, we set* $\rho(P, C, \omega) = 0$.

Then we have the following results.

Theorem 1 *Let* $[P, C_0]$ *be internally stable and*

$$\kappa(C_0, C_1, \omega) < \rho(P, C_0, \omega) \quad \forall \omega\,.$$

Then the closed loop system $[P, C_1]$ *is internally stable if and only if the pair* C_0, C_1 *satisfies Condition* $\mathcal{C}$.
Proof *See* [12, page 136]. □

Theorem 2 *Let* $[P, C_0]$ *be internally stable and* C_0, C_1 *satisfy Condition* $\mathcal{C}$. *Then*

$$\rho(P, C_1, \omega) \geq \rho(P, C_0, \omega) - \kappa(C_0, C_1, \omega)\,.$$

Proof *See* [12, page 137]. □

The proposition below links the modifications of the controller to the corresponding changes which occur in the closed loop transfer function.

Theorem 3 *Let* $[P, C_0]$ *and* $[P, C_1]$ *be internally stable. Then*

$$\kappa(C_0, C_1, \omega) \leq \bar{\sigma}\left(T(P, C_0) - T(P, C_1), \omega\right) \leq \frac{\kappa(C_0, C_1, \omega)}{\rho(P, C_0, \omega)\rho(P, C_1, \omega)}$$

Proof *See* [12, page 159]. □

3 Initial assumptions

We assume that the exact transfer function of the plant P is unknown. However, we assume that the plant is operating in feedback connection with a known stabilising controller C_0 (as a particular case, if the plant is stable, this controller could be $C_0 = 0$) and that, on the basis of data obtained in this operating condition, a model P_0, which approximates P in a closed loop sense, has been identified. More precisely, we make the following assumptions.

Identification Assumptions

A.1 The controller C_0 stabilises both P_0 and P.
A.2 P_0 is such that

$$\bar{\sigma}\left(T(P, C_0) - T(P_0, C_0), \omega\right) \leq \varepsilon_\omega \rho(P_0, C_0, \omega) \quad \forall \omega$$

where ε_ω is known and $0 \leq \varepsilon_\omega < 1$.

In Assumption A.2 we basically require that the modelled closed loop $[P_0, C_0]$ captures the approximate frequency domain behaviour of the real closed loop $[P, C_0]$. An expression of $\bar{\sigma}\left(T(P, C_0) - T(P_0, C_0), \omega\right)$ in terms of frequency responses is given in the following lemma.

Lemma 4 *Let P_0, P and C_0 satisfy Assumptions A.1 and $C_S(P, C_0) = P(I - C_0 P)^{-1} C_0$ denote the complementary sensitivity of the closed loop system $[P, C_0]$ then*

$$\begin{aligned}&\bar{\sigma}\left(T(P, C_0) - T(P_0, C_0), \omega\right) = \\ &\bar{\sigma}\left([I + C_0^* C_0]^{\frac{1}{2}} [C_S(P, C_0) - C_S(P_0, C_0)][I + (C_0^* C_0)^{-1}]^{\frac{1}{2}}, \omega\right)\end{aligned}$$

Proof *See [4, Lemma 3.1].*

In the SISO case the right hand side of the equation above becomes $\left[|C_0(\omega)| + \frac{1}{|C_0(\omega)|}\right] |C_S(P, C_0, \omega) - C_S(P_0, C_0, \omega)|$. It is clear then that $\bar{\sigma}\left(T(P, C_0) - T(P_0, C_0), \omega\right)$ is the weighted mismatch between the frequency responses of the complementary sensitivities of $[P, C_0]$ and of $[P_0, C_0]$ where the weighting term is known and depends on the controller in the loop. If, from the identification procedure, one obtains a bound, say Δ_ω, directly on the closed loop error $\bar{\sigma}\left(T(P, C_0) - T(P_0, C_0), \omega\right)$, then, since $\rho(P_0, C_0, \omega)$ is known, ε_ω can be calculated as $\varepsilon_\omega = \frac{\Delta_\omega}{\rho(P_0, C_0, \omega)}$. Notice that a poorly designed C_0 will generally lead to a small value of $\rho(P_0, C_0, \omega)$ for some ω. Poor designs then require better modelling of P by P_0 for the assumptions to be fulfilled, by forcing a smaller value for the left side of the inequality.
In the redesign of the controller, our first objective is safety. We wish to obtain a controller C_1 which is guaranteed to realize a certain level of the stability margin when it is connected to the unknown plant P.
We have the following result.

Lemma 5 *Let P_0, P and C_0 satisfy Assumptions A.1-A.2 and assume that C_1 is a new controller that also stabilises P_0, then the condition*

$$\kappa(C_0, C_1, \omega) \leq (\alpha_\omega - \varepsilon_\omega)\ \rho(P_0, C_0, \omega) \quad \forall \omega \tag{1}$$

guarantees

$$\rho(P, C_1, \omega) \geq (1 - \alpha_\omega)\ \rho(P_0, C_0, \omega) \quad \forall \omega, \tag{2}$$

where $\alpha_\omega \in [\varepsilon_\omega, 1)\ \forall \omega$ is a pointwise upper bound on the percentage allowable degradation in the robust stability margin of $[P, C_1]$ when compared to that of $[P_0, C_0]$.
Proof *See Appendix.*

In the reminder of the paper we use the term safe controller change to denote a controller change that satisfies (1).

4 Safe reference models

We now assume that a controller C_* has been designed using the model P_0 through some design method, and that the closed loop transfer function $T_* = T(P_0, C_*)$ has the desired performance. We also assume that the controller C_* is sufficiently different from C_0 that $C_1 = C_*$ does *not* satisfy the inequality (1). Therefore, it is not safe to implement directly C_* on the real plant. In this section, we define, on the basis of the knowledge of T_* and the identification assumptions, a simple model reference control problem, with intermediate reference model $T_{*,1}$, such that the solution controller C_1 (a) satisfies the safety condition (1) and (b) gives a nominal closed loop $[P_0, C_1]$ with better performance than $[P_0, C_0]$. In our derivation, we will also make use of the following assumption which allows some extra freedom when solving the intermediate model reference problem involving $T_{*,1}$. This extra freedom will be used later for controller order reduction purposes. After the assumption, we describe how $T_{*,1}$ may be chosen.

Control Design Assumption

A.3 Given a temporary intermediate reference model $T_{*,1}$ and a nominal plant transfer function P_0, the controller C_1 is designed in such a way that $[P_0, C_1]$ is stable and the following inequality is satisfied:

$$\bar{\sigma}\left(T(P_0, C_1) - T_{*,1}, \omega\right) \leq c_\omega \, \bar{\sigma}\left(T(P_0, C_0) - T_{*,1}, \omega\right) \quad \forall \omega$$

where $c_\omega \in [0\,, 1]$ is known.

This assumption is not very restrictive: it is trivially satisfied by choosing $c_\omega = 1$ and $C_1 = C_0$. For $c_\omega < 1$, it says that $T(P_0, C_1)$ is closer to $T_{*,1}$ than is $T(P_0, C_0)$, i.e. C_1 does a better job of achieving a closed-loop like $T_{*,1}$ than C_0. As will be shown in Section 6, it is possible to choose $T_{*,1}$ in a parameterised way so that there exists a controller $C_{*,1}$ such that $T_{*,1}$ is exactly attainable for the model P_0, i.e. $T_{*,1} = T(P_0, C_{*,1})$. We remark that there are practical advantages in considering situations where $T_{*,1} \neq T(P_0, C_1)$. For instance, it may well be the case that a low order controller C_1 is desired and the degree constraint makes impossible the exact achieving of $T_{*,1}$. One could initially find $C_{*,1}$ with $T_{*,1} = T(P_0, C_{*,1})$ and then find a low order approximation C_1 of $C_{*,1}$, which would need to obey the inequality of Assumption A.3.
We are now in the position to state the characteristics of those $T_{*,1}$ for which safety in controller change and nominal performance improvement are guaranteed.

Theorem 6 *Let P_0, P and C_0 satisfy Assumptions A.1-A.2 and $\gamma_\omega \in [c_\omega, 1]$.*

If $T_{*,1}$ satisfies the following two conditions:

$$\bar{\sigma}\left(T_{*,1} - T(P_0, C_0), \omega\right) \leq \frac{\alpha_\omega - \varepsilon_\omega}{1 + c_\omega}\, \rho(P_0, C_0, \omega) \quad \forall\omega \tag{3}$$

$$\bar{\sigma}\left(T_{*,1} - T_*, \omega\right) \leq \frac{\gamma_\omega - c_\omega}{1 + c_\omega}\, \bar{\sigma}\left(T(P_0, C_0) - T_*, \omega\right) \quad \forall\omega \tag{4}$$

then a controller C_1 that satisfies Assumption A.3 satisfies also the inequality (nominal performance improvement condition)

$$\bar{\sigma}\left(T(P_0, C_1) - T_*, \omega\right) \leq \gamma_\omega \bar{\sigma}\left(T(P_0, C_0) - T_*, \omega\right) \quad \forall\omega \tag{5}$$

and the safety condition (1).
Proof *See Appendix.*

The inequality (5) is a bound on nominal performance improvement between $T(P_0, C_0)$ and $T(P_0, C_1)$. In Section 6 we will construct a possible set of reference models $T_{*,1}$ that satisfy (3) and (4).

5 Guaranteed performance improvement for the unknown plant

In this section, we present additional conditions under which the unknown physical closed loop system $[P, C_1]$, once again with the guaranteed bound on its stability margin given by (2), also offers guaranteed performance improvement in the sense of moving $T(P, C_1)$ closer to T_* than $T(P, C_0)$, similarly to what was previously achieved for $T(P_0, C_i)$ in equation (5). Such a result comes at some cost; not surprisingly, the quality of identification has to be strengthened. It will be shown that, provided some additional inequalities in the identification assumptions hold, there exist some choices of parameters α_ω and γ_ω such that the controller C_1 designed as required in Theorem 6 attains also performance improvement on the unknown plant P.
In this section, we consider two cases: *improvement of the worst-case performance* and *guaranteed performance improvement for the unknown plant.*
To start with, we have the following result.

Lemma 7 *Suppose the hypotheses of Theorem 6 are fulfilled, including the inequality conditions in the theorem statement, and let $\eta_\omega \in (\gamma_\omega, 1]$. If the following additional inequality holds*

$$\alpha_\omega \leq 1 + \frac{\varepsilon_\omega}{2} - \sqrt{\frac{\varepsilon_\omega^2}{4} + \frac{1}{\eta_\omega - \gamma_\omega} \frac{\varepsilon_\omega}{\rho(P_0, C_0, \omega)} \frac{1}{\bar{\sigma}\left(T(P_0, C_0) - T_*, \omega\right)}} \quad \forall\omega \tag{6}$$

then the controller C_1 satisfies also the inequality

$$\bar{\sigma}\left(T(P, C_1) - T_*, \omega\right) \leq \eta_\omega \bar{\sigma}\left(T(P_0, C_0) - T_*, \omega\right) \quad \forall\omega\,. \tag{7}$$

Proof *See Appendix.*

Then, in order to quantify improvement of worst-case performance, notice that for the initial controller C_0, due to Assumption A.2, we have

$$\bar{\sigma}(T(P,C_0)-T_*,\omega) \leq \bar{\sigma}(T(P_0,C_0)-T_*,\omega)+\varepsilon_\omega \rho(P_0,C_0,\omega)$$

which can be written also as

$$\begin{aligned} &\bar{\sigma}(T(P,C_0)-T_*,\omega) \leq \bar{\eta}_\omega^+ \bar{\sigma}(T(P_0,C_0)-T_*,\omega) \\ &\bar{\eta}_\omega^+ = 1+\frac{\varepsilon_\omega \rho(P_0,C_0,\omega)}{\bar{\sigma}(T(P_0,C_0)-T_*,\omega)}\,. \end{aligned} \tag{8}$$

The above inequality characterises the worst case performance in the case of controller C_0. Therefore, we state that C_1 attains *improvement of the worst case performance* if the inequality (7) is satisfied for some $\eta_\omega \leq \bar{\eta}_\omega^+$.
Notice now that by using a lower bound on $\bar{\sigma}\left(T(P,C_0)-T_*,\omega\right)$ we can obtain also guaranteed performance improvement for the unknown plant. In fact we can write

$$\bar{\sigma}(T(P,C_0)-T_*,\omega) \geq \bar{\sigma}(T(P_0,C_0)-T_*,\omega)-\varepsilon_\omega \rho(P_0,C_0,\omega)$$

which can be written also as

$$\begin{aligned} &\bar{\sigma}(T(P,C_0)-T_*,\omega) \geq \bar{\eta}_\omega^- \bar{\sigma}(T(P_0,C_0)-T_*,\omega) \\ &\bar{\eta}_\omega^- = 1-\frac{\varepsilon_\omega \rho(P_0,C_0,\omega)}{\bar{\sigma}(T(P_0,C_0)-T_*,\omega)}\,. \end{aligned} \tag{9}$$

Then, assuming that (7) is satisfied for some η_ω, from (7) and (9) we obtain

$$\bar{\sigma}\left(T(P,C_1)-T_*,\omega\right) \leq \frac{\eta_\omega}{\bar{\eta}_\omega^-}\bar{\sigma}\left(T(P,C_0)-T_*,\omega\right) \quad \forall\omega \tag{10}$$

Notice that on the right hand side of the above inequality now the unknown plant P appears instead of P_0. Therefore, we can state that C_1 attains *guaranteed performance improvement for the unknown plant* if the inequality (7) is satisfied for some $\eta_\omega \leq \bar{\eta}_\omega^-$. In this case the performance improvement is measured by (10).
In the sequel, we derive conditions under which inequality (6) is satisfied, and therefore (7) is also true, with $\eta_\omega \leq \bar{\eta}_\omega^+$ and $\eta_\omega \leq \bar{\eta}_\omega^-$ respectively. Obviously conditions for the first case are less strict than conditions for the second case since $\bar{\eta}_\omega^- \leq \bar{\eta}_\omega^+$.
We have the following results.

Theorem 8 *Let P_0, P and C_0 satisfy Assumptions A.1-A.2. If the following additional inequality holds*

$$1-\varepsilon_\omega \geq \frac{\varepsilon_\omega}{\rho(P_0,C_0,\omega)}\frac{1}{\left[\bar{\sigma}(T(P_0,C_0)-T_*,\omega)+\varepsilon_\omega \rho(P_0,C_0,\omega)\right]} \quad \forall\omega \tag{11}$$

then there exist: $\gamma_\omega \in [0,1]$ *that satisfies*

$$\gamma_\omega \leq \bar{\eta}_\omega^+ - \frac{\varepsilon_\omega}{1-\varepsilon_\omega}\frac{1}{\rho(P_0,C_0,\omega)}\frac{1}{\bar{\sigma}(T(P_0,C_0)-T_*,\omega)} \quad \forall\omega\,, \tag{12}$$

$c_\omega \in [0,\gamma_\omega]$, η_ω *that satisfies* $\eta_\omega \leq \bar{\eta}_\omega^+$ *and*

$$\eta_\omega \geq \gamma_\omega + \frac{\varepsilon_\omega}{1-\varepsilon_\omega}\frac{1}{\rho(P_0,C_0,\omega)}\frac{1}{\bar{\sigma}(T(P_0,C_0)-T_*,\omega)}\ \forall\omega\,, \tag{13}$$

and $\alpha_\omega \in [\varepsilon_\omega, 1)$ *that satisfies inequality (6) such that the controller* C_1 *designed under the hypotheses of Theorem 6, including the inequality conditions in the theorem statement, satisfies also the inequality (7) and hence* C_1 *guarantees improvement in worst-case performance.*
Proof *See Appendix.*

Theorem 9 *Let* P_0, P *and* C_0 *satisfy Assumptions A.1-A.2. If the following additional inequality holds*

$$1-\varepsilon_\omega \geq \frac{\varepsilon_\omega}{\rho(P_0,C_0,\omega)}\frac{1}{[\bar{\sigma}(T(P_0,C_0)-T_*,\omega)-\varepsilon_\omega\rho(P_0,C_0,\omega)]} \quad \forall\omega \tag{14}$$

then there exist: $\gamma_\omega \in [0,1]$ *that satisfies*

$$\gamma_\omega \leq \bar{\eta}_\omega^- - \frac{\varepsilon_\omega}{1-\varepsilon_\omega}\frac{1}{\rho(P_0,C_0,\omega)}\frac{1}{\bar{\sigma}(T(P_0,C_0)-T_*,\omega)} \quad \forall\omega\,, \tag{15}$$

$c_\omega \in [0,\gamma_\omega]$, η_ω *that satisfies* $\eta_\omega \leq \bar{\eta}_\omega^-$ *and (13), and* $\alpha_\omega \in [\varepsilon_\omega, 1)$ *that satisfies inequality (6) such that the controller* C_1 *designed under the hypotheses of Theorem 6, including the inequality conditions in the theorem statement, satisfies also the inequality (7) and hence* C_1 *guarantees performance improvement for the unknown plant given by (10).*
Proof *See Appendix*

Conditions (11) and (14) can be seen as an additional restriction on the Assumption A.2. It can be easily seen that they are upper bounds on ε_ω which are satisfied for ε_ω sufficiently small.
Notice from (14) that for performance improvement on the real plant one needs $\bar{\sigma}(T(P_0,C_0)-T_*,\omega)-\varepsilon_\omega\rho(P_0,C_0,\omega) > 0$, i.e. the distance from the desired target must be greater than the identification error. If this is not the case only improvement in worst case performance can be guaranteed.

6 A set of safe reference models

In this section, we will consider a set of possible reference models $T_{*,1}$ that satisfy (3) and (4). To this end, let $T_{*,1}$ be parameterised as:

$$T_{*,1} = B\,T_* + (1 - B)T(P_0, C_0) \tag{16}$$

where $B \in \mathscr{RH}_\infty$ is a SISO transfer function.
Notice that for any $T_{*,1}$ given by (16) there always exists a controller $C_{*,1}$ such that $T_{*,1} = T(P_0, C_{*,1})$. Indeed, we have the following result.

Theorem 10 *Given a reference model $T_{*,1}$ in the form (16), there exists a controller $C_{*,1}$ such that $T(P_0, C_{*,1}) = T_{*,1}$. Defining $S_0 = [I - C_0P_0]^{-1}$ and $S_* = [I - C_*P_0]^{-1}$, this controller $C_{*,1}$ is given by:*

$$C_{*,1} = [S_0 + B(S_* - S_0)]^{-1}[S_0C_0 + B(S_*C_* - S_0C_0)]\,. \tag{17}$$

Proof *See Appendix.*

Using parameterisation (16), the safety and nominal performance improvement conditions (3) and (4) on $T_{*,1}$ can be translated into conditions on B. Indeed, we obtain that (3) and (4) are respectively equivalent to

$$|B(\omega)| \leq \frac{\alpha_\omega - \varepsilon_\omega}{1 + c_\omega}\,\frac{\rho(P_0, C_0, \omega)}{\bar{\sigma}\left(T_* - T(P_0, C_0), \omega\right)} \quad \forall\omega \tag{18}$$

$$|1 - B(\omega)| \leq \frac{\gamma_\omega - c_\omega}{1 + c_\omega} \quad \forall\omega\,. \tag{19}$$

In this section, we will illustrate how transfer functions $B \in \mathscr{RH}_\infty$, which satisfy (18) and (19), can be constructed.
To start with, let us denote E and F two scalar transfer functions with $E, E^{-1} \in \mathscr{RH}_\infty$ and $F, F^{-1} \in \mathscr{RH}_\infty$ such that $|E|^{-1}$ and $|F|^{-1}$ approximate from below the right hand sides of (18) and (19) respectively. Notice that such transfer functions can be easily found with standard techniques. Moreover, by increasing the order of the transfer functions, the approximation errors can be made arbitrarily small.
Now, we have that (18) can be equivalently rewritten as $|B(\omega)| \leq \left|E^{-1}(\omega)\right| \quad \forall\omega$ which is equivalent to $\|BE\|_\infty \leq 1$. In a similar way, inequality (19) can be equivalently rewritten as $\|F - FB\|_\infty \leq 1$. Then the following condition, while not equivalent to (18) and (19), certainly implies (18) and (19):

$$\left\| \begin{bmatrix} 0 \\ F \end{bmatrix} + \begin{bmatrix} E \\ -F \end{bmatrix} B \right\|_\infty \leq 1\,. \tag{20}$$

Any $B \in \mathscr{RH}_\infty$ satisfying (20) defines a $T_{*,1}$ by (16) for which a safe, nominal performance improving $C_{*,1}$ can be found. It turns out that the problem of finding all the B satisfying (20) is a model matching problem which can be solved as shown in [7]. The solution, when it exists (see condition (21) below), is provided in the following theorem.

Theorem 11 *Let E, F be scalar transfer functions with $E, E^{-1} \in \mathcal{R}H_\infty$, $F, F^{-1} \in \mathscr{RH}_\infty$ be such that*

$$|E(\omega)|^{-2} + |F(\omega)|^{-2} > 1 \quad \forall\omega\,. \tag{21}$$

Then, the set of all transfer functions $B \in \mathscr{RH}_\infty$ satisfying (20) is given by

$$B = B_1 B_2^{-1}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \Xi^{-1} \begin{bmatrix} U \\ 1 \end{bmatrix} \text{ with } U \in \mathscr{RH}_\infty, \|U\|_\infty \le 1, \tag{22}$$

where Ξ is a (2×2) unimodular transfer function matrix in $\mathscr{RH}_\infty$ with a unimodular (11)-element that satisfies

$$\left(\begin{matrix} \begin{bmatrix} E \\ -F \end{bmatrix} & \begin{bmatrix} 0 \\ F \end{bmatrix} \\ 0 & 1 \end{matrix}\right)^{\sim} \begin{bmatrix} I_2 & 0 \\ 0 & -1 \end{bmatrix} \left(\begin{matrix} \begin{bmatrix} E \\ -F \end{bmatrix} & \begin{bmatrix} 0 \\ F \end{bmatrix} \\ 0 & 1 \end{matrix}\right) = \Xi^{\sim} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Xi. \tag{23}$$

Proof *See Appendix.*

The following comments are in order.
The condition (21) for the existence of the parameterisation (22) is a constraint on the choice of γ_ω and c_ω given ε_ω and α_ω. It can be written as:

$$\gamma_\omega \ge c_\omega + (1 + c_\omega)\sqrt{\left[1 - \frac{(\alpha_\omega - \varepsilon_\omega)^2}{(1 + c_\omega)^2} \frac{\rho(P_0, C_0, \omega)^2}{\bar{\sigma}\left(T_* - T(P_0, C_0), \omega\right)^2}\right]^+} \quad \forall\omega \tag{24}$$

where $[\,\cdot\,]^+$ denotes $\max(0, \cdot)$.
Now, in the case of nominal performance improvement, one has to select $\gamma_\omega \in [c_\omega, 1]$. Notice that for $c_\omega = 0$ the right hand side of (24) is always smaller than 1. Therefore a possible strategy is to select the desired γ_ω that satisfies (24) for $c_\omega = 0$ and then increase c_ω while (24) remains true. In the case where one wants to fulfil also the conditions on real performance improvement of Section 5 care must be exercised since, there, an upper bound on γ_ω was imposed. Let us consider performance improvement on the real plant. In this case one needs that the right hand side of (24) is less than or equal to the right hand side of (15). Let us consider again the case $c_\omega = 0$, then this condition can be written as a lower bound on α_ω

$$\alpha_\omega \ge \varepsilon_\omega + \left[\frac{\rho(P_0, C_0, \omega)}{\bar{\sigma}\left(T_* - T(P_0, C_0), \omega\right)}\right]^{-1} \sqrt{[1 - Q_\omega^2]^+} \quad \forall\omega \tag{25}$$

where Q_ω denotes the right hand side of (15).
It can be easily seen that the entire set of conditions for performance improvement on the real plant and for the existence of parameterisation (22) can be fulfilled for ε_ω sufficiently small. Indeed for $\varepsilon_\omega = 0$ the upper bounds become equals to one and the lower bounds become equals to zero.
In general, the matrix Ξ can be found with techniques given in [7]. In the particular case in which $\gamma_\omega = \gamma$ and $c_\omega = c$ are constant, the explicit expression of Ξ can be obtained after some algebraic manipulations. In this case the parameterisation (22) takes the form

$$B = \left[\frac{\bar{\gamma}}{\sqrt{1-\bar{\gamma}^2}} RU + \frac{1}{1-\bar{\gamma}^2} \right]^{-1} \tag{26}$$

where $\bar{\gamma} = \frac{\gamma - c}{1+c}$ and $R, R^{-1} \in \mathscr{RH}_\infty$ is the solution to the spectral factorisation problem

$$R^*(\omega)R(\omega) = \left(\frac{1}{1-\bar{\gamma}^2} - |E(\omega)|^2 \right) . \tag{27}$$

The transfer function R can be obtained with standard techniques.
Once $T_{*,1}$ has been obtained, one can obtain $C_{*,1}$ through (17). In general, any algorithm for controller order reduction can then be used to obtain C_1. The important issue is that the final controller C_1 must satisfy the Assumption A.3. The reader is referred to [10, Section 4.3] for a controller reduction algorithm with a priori guaranteed bounds on closed loop performance.

7 Simulation example

In this section we illustrate the results derived in the paper with a numerical example. We will assume that the transfer function of the true plant, actually unknown to the designer, is given by

$$P(z) = \frac{0.336z^3}{(z-0.6)^2(z^2+0.6z+0.5)} .$$

In the following, we will illustrate two design cases: a change with nominal performance improvement only and a change with guaranteed performance improvement on the real plant. The first case requires a less accurate model of P than the second case.

a) Safe controller change with nominal performance improvement In this case we assume that the model P_0 is given by

$$P_0(z) = \frac{0.3245z}{(z-0.705)(z+0.1)} .$$

The bode diagrams of P and P_0 are compared in Fig. 2-a. The initial stabilising controller is

$$C_0(z) = -0.49307\frac{(z-0.705)(z+0.1)}{(z-1)(z-0.36)}\ .$$

The controller $C_0(z)$ has been designed with Internal Model Control (IMC) method for a reference model given by $T_0(z) = \dfrac{0.16z}{(z-0.6)^2}$.

The stability margin of the initial nominal closed loop $\rho(P_0, C_0)$, the assumed bound $\varepsilon_\omega \rho(P_0, C_0, \omega)$ on the closed loop identification error and the actual identification error are displayed in Fig. 2-b. Notice that in the high frequency region the uncertainty is almost 30% of the nominal stability margin.

We assume that the objective of the controller change is to enlarge the bandwidth. The desired target controller is

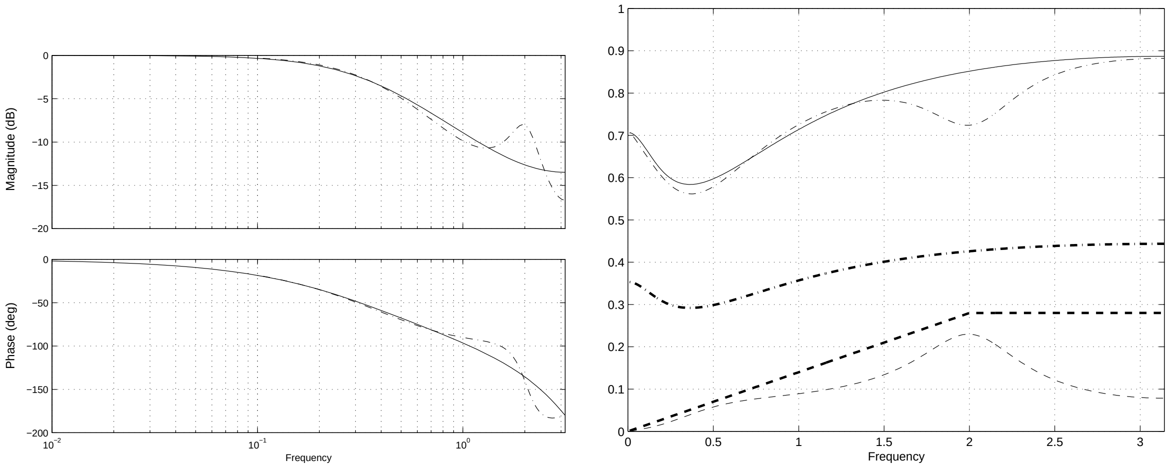
$$C_*(z) = -0.77042\frac{(z-0.705)(z+0.1)}{(z-1)(z-0.25)}$$

which has been obtained with IMC for a reference model T_* given by $T_*(z) = \dfrac{0.25z}{(z-0.5)^2}$. In order to obtain a safe change we select $\alpha_\omega = 0.5$. This means that we allow 50% of stability margin degradation on the real plant with respect to $\rho(P_0, C_0)$ - see Fig. 2-b. As it is displayed in Fig. 3-a the controller C_* does not satisfy the safety condition (1) for the selected bound.

In order to perform a safe controller change the parameters $\gamma_\omega = 1$ and $c_\omega = 0.04$ (i.e. an allowed closed loop performance degradation due to controller reduction less than 4%) has been selected. These values satisfy condition (24) for the existence of parameterisation (22). Since γ and c have chosen to be constant the parameterisation (26) has been considered. The transfer function R in (26) has McMillan degree equal to 2 and has been designed by fitting the square root of right hand side of (27). As for the choice of U in (26), the case $U = const$ with $const \in [-1, 1]$ has been considered. It turned out that the best achievements in performance (i.e. the smallest $\bar{\sigma}\,(T_{*,1} - T_*, \omega)$) occurred for negative values of *const*. In the following we illustrate the choice $U = -0.9$. By choosing $U = -0.9$, we obtained $T_{*,1}$ with McMillan degree equal to 8 and the corresponding $C_{*,1}$ with McMillan degree equal to 6. In this case, by reducing the controller $C_{*,1}$ we could find a controller C_1 with McMillan degree equal to 3 satisfying the Assumption A.3 . The controller C_1 is given by

$$C_1(z) = -0.58245\frac{(z+0.02205)(z-0.6993)(z-0.9463)}{(z-0.3393)(z-0.9564)(z-1)}$$

The stability margin of the real closed loop system $\rho(P, C_1, \omega)$ is shown in Fig. 2-b. The performance improvement between $T(P_0, C_0)$ and $T(P_0, C_1)$ is displayed in Fig. 3-b. As it is also displayed in Fig. 3-b, in this case we eventually obtained also performance improvement on the real plant. This improvement

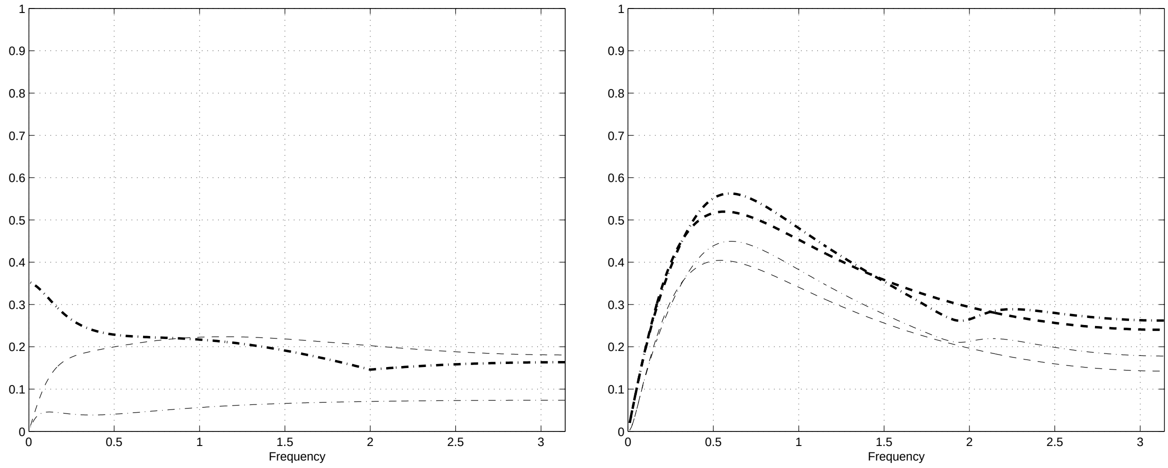


(a) (b)

Fig. 2. **(a)** Bode diagrams of P (dash-dotted) and P_0 (continuous). **(b)** The bound on the closed loop identification error $\varepsilon_\omega \rho(P_0, C_0, \omega)$ (dash bold), and $\bar{\sigma}\left(T(P, C_0) - T(P_0, C_0), \omega\right)$ (dash). The stability margins $\rho(P_0, C_0, \omega)$ (continuous), $\rho(P, C_1, \omega)$ (dash-dot) and the guaranteed stability margin $(1 - \alpha_\omega)\rho(P_0, C_0, \omega)$ (dash-dot bold).

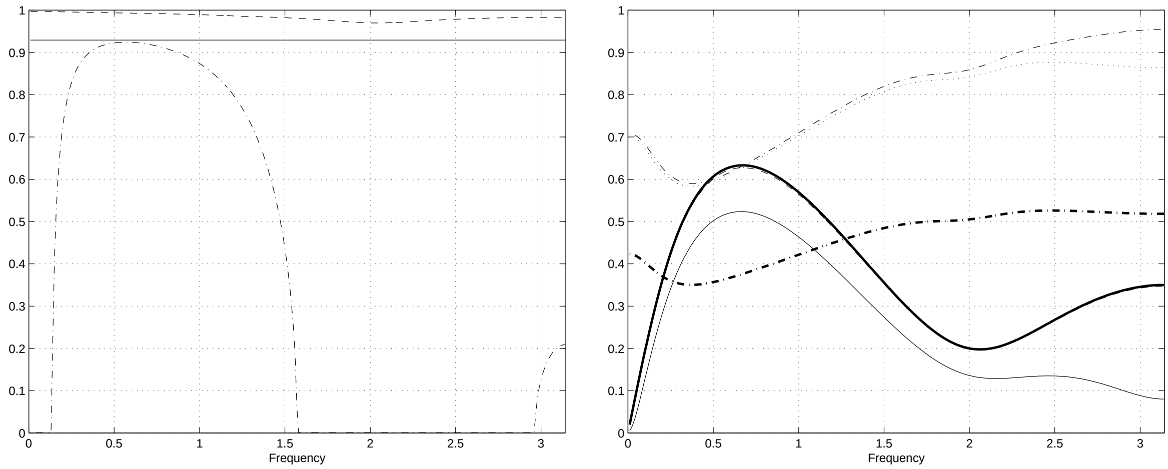
was not guaranteed for this design case, but safety in testing the controller C_1 on the real plant was assured.

b) Safe controller change with guaranteed performance on the real plant In this case we assume a 4th order nominal model P_0. The initial controller C_0 and the desired controller C_* have been designed as in the previous case. They have McMillan degree equal to 4. The closed loop identification bound in this design case must be more strict. The bound has the same shape as in the previous case but now it is approximately 0.5% of the stability margin in the high frequency region. This value is lowered also by the additional constraint that we want to select a constant γ_ω in order to use parameterisation (26) without this additional constraint we could have assumed a bound up to 3% of the stability margin. Assuming this uncertainty we could find valid values for all the design parameters. We selected $\alpha_\omega = 0.4$ (for this values of α_ω the change from C_0 to C_* indeed is not a safe change), $\gamma_\omega = 0.93$ and $c_\omega = 0.001$. The choice of γ_ω is illustrated in Fig. 4-a. The parameter η_ω was selected slightly smaller than its upper bound in order to force a higher value in the right hand side of (6). The parameterisation (26) was considered, also in this case $U = -0.9$ gave the best performance achievement on the range $[-1, 1]$ (in this case however performance on the real closed loop was tested). By selecting $U = -0.9$, we obtained $T_{*,1}$ with McMillan degree equal to 14 and the corresponding $C_{*,1}$ with McMillan degree equal to 10. In this case, by reducing the controller $C_{*,1}$ we could find a controller C_1 with McMillan degree equal to 6 satisfying the Assumption A.3 . The achieved improvement on the real plant in displayed in Fig. 4-b.



(a) (b)

Fig. 3. **(a)** The chordal distances $\kappa(C_0, C_*, \omega)$ (dash), $\kappa(C_0, C_1, \omega)$ (dash dot) and the safety bound $(\alpha_\omega - \varepsilon_\omega)\ \rho(P_0, C_0, \omega)$ (dash dot bold). **(b)** The distances $\bar{\sigma}\,(T(P_0, C_0) - T_*, \omega)$ (dash bold), $\bar{\sigma}\,(T(P_0, C_1) - T_*, \omega)$ (dash), and the distances $\bar{\sigma}\,(T(P, C_0) - T_*, \omega)$ (dash dot bold) and $\bar{\sigma}\,(T(P, C_1) - T_*), \omega)$ (dash dot).



(a) (b)

Fig. 4. **(a)** Upper bound (dash), lower bound (dash dot) and the actual value of γ_ω (continuous). **(b)** The stability margins $\rho(P_0, C_0, \omega)$ (dot), $\rho(P, C_1, \omega)$ (dash-dot) and the guaranteed stability margin $(1 - \alpha_\omega)\rho(P_0, C_0, \omega)$ (dash-dot bold). The distance $\bar{\sigma}\,(T(P, C_0) - T_*, \omega)$ (continuous bold), the guaranteed bound on performance improvement $\frac{\eta_\omega}{\bar{\eta}_\omega}\bar{\sigma}\,(T(P, C_0) - T_*, \omega)$ (dash), and $\bar{\sigma}\,(T(P, C_1) - T_*), \omega)$ (continuous).

8 Conclusions

In this paper, we have proposed an approach to the design of safe controller changes which is based on the use of closed loop models. In our approach we assume some known bounds on the error between the modelled closed loop and the actual closed loop. We have shown that safe controller changes can

be obtained as the solution of a suitable model reference control problem. A practical procedure to construct the reference model through a particular parameterisation of the reference model has also been provided. The choice of the optimal parameters in the parameterisation of the reference model and the extension of this procedure to more general parameterisations will be the objective of future work.
In the paper, a number of identification assumptions has been stated. These assumptions can be guidelines for the design of the identification experiment.

References

[1] B.D.O. Anderson, T.S. Brinsmead, F. De Bruyne, J. Hespana, D. Liberzon, and A.S. Morse. Multiple model adaptive control. Part 1: Finite controller coverings. *International Journal of Robust and Nonlinear Control (Special Issue: George Zames Commemorative Issue.)*, 10(10-11):909–929, 2000.

[2] B.D.O. Anderson, T.S. Brinsmead, D. Liberzon, and A.S. Morse. Multiple model adaptive control with safe switching. *International Journal of Adaptive Control and Signal Processing*, 15(5):445–470, 2001.

[3] B.D.O. Anderson and M. Gevers. Fundamental Problems in Adaptive Control. In *Perspectives in Control, Theory and Applications*, pages 9–21. D. Normand-Cyrot (Ed), Springer-Verlag, 1998.

[4] A. Dehghani, A. Lanzon, and B.D.O. Anderson. An $\mathscr{H}_\infty$ algorithm for the windsurfer approach to adaptive robust control. *International Journal of Adaptive Control and Signal Processing*, 18(8):607–628, 2004.

[5] J.C. Doyle. Guaranteed margins for LQG regulators. *IEEE Trans. Automatic Control*, 23(4):756–757, 1978.

[6] M. Gevers. Modelling, Identification and Control. In *Iterative Identification and Control*, pages 3–16. P. Albertos and A. Sala (Eds.), Springer, 2002.

[7] M. Green, K. Glover, D. Limebeer, and J. Doyle. A J-spectral factorization approach to $\mathscr{H}_\infty$ control. *SIAM Journal of Control and Optimization*, 28(6):1350–1371, November 1990.

[8] J. Hespanha, D. Liberzon, A.S. Morse, B.D.O. Anderson, T.S. Brinsmead, and F. De Bruyne. Multiple model adaptive control. Part 2: switching. *International Journal of Robust and Nonlinear Control*, 11(5):479–496, 2001.

[9] L. Ljung. *System Identification: Theory for the user.* Prentice Hall, 1999.

[10] G. Obinata and B.D.O. Anderson. *Model Reduction for Control System Design.* Springer Verlag, 2000.

[11] P.M.J. Van den Hof and R. Schrama. Identification and Control - closed loop issues. *Automatica*, 31(12):1751–1770, 1995.

[12] G. Vinnicombe. *Uncertainty and Feedback - $\mathscr{H}_\infty$ loop-shaping and the ν-gap metric*. Imperial College Press, 2001.

[13] Z. Zang, R.R. Bitmead, and M. Gevers. Iterative weighted least-squares identification and weighted LQG control. *Automatica*, 31(11):1577–1594, 1995.

A Proofs

Proof of Lemma 5 By Assumption A.2 and Theorem 3, we have

$$\kappa(P, P_0, \omega) \leq \varepsilon_\omega \rho(P_0, C_0, \omega)\,. \tag{A.1}$$

Moreover, due to the "only if" implication in Theorem 1, we have that P and P_0 satisfy Condition $\mathcal{C}$. Therefore, by applying Theorem 2 and (A.1), we obtain that $\rho(P, C_0, \omega)$ satisfies

$$\rho(P, C_0, \omega) \geq \rho(P_0, C_0, \omega) - \varepsilon_\omega \rho(P_0, C_0, \omega). \tag{A.2}$$

Now, we can use this lower bound on $\rho(P, C_0, \omega)$ to compute a lower-bound to the stability margin $\rho(P, C_1, \omega)$ of the closed loop system formed by P and a new controller C_1.
Since condition (1) implies that $\kappa(C_0, C_1, \omega) < \rho(P_0, C_0, \omega)$ and since C_1 is assumed to stabilise P_0, then, again due to the "only if" implication in Theorem 1, C_0 and C_1 satisfy Condition $\mathcal{C}$.
Hence, by using again Theorem 2 and (A.2), we obtain that $\rho(P, C_1, \omega)$ satisfies

$$\rho(P, C_1, \omega) \geq (1 - \varepsilon_\omega)\, \rho(P_0, C_0, \omega) - (\alpha_\omega - \varepsilon_\omega)\ \rho(P_0, C_0, \omega) \tag{A.3}$$

from which we obtain (2). □

Proof of Theorem 6 For the safety constraint, we shall introduce an inequality that implies (1) but in which the distance between the closed loop transfer functions appears (instead of the distance between the controllers directly). This can be done by means of Theorem 3, in fact we have that (1) is implied by

$$\bar{\sigma}\left(T(P_0, C_0) - T(P_0, C_1), \omega\right) \leq (\alpha_\omega - \varepsilon_\omega)\ \rho(P_0, C_0, \omega) \quad \forall \omega\,. \tag{A.4}$$

Inequality (A.4) thus also guarantees safety. Now notice that Assumption A.3 allows us to link $\bar{\sigma}\left(T(P_0, C_0) - T(P_0, C_1), \omega\right)$ to the reference model $T_{*,1}$ prior to the design of controller C_1. In fact, we can write

$$\begin{aligned} \bar{\sigma}\left(T(P_0, C_0) - T(P_0, C_1), \omega\right) &\leq \bar{\sigma}\left(T(P_0, C_1) - T_{*,1}, \omega\right) + \bar{\sigma}\left(T_{*,1} - T(P_0, C_0), \omega\right) \\ &\leq (1 + c_\omega)\, \bar{\sigma}\left(T_{*,1} - T(P_0, C_0), \omega\right)\,. \end{aligned} \tag{A.5}$$

and we obtain that (A.4) is implied by (3). Therefore, if $T_{*,1}$ is chosen so that (3) is satisfied, then safety is guaranteed.
Now, for nominal performance improvement, we require inequality (5) to be satisfied. We shall also use Assumption A.3 in order to express quantities in terms of $T_{*,1}$. Since

$$\begin{aligned}\bar{\sigma}\left(T(P_0,C_1)-T_*,\omega\right) &\leq \bar{\sigma}\left(T(P_0,C_1)-T_{*,1},\omega\right)+\bar{\sigma}\left(T_{*,1}-T_*,\omega\right)\\ &\leq c_\omega\,\bar{\sigma}\left(T(P_0,C_0)-T_{*,1},\omega\right)+\bar{\sigma}\left(T_{*,1}-T_*,\omega\right)\end{aligned}$$

and

$$\bar{\sigma}\left(T(P_0,C_0)-T_{*,1},\omega\right)\leq \bar{\sigma}\left(T(P_0,C_0)-T_*,\omega\right)+\bar{\sigma}\left(T_*-T_{*,1},\omega\right)\ ,$$

it follows that inequality (5) is implied by (4). □

Proof of Lemma 7 Let us consider the left hand side of inequality (7). We can write

$$\bar{\sigma}(T(P,C_1)-T_*,\omega)\leq \bar{\sigma}(T(P,C_1)-T(P_0,C_1),\omega)+\bar{\sigma}(T(P_0,C_1)-T_*,\omega)\,.$$

The last term of the right hand side of the above inequality is bounded in (5). As for the first term, by using Theorems 2, 3 we have

$$\begin{aligned}\bar{\sigma}(T(P,C_1)-T(P_0,C_1),\omega) &\leq \frac{\kappa(P,P_0,\omega)}{\rho(P,C_1,\omega)\rho(P_0,C_1,\omega)}\\ &\leq \frac{\bar{\sigma}(T(P,C_0)-T(P_0,C_0),\omega)}{\rho(P,C_1,\omega)[\rho(P_0,C_0,\omega)-\kappa(C_0,C_1,\omega)]}\end{aligned}$$

Moreover, by using (1), (2) and Assumption A.2 we obtain

$$\bar{\sigma}(T(P,C_1)-T(P_0,C_1),\omega)\leq \frac{\varepsilon_\omega}{1-\alpha_\omega}\frac{1}{1-(\alpha_\omega-\varepsilon_\omega)}\frac{1}{\rho(P_0,C_0,\omega)}\,.$$

Therefore, we have

$$\begin{aligned}\bar{\sigma}\left(T(P,C_1)-T_*,\omega\right) &\leq \frac{\varepsilon_\omega}{1-\alpha_\omega}\frac{1}{1-(\alpha_\omega-\varepsilon_\omega)}\frac{1}{\rho(P_0,C_0,\omega)}\\ &\quad+\gamma_\omega\bar{\sigma}\left(T(P_0,C_0)-T_*,\omega\right)\ .\end{aligned}\tag{A.6}$$

Now, we can state that if the right hand side of (A.6) is smaller or equal to the right hand side of (7) then (7) itself is implied. This condition can be written as

$$\alpha_\omega^2-(2+\varepsilon_\omega)\alpha_\omega+(1+\varepsilon_\omega)\geq \frac{1}{\eta_\omega-\gamma_\omega}\frac{\varepsilon_\omega}{\rho(P_0,C_0,\omega)}\frac{1}{\bar{\sigma}\left(T(P_0,C_0)-T_*,\omega\right)}\quad \forall\omega$$

On the interval $\alpha_\omega\in[\varepsilon_\omega,1)$ the left hand side of the above inequality, as a function of α_ω, decreases monotonically from $(1-\varepsilon_\omega)$ to 0. If the right hand side belongs to this range of values then the inequality is satisfied for the values of α_ω given by (6). $\square$

Proof of Theorem 8 The condition for existence of $\alpha_\omega \in [\varepsilon_\omega, 1)$ satisfying (6) is that ε_ω is smaller or equal to the right hand side the inequality. This gives inequality (13). Now, for worst-case performance improvement we want also $\eta_\omega \leq \bar{\eta}_\omega^+$. Obviously, this is possible if the right hand side of (13) is less or equal $\bar{\eta}_\omega^+$. This gives inequality (12). Since γ_ω is a positive number we need that the right hand side of (12) is positive. This is exactly condition (11). $\square$

Proof of Theorem 9 From the proof of Theorem 8 we have that η_ω must satisfy inequality (13). Therefore we can choose $\eta_\omega \leq \bar{\eta}_\omega^-$ if the right hand side of (13) is smaller than $\bar{\eta}_\omega^-$. This gives inequality (15). Since γ_ω is a positive number the right hand side of (15) must be positive. This is exactly condition (14). $\square$

Proof of Theorem 10 The proof is constructive. First, note that

$$T(P_o, C_{*,1}) = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & P_o \\ C_{*,1} & I \end{bmatrix}^{-1} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{aligned} T_{*,1} = B \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & P_o \\ C_* & I \end{bmatrix}^{-1} + B \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ + (1-B) \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & P_o \\ C_o & I \end{bmatrix}^{-1} + (1-B) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Then, for $T(P_0, C_{*,1}) = T_{*,1}$, we need:

$$\begin{bmatrix} I & P_o \\ C_{*,1} & I \end{bmatrix}^{-1} = B \begin{bmatrix} I & P_o \\ C_* & I \end{bmatrix}^{-1} + (1-B) \begin{bmatrix} I & P_o \\ C_o & I \end{bmatrix}^{-1} \tag{A.7}$$

$$\Updownarrow$$

$$\begin{aligned} \begin{bmatrix} I & 0 \\ 0 & (I - C_{*,1}P_o)^{-1} \end{bmatrix} = B \begin{bmatrix} I & 0 \\ 0 & S_* \end{bmatrix} \begin{bmatrix} I & 0 \\ C_{*,1} - C_* & I \end{bmatrix} \\ + (1-B) \begin{bmatrix} I & 0 \\ 0 & S_o \end{bmatrix} \begin{bmatrix} I & 0 \\ C_{*,1} - C_o & I \end{bmatrix} \end{aligned} \tag{A.8}$$

The equivalence (A.7)⇔(A.8) follows through premultiplication by $\begin{bmatrix} I & P_o \\ 0 & I \end{bmatrix}$ and postmultiplication by $\begin{bmatrix} I & 0 \\ C_{*,1} & I \end{bmatrix}$ of statement (A.7). Then, it is clear that (17) is necessary and sufficient for statement (A.8) to hold. □

Proof of Theorem 11 The result follows from [7, Theorem 2.4]. Let us show where condition (21) comes from. On expanding the left hand side of equation (23) we get $\begin{pmatrix} |E|^2 + |F|^2 & -|F|^2 \\ -|F|^2 & |F|^2 - 1 \end{pmatrix}$ which is similar to $\begin{pmatrix} |E|^2 + |F|^2 & 0 \\ 0 & \frac{1-|E|^{-2}-|F|^{-2}}{|E|^{-2}+|F|^{-2}} \end{pmatrix}$. For equation (23) to have a solution, we must have one positive eigenvalue and one negative eigenvalue, hence the necessity for inequality (21). □