Minimal Gyrator Lossless Synthesis

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Abstract—**A synthesis procedure is presented for a lossless positive real impedance matrix $Z(s)$. The procedure, based on a choice of a suitable state-space representation for $Z(s)$, uses the minimum possible number of reactive elements and, simultaneously, the minimum possible number of gyrators.**

I. Introduction

Suppose there is prescribed an $m \times m$ rational positive real matrix $Z(s)$. Numerous procedures, see [1], have been developed for synthesizing $Z(s)$, i.e., delineating an m-port network, the impedance matrix of which is the prescribed $Z(s)$. Of particular interest are those procedures which result in a minimal number of one kind of element.

In this paper, we tackle the problem of synthesizing a *lossless* rational positive real $Z(s)$, subject to two minimality constraints: the number of reactive elements must be the minimum possible, namely $\delta[Z(s)]$ [the degree of $Z(s)$], and the number of gyrators must be the minimum possible, namely $\frac{1}{2}$ of the normal rank $[Z'(s)-Z(s)]$. That $\delta[Z(s)]$ is the minimal number of reactive elements required is a well-known result and syntheses using this number, but not incorporating the minimal gyrator constraint, are well known for lossless and lossy $Z(s)$; see [1]. That $\frac{1}{2}$ of the normal rank $[Z'(s)-Z(s)]$ gyrators is the minimum possible number is less well known. Certainly this is readily established for any constant $Z(s)$, lossless or not; also $\frac{1}{2}$ of the normal rank $[Z'(s)-Z(s)]$ has been known for some time to be a *lower bound* on the number of gyrators [2]. That the lower bound is actually achievable has only recently been established for lossless $Z(s)$ in [3] and for lossy $Z(s)$ in [4].

The lossless minimal gyrator synthesis of [3] proceeds via frequency-domain techniques and simultaneously uses the minimal number of reactive elements. The lossy synthesis of [4] draws heavily on the lossless synthesis of [3] and, in general, does not use the minimal number of reactive elements. Lossy synthesis using a minimal number of reactive elements and gyrators has yet to be achieved.

This paper differs from [3] in that the synthesis procedure here is a state-space one. The synthesis problem in essence becomes one of describing the impedance to be synthesized via state-space equations in a properly selected coordinate basis. Once such a description is obtained, synthesis is virtually immediate.

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Lossless state-space synthesis has been discussed in [5], where procedures are given both for general nonreciprocal synthesis, as well as for reciprocal synthesis of a symmetric impedance matrix. In a sense, the methods of this paper constitute a refinement of those of [5].

An outline of the paper is as follows. In Section II we indicate a preliminary simplification of the synthesis problem. In Section III we review material from [5] and present the main problem in state-space terms, and in Section IV the problem is solved and the solution procedure summarized. The technique for solution rests on an interesting result describing a property of pairs of skew matrices, proved in the Appendix. Section V contains conclusions.

We remind the reader that solution of the impedance matrix synthesis problem effectively includes solution of the scattering matrix synthesis problem and the hybrid matrix synthesis problem. At times, therefore, we shall talk in terms of these other matrices without loss of generality.

II. Preliminary Points

First, observe that in the problem of synthesizing a prescribed $m \times m$ positive real $Z(s)$, we can assume without loss of generality that 1) $Z(\infty)$ is finite and nonsingular, and 2) that $Z(s)$ has no pole at the origin. If this is not the case, a multiport Cauer extraction process, as set out in [1], will produce a new $Z(s)$; call it $\hat{Z}(s)$, with these properties, or a constant $\hat{Z}(s)$. In the latter instance, minimal gyrator synthesis is trivial. In the first instance, a minimal reactive element, minimal gyrator synthesis (in brief, a minimal synthesis) of $\hat{Z}(s)$ yields a minimal synthesis for $Z(s)$.

The second point we make relates to the synthesis of constant hybrid matrices. Suppose M is a hybrid matrix describing an m-port via

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = M \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} \tag{1}$$

with v_1 and i_1 voltage and current m_1-vectors, respectively, and v_2 and i_2 voltage and current m_2-vectors, respectively, with $m_1+m_2=m$. Then the minimum number of gyrators in a synthesis of M is $\frac{1}{2}$ the rank $(\Sigma M - M'\Sigma)$ where $\Sigma = I_{m_1} \oplus -I_{m_2}$. (Here, $\oplus$ denotes the direct sum operation.)

III. The State-Space Minimal Synthesis Problem

Let $Z(s)$ be an $m \times m$ lossless impedance matrix with $Z(\infty)$ finite and let $\{F, G, H, J\}$ be a quadruple of matrices constituting a minimal realization of $Z(s)$ in

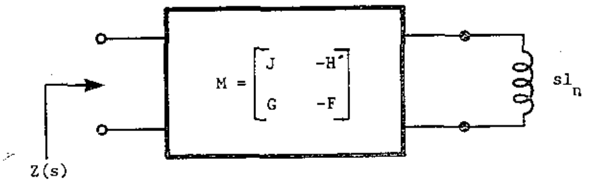

Fig. 1. Synthesis of $Z(s)$ from a synthesis of M.

the sense that

$$Z(s) = J + H'(sI - F)^{-1}G \tag{2}$$

with F of minimal dimension, say, n. Such a quadruple is easily found from $Z(s)$; see [7]. In tackling the synthesis problem, we can, as noted in Section II, assume that J is nonsingular (because $J = Z(\infty)$, which can be taken nonsingular) and that F is nonsingular (because no element of $Z(s)$ need possess a pole at $s=0$). In addition, we are justified in assuming in the first place that $Z(\infty)$ is finite.

Consider now the matrix

$$M = \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix}. \tag{3}$$

Suppose M can be synthesized as the impedance matrix of a nondynamic lossless network. Then a synthesis of $Z(s)$ would follow by terminating the last n-ports of this network in unit inductors (see Fig. 1). (The calculation is easily done; see [5].) More generally, suppose that M is the hybrid matrix of an $(m+n)$-port nondynamic lossless network, with the first $m+n_1$-ports current excited and the last n_2 $(=n-n_1)$-ports voltage excited. A synthesis of $Z(s)$ would follow by terminating ports $m+1$ through $m+n_1$ in unit inductors, and ports $m+n_1+1$ through $m+n$ in unit capacitors.

These syntheses would use n-reactive elements; this is the minimum possible because n is the dimension of F and F, being minimal, has dimension equal to $\delta[Z(s)]$. However, validity of the synthesis procedures rests on M being synthesizable by a nondynamic lossless network, or equivalently, on M being skew.

Next, for a synthesis of $Z(s)$ achieved by this method to use the minimal number of gyrators, we should require the number of gyrators used in the synthesis of M to be $\frac{1}{2}$ of the normal rank $[Z(s) - Z'(s)]$. The minimum number of gyrators used in the synthesis of M is, as noted in Section II,

$$g = \tfrac{1}{2} \text{ rank } [\Sigma M - M'\Sigma] \tag{4}$$

where

$$\Sigma = I_m \oplus I_{n_1} \oplus (-I_{n_2}). \tag{5}$$

Also,

$$\tfrac{1}{2}m \geq \tfrac{1}{2} \text{ normal rank } [Z(s) - Z'(s)]$$
$$\geq \tfrac{1}{2} \text{ normal rank } [Z(\infty) - Z'(\infty)]$$

and

$$\tfrac{1}{2} \text{ normal rank } [Z(\infty) - Z'(\infty)] = \tfrac{1}{2} \text{ rank } J = \tfrac{1}{2}m.$$

The condition for minimal gyrator synthesis therefore becomes

$$\text{rank } [\Sigma M - M'\Sigma] = \text{rank } J = m. \tag{6}$$

Combining these ideas together, we can state the main problem as follows.

Algebraic Statement of the Minimal Synthesis Problem

Let $Z(s)$ be an $m \times m$ lossless positive real matrix of degree n with $Z(\infty)$ finite and nonsingular and with no element of $Z(s)$ possessing a pole at $s=0$. Find a minimal realization $\{F, G, H, J\}$ of $Z(s)$ such that

$$M = \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix} \tag{3}$$

is skew and such that rank $[\Sigma M - M'\Sigma] = $ rank $J = m$, where

$$\Sigma = I_m \oplus I_{n_1} \oplus (-I_{n_2}) \tag{5}$$

for some nonnegative integers n_1 and $n_2 = n - n_1$.

Using the approach of [5], we can immediately ensure satisfaction of the skew constraint on M.

Proposition 1: Let $Z(s)$ be an $m \times m$ lossless impedance matrix with $Z(\infty) < \infty$. Among the minimal realizations $\{F, G, H, J\}$ of $Z(s)$ there exist realizations for which M in (12) is skew or

$$F + F' = 0 \qquad G = H \qquad J + J' = 0. \tag{7}$$

Proof: Let $\{F_1, G_1, H_1, J\}$ be an arbitrary minimal realization of $Z(s)$. Because $Z(s)$ is lossless positive real, there exists a positive definite symmetric P (see [8]) such that

$$PF_1 + F_1'P = 0 \qquad PG_1 = H_1. \tag{8}$$

(Also $J + J' = 0$.) The matrix P is readily computable from (8). Let T be any matrix such that $T'T = P$. Define a new minimal realization of $Z(s)$ by

$$F = TF_1T^{-1} \qquad G = TG_1 \qquad H = (T^{-1})'H_1. \tag{9}$$

Then (8) and the definitions of T, F, G, and H immediately imply (7).

The use of this proposition brings us to the point where M is skew, but most probably rank $(\Sigma M - M'\Sigma) \neq m$ for some suitable Σ.

Now notice that if V is any real orthogonal matrix, a minimal realization of $Z(s)$ is provided by $\{V'FV, V'G, V'H = V'G, J\}$ and the M-matrix associated with the transformed realization, call it $\hat{M}$, is still skew. This means that we can seek to achieve the rank constraint on rank $(\Sigma\hat{M} - \hat{M}'\Sigma)$ by suitable choice of V, without losing the skew constraint on $\hat{M}$. In formal terms, the problem is as follows.

Algebraic Restatement of the Minimal Synthesis Problem

With conditions on $Z(s)$ as before, let $\{F, G, H, J\}$ be a minimal realization satisfying (7). Find an $n \times n$ orthogonal matrix V such that with

$$\hat{M} = \begin{bmatrix} J & -G'V \\ V'G & -V'FV \end{bmatrix} \tag{10}$$

rank $(\Sigma\hat{M} - \hat{M}'\Sigma) = \text{rank } J = m$, where Σ is defined by (5) for some nonnegative n_1 and n_2 satisfying $n_1 + n_2 = n$.

We show how to find V in Section IV.

IV. Solution of the Synthesis Problem

The key to solving the synthesis problem lies in the following result. A constructive proof will be found in the Appendix.

Proposition 2: Let S_A and S_B be two real skew matrices of dimension $2r \times 2r$. Then there exists a real orthogonal V such that

$$V'S_AV = \begin{bmatrix} 0_{r\times r} & \hat{S}_{A12} \\ -\hat{S}_{A12}' & \hat{S}_{A22} \end{bmatrix}$$

$$V'S_BV = \begin{bmatrix} \hat{S}_{B11} & \hat{S}_{B12} \\ -\hat{S}_{B12}' & 0_{r\times r} \end{bmatrix}. \tag{11}$$

With F, G, H, and J as described in the subsection Algebraic Restatement of the Minimal Synthesis Problem, we shall take in Proposition 4:

$$S_A = GJ^{-1}G' - F \qquad S_B = F \qquad r = \frac{n}{2}\cdot \tag{12}$$

(Notice that the dimension n of F must be even, since F is both skew and nonsingular.) The main result is as follows.

Proposition 3: With quantities as defined above, the matrix

$$\hat{M} = \begin{bmatrix} J & -G'V \\ V'G & -V'FV \end{bmatrix} \tag{10}$$

is such that rank $(\Sigma\hat{M} - \hat{M}'\Sigma) = \text{rank } J = m$, where

$$\Sigma = I_m \oplus I_r \oplus -I_r, \qquad r = \frac{n}{2}\cdot \tag{13}$$

Proof: Define $\hat{F}$ and $\hat{G}$ by $\hat{F} = V'FV$ and $\hat{G} = V'G$. Partition $\hat{F}$ and $\hat{G}$ into submatrices with $r = n/2$ rows as

$$\hat{F} = \begin{bmatrix} \hat{F}_{11} & \hat{F}_{12} \\ -\hat{F}_{12}' & \hat{F}_{22} \end{bmatrix} \qquad \hat{G} = \begin{bmatrix} \hat{G}_1 \\ \hat{G}_2 \end{bmatrix}. \tag{14}$$

The fact that the top-left $r \times r$ submatrix of $V'S_AV$ and the bottom-right $r \times r$ submatrix of $V'S_BV$ are zero implies

$$\hat{G}_1J^{-1}\hat{G}_1' - \hat{F}_{11} = 0 \qquad \hat{F}_{22} = 0.$$

Accordingly,

$$\hat{M} = \begin{bmatrix} J & -\hat{G}_1' & -\hat{G}_2' \\ \hat{G}_1 & -\hat{G}_1J^{-1}\hat{G}_1' & -\hat{F}_{12} \\ \hat{G}_2 & \hat{F}_{12}' & 0 \end{bmatrix}.$$

It follows that

$$\Sigma\hat{M} - \hat{M}'\Sigma = 2\begin{bmatrix} J & -\hat{G}_1' \\ \hat{G}_1 & -\hat{G}_1J^{-1}\hat{G}_1' \end{bmatrix} \oplus 0_r$$

$$= 2\begin{bmatrix} I \\ \hat{G}_1J^{-1} \end{bmatrix} J[I \quad -J^{-1}\hat{G}_1'] \oplus 0_r$$

and this matrix evidently has rank J.

A summary of the synthesis procedure is as follows.

1) Carry out preliminary simplifications to reduce the synthesis problem to one of synthesizing a lossless positive real $Z(s)$ with $Z(\infty)$ finite and nonsingular and no element of $Z(s)$ possessing a pole at $s=0$.

2) Find a minimal realization $\{F_1, G_1, H_1, J\}$ of $Z(s)$.

3) Solve the equations $PF_1 + F_1'P = 0$, $PG_1 = H_1$ for P, and find a matrix T such that $T'T = P$. Define a new minimal realization $\{F, G, H, J\}$ for $Z(s)$ via $F = TF_1T^{-1}$, $G = TG_1$, $H = (T^{-1})'H_1$.

4) Find an orthogonal matrix V such that $V'(GJ^{-1}G' - F)V$ and $V'FV$ have $r \times r$ zero blocks in, respectively, the top-left and bottom-right corners. Here $r = n/2$, where n is the dimension of F.

5) Synthesize the lossless hybrid matrix $\hat{M}$ of (10) with rank J gyrators, terminate ports $(m+1)$ through $m + n/2$ in unit inductors, and ports $m + n/2 + 1$ through $m+n$ in unit capacitors. A minimal gyrator synthesis of $Z(s)$ results.

V. Conclusions and Remarks

The correct way to view the synthesis procedure is merely to regard it as the selection of an appropriate coordinate basis for a state-space realization of $Z(s)$. In this coordinate basis, the matrices of the state-space realization must, as analysis shows, possess certain properties. Given an arbitrary realization, one is faced with the problem of transforming the coordinate basis to ensure satisfaction of these properties.

In this paper, this transformation is done in two steps. At the first step, the property is obtained which reflects the lossless character of the nondynamic network with hybrid matrix M. At the second step, the minimal gyrator property is obtained; at the same time, the lossless property is retained.

There are two significant variants on the procedure we have discussed. First one could at any stage of the preliminary procedures expand $Z_i(s)$ in the form

$$Z_i(s) = J + \frac{A_0}{s} + \sum_{i=1}^{p} \frac{A_is + B_i}{s^2 + \omega_i^2} + \sum_{i=p+1}^{q} \frac{A_is}{s^2 + \omega_i^2}$$

with $B_1, B_2, \cdots, B_p$ nonzero. One can then reduce the problem of giving a minimal synthesis of $Z_i(s)$ to one of giving a minimal synthesis of

$$Z_{i+1}(s) = J + \sum_{i=1}^{p} \frac{A_i s + B_i}{s^2 + \omega_i^2}$$

which has a lower degree. (A minimal synthesis of $Z_i(s)$ will follow by series connection of a minimal synthesis $Z_{i+1}(s)$ and

$$\sum_{i=p+1}^{q} A_i s(s^2 + \omega_i^2)^{-1}$$

with the latter possessing a reciprocal synthesis [1].)

Second, it is not essential to eliminate a pole at the origin in any element of $Z(s)$. If such a pole is present, the F-matrix in any minimal realization of $Z(s)$ will have an odd dimension; then a variant on Proposition 2, discussed in the Appendix, may be used. Discussion was avoided in the text to restrict the bookkeeping of indices.

Finally, we remark on the almost obvious: lossy state-space synthesis with a minimum number of gyrators has yet to be achieved, as does any lossy synthesis, frequency-domain or state-space, which uses both the minimum number of reactive elements and the minimum number of gyrators.

Appendix

Our main aim here is to prove the following result.

Proposition 2: Let S_A and S_B be two real skew matrices of dimension $2r \times 2r$. Then there exists a real orthogonal V such that

$$V'S_A V = \begin{bmatrix} 0_{r\times r} & \hat{S}_{A12} \\ -\hat{S}_{A12}' & \hat{S}_{A22} \end{bmatrix}$$

$$V'S_B V = \begin{bmatrix} \hat{S}_{B11} & \hat{S}_{B12} \\ -\hat{S}_{B12}' & 0_{r\times r} \end{bmatrix}. \quad \text{(A-1)}$$

The theorem will be proved in several stages. First we shall indicate a procedure for constructing a set of $2r$-dimensional vectors x_i, $i = 1, 2, \cdots, 2r$. Then we shall show the set is orthonormal. Finally, we shall show that V can be constructed from the x_i so that (A-1) holds. Equation (A-9) is the formula for V.

The Constructive Procedure: We attempt to define a sequence of $2r$-vectors as follows:

$$x_1 = \text{arbitrary vector of unit length} \quad \text{(A-2)}$$

$$x_2 = \alpha_{20} S_A x_1 \quad \text{(A-3)}$$

where α_{20} is a nonzero constant chosen to normalize $S_A x_1$.

$$x_3 = \alpha_{30} S_B x_2 + \alpha_{31} x_1 \quad \text{(A-4)}$$

where α_{30} ($\neq 0$) and α_{31} are chosen so that x_3 has unit length and is orthogonal to x_1. (A Gram–Schmidt procedure is applied to $S_B x_2$.)

$$x_4 = \alpha_{40} S_A x_3 + \alpha_{42} x_2. \quad \text{(A-5)}$$

(A Gram–Schmidt procedure is applied to $S_A x_3$, so that x_4 has unit length and is orthogonal to x_2.) More generally,

$$x_{2i+1} = \alpha_{2i+1,0} S_B x_{2i} + \alpha_{2i+1,1} x_1 + \alpha_{2i+1,3} x_3 + \cdots + \alpha_{2i+1,2i-1} x_{2i-1}, \qquad \alpha_{2i+1,0} \neq 0 \quad \text{(A-6)}$$

$$x_{2i+2} = \alpha_{2i+2,0} S_A x_{2i+1} + \alpha_{2i+2,2} x_2 + \alpha_{2i+2,4} x_4 + \cdots + \alpha_{2i+2,2i} x_{2i}, \qquad \alpha_{2i+2,0} \neq 0. \quad \text{(A-7)}$$

Here, of course, x_{2i+1} is of unit length and is orthogonal to $x_1, x_3, \cdots, x_{2i-1}$, being derived from $S_B x_{2i}$ via a Gram–Schmidt procedure; similar remarks hold for x_{2i+2}.

The only way the above procedure will fail is if for some i, $S_B x_{2i}$ is a linear combination of $x_1, \cdots, x_{2i-1}$ or $S_A x_{2i+1}$ is a linear combination of $x_2, \cdots, x_{2i}$. If this occurs, x_{2i+1} or x_{2i+2} as the case may be is taken to be any vector orthonormal to all previously selected x_j.

Lemma 1: Suppose the x_i are defined as above. Then they constitute an orthonormal set.

Proof: Clearly each x_i has unit length and clearly $x_1, x_3, \cdots, x_{2r-1}$ forms an orthonormal set, as does $x_2, x_4, \cdots, x_{2r}$. It is therefore necessary and sufficient to show that x_{2i+1} and x_{2j} are perpendicular for all i and j. This follows by induction. Clearly x_1 and x_2 are perpendicular because S_1 is skew (or, if $S_1 x_1 = 0$, because x_2 is chosen perpendicular to x_1). Suppose that it has been shown that for all $i, j \leq k$, x_{2i+1} and x_{2j} are orthogonal. We shall show that x_{2k+2} is perpendicular to $x_1, x_3, \cdots, x_{2k+1}$ and that x_{2k+3} is perpendicular to $x_2, x_4, \cdots, x_{2k+2}$. This will mean that for all $i, j \leq k+1$, x_{2i+1} and x_{2j} are perpendicular and constitute the recursive step of the inductive proof.

Either x_{2k+2} is given by a formula of the type (A-7) or it is selected *a priori* orthogonal to $x_1, x_2, \cdots, x_{2k+1}$. In the latter case there is nothing to prove. Suppose therefore that

$$x_{2k+2} = \alpha_{2k+2,0} S_A x_{2k+1} + \alpha_{2k+2,2} x_2 + \cdots + \alpha_{2k+2,2k} x_{2k}. \quad \text{(A-8)}$$

Then for all $i \leq k$, the inductive hypothesis yields

$$x_{2i+1}' x_{2k+2} = \alpha_{2k+2,0} x_{2i+1}' S_A x_{2k+1}.$$

If $i = k$, this is zero by the skew nature of S. If $i < k$ and x_{2i+2} is determined via a formula of the type (A-7), then

$$\begin{aligned} x_{2i+1}' x_{2k+2} &= -\alpha_{2k+2,0} x_{2k+1}' S_A x_{2i+1} \\ &= -\frac{\alpha_{2k+2,0}}{\alpha_{2i+2,0}} x_{2k+1}' [x_{2i+2} - \alpha_{2i+2,2} x_2 - \cdots - \alpha_{2i+2,2i} x_{2i}] \\ &= 0 \end{aligned}$$

by the inductive hypothesis. If x_{2i+2} is not determined via a formula of the type (A-7), then $S_A x_{2i+1}$ is a weighted linear combination of $x_2, x_4, \cdots, x_{2i}$ and the inductive hypothesis again yields $x_{2i+1}'x_{2k+2}=0$. Consequently, in all cases, x_{2k+2} is orthogonal to $x_1, x_3, \cdots, x_{2k+1}$. In a similar way, x_{2k+3} is shown to be orthogonal to $x_2, x_4, \cdots, x_{2k+2}$.

Lemma 2: The vector $S_B x_{2r}$ is normal to $x_2, x_4, \cdots, x_{2r-2}, x_{2r}$.

The proof follows by minor modification of the proof of Lemma 1.

Proof of Proposition 2: With the x_i chosen according to the constructive procedure, we define

$$V = [x_1\, x_3 \cdots x_{2r-1}\, x_2\, x_4 \cdots x_{2r}]. \qquad \text{(A-9)}$$

Certainly V is orthogonal. Observe also that

$$S_A V = [S_A x_1\, S_A x_3 \cdots S_A x_{2r-1}\, S_A x_2 \cdots S_A x_{2r}].$$

Now it follows that for some scalar constants

$$S_A x_{2i+1} = \sum_{j=1}^{i+1} \gamma_{ij} x_{2j}$$

if x_{2i+2} is given by (A-7) and by the same formula with $\gamma_{i,i+1}=0$ otherwise. Hence, for suitably defined $\hat{S}_{A12}$ and $\hat{S}_{A22}$,

$$S_A V = \left[\gamma_{01} x_2 \;\; \sum_{j=1}^{2} \gamma_{ij} x_{2j} \cdots \sum_{j=1}^{r} \gamma_{r-1,j} x_{2j}\; S_A x_2 \cdots S_A x_{2r}\right]$$
$$= [x_1\, x_3 \cdots x_{2r-1}\, x_2 \cdots x_{2r}] \begin{bmatrix} 0 & \hat{S}_{A12} \\ -\hat{S}_{A12}' & \hat{S}_{A22} \end{bmatrix}$$

from which the first equation in (A-1) follows. The second follows in virtually the same way, except that use needs to be made of the fact that

$$S_B x_{2r} = \sum_{j=0}^{r-1} \delta_{rj} x_{2j+1}.$$

This follows from Lemma 2.

Two other points related to Proposition 2 are worth noting. First if S_A and S_B have dimension $2r+1$, a similar result holds with

$$V'S_A V = \begin{bmatrix} 0_{(r+1)\times(r+1)} & \hat{S}_{A12} \\ -\hat{S}_{A12}' & \hat{S}_{A22} \end{bmatrix}$$
$$V'S_B V = \begin{bmatrix} \hat{S}_{B11} & \hat{S}_{B12} \\ -\hat{S}_{B12}' & 0_{r\times r} \end{bmatrix}.$$

Second, the proposition provides a very quick proof of the result that for two real skew matrices S_A and S_B of the same dimensions, all eigenvalues of $S_A S_B$ are of even multiplicity, except perhaps the zero eigenvalue [9]. When S_A and S_B are of dimension $2r$, we have

$$V'S_A S_B V = V'S_A V V' S_B V = \begin{bmatrix} -\hat{S}_{A12}\hat{S}_{B12}' & 0_{r\times r} \\ C & -\hat{S}_{A12}'\hat{S}_{B12} \end{bmatrix}$$

where C is an unimportant matrix. The eigenvalues of $S_A S_B$ are then the eigenvalues of $-\hat{S}_{A12}\hat{S}_{B12}'$ and of $-\hat{S}_{A12}'\hat{S}_{B12}$. The latter are the eigenvalues of the transpose $-\hat{S}_{B12}'\hat{S}_{A12}$, which are also the eigenvalues of $-\hat{S}_{A12}\hat{S}_{B12}'$, the two matrices of the product being square, see [10, p. 95]. A minor modification holds in case S_A and S_B have odd dimension.

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