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The Reduced Hermite Criterion with Application to Proof of the Liénard–Chipart Criterion

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***Abstract*—A new proof of the Liénard–Chipart criterion is given that is based on the reduced Hermite criterion. It is proved from first principles.**

I. Introduction

Let $F(s)$ be a real polynomial in a variable s of degree n:

$$F(s) = \sum_{k=0}^{n} a_k s^k, \qquad a_n > 0. \tag{1}$$

Tests to determine whether all zeros of $F(s)$ have negative real parts are numerous. Among these are the Hurwitz test [1]–[3] and its modification, the Liénard–Chipart test [3], [4]. Both tests involve the Hurwitz determinants $\Delta_1, \Delta_2, \cdots, \Delta_n$, which are the successive principal minors of the $n \times n$ Hurwitz matrix

$$H = \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ a_n & a_{n-2} & a_{n-4} & \cdots \\ 0 & a_{n-1} & a_{n-3} & \cdots \\ 0 & a_n & a_{n-2} & \cdots \\ 0 & 0 & a_{n-1} & \cdots \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \end{bmatrix}. \tag{2}$$

Here, $a_i = 0$ if $i > n$ or $i < 0$.

The Hurwitz result states that $F(s)$ has all its zeros with negative real parts (in brief, $F(s)$ is Hurwitz) if and only if $\Delta_i > 0$, $i = 1,2,\cdots,n$. The Liénard–Chipart criterion says that $F(s)$ is Hurwitz if and only if any one of the four following sets of inequalities hold:

$$\Delta_1 > 0,\ \Delta_3 > 0,\ \cdots,\ a_0 > 0,\ a_2 > 0,\ a_4 > 0\ \cdots \tag{3}$$

$$\Delta_1 > 0,\ \Delta_3 > 0,\ \cdots,\ a_0 > 0,\ a_1 > 0,\ a_3 > 0,\ \cdots \tag{4}$$

$$\Delta_2 > 0,\ \Delta_4 > 0,\ \cdots,\ a_0 > 0,\ a_2 > 0,\ a_4 > 0\ \cdots \tag{5}$$

$$\Delta_2 > 0,\ \Delta_4 > 0,\ \cdots,\ a_0 > 0,\ a_1 > 0,\ a_3 > 0,\ \cdots. \tag{6}$$

This statement of the criterion, drawn from [3], extends slightly beyond the original statement of [4] in that [4] requires $a_i > 0$ for all i in each condition.

We aim to give a new proof of the Liénard–Chipart result. This proof will follow from a modification of another stability criterion —the Hermite criterion [3], [5]. We term this modification the reduced Hermite criterion; it is of independent interest.

Other material dealing with the Liénard–Chipart criterion and the reduced Hermite criterion can be found in [6]–[13]. In [6] and [7], it is argued that the two criteria are the same, given all $a_i > 0$. In [8], the Liénard–Chipart criterion is derived with the aid of resultants with inequalities approximately on only half of the coefficients being required; yet another derivation, using continued fractions, is contained in [9]. Using a matrix formulation of theorems involving resultants, Barnett [10] builds on [8] to get a new formulation of the Liénard–Chipart criterion. The equivalence of the reduced Hermite and Liénard–Chipart criterion is noted also in [11], and equivalence to the reduced Markov criterion of [12] is given in [13]. Finally, we note that Liénard and Chipart [4] present a result which is essentially equivalent to the reduced-order Hermite criterion, with the exception of requiring that $a_i > 0$ for all i.

The outline of the paper is as follows. The Hermite criterion and the reduced Hermite criterion are stated in Section II. In Section III we give a proof of the reduced Hermite criterion from first principles, and then we indicate in Section IV the existence of formulas connecting the Hurwitz determinants to the minors of a Hermite matrix; this immediately exhibits the equivalence of the Liénard–Chipart and reduced Hermite criteria. Section V contains concluding remarks.

II. The Hermite Criterion and the Reduced Hermite Criterion

We define the $n \times n$ Hermite matrix P associated with the polynomial $F(s)$ in (1) via

$$\begin{aligned} p_{ij} &= \sum_{k=1}^{i} (-1)^{k+i} a_{n-k+1} a_{n-i-j+k}, && j \geq i, j+i \text{ even} \\ &= p_{ji}, && j < i, j+i \text{ even} \\ &= 0, && j+i \text{ odd}. \end{aligned} \tag{7}$$

It is helpful to note the structure of this matrix by looking at two examples. For $n = 5$, one has

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$$P = \begin{bmatrix} a_5a_4 & 0 & a_5a_2 & 0 & a_5a_0 \\ 0 & -a_5a_2 + a_4a_3 & 0 & -a_5a_0 + a_4a_1 & 0 \\ a_5a_2 & 0 & a_5a_0 - a_4a_1 + a_3a_2 & 0 & a_3a_0 \\ 0 & -a_5a_0 + a_4a_1 & 0 & -a_3a_0 + a_2a_1 & 0 \\ a_5a_0 & 0 & a_3a_0 & 0 & a_1a_0 \end{bmatrix}.$$

For $n = 6$, one has

$$P = \begin{bmatrix} a_6a_5 & 0 & a_6a_3 & 0 & a_6a_1 & 0 \\ 0 & -a_6a_3 + a_5a_4 & 0 & -a_6a_1 + a_5a_2 & 0 & a_5a_0 \\ a_6a_3 & 0 & a_6a_1 - a_5a_2 + a_4a_3 & 0 & -a_5a_0 + a_4a_1 & 0 \\ 0 & -a_6a_1 + a_5a_2 & 0 & a_5a_0 - a_4a_1 + a_3a_2 & 0 & a_3a_0 \\ a_6a_1 & 0 & -a_5a_0 + a_4a_1 & 0 & -a_3a_0 + a_2a_1 & 0 \\ 0 & a_5a_0 & 0 & a_3a_0 & 0 & a_1a_0 \end{bmatrix}.$$

The Hermite criterion states that $F(s)$ is Hurwitz if and only if P is positive definite (in brief, $P > 0$). An immediate proof of this result has been obtained by Parks [14]; it is as follows. Define the matrix A and vector b by

$$A = \begin{bmatrix} -\frac{a_{n-1}}{a_n} & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-3}}{a_n} & \cdots & -\frac{a_0}{a_n} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & & \cdot & \\ & & & \cdot & \\ & & & \cdot & \end{bmatrix}, \quad b = \begin{bmatrix} a_{n-1} \\ 0 \\ a_{n-3} \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}. \tag{8}$$

Then

$$PA + A'P = -bb'. \tag{9}$$

If P is positive definite it follows that $x'Px$ is a Lyapunov function establishing stability (actually, asymptotic stability) of $\dot{x} = Ax$; since $\det |sI - A| = F(s)$, the Hurwitz nature of $F(s)$ is immediate. The argument is also reversible to prove the converse.

The reduced-order Hermite criterion is stated in terms of two symmetric matrices C and D defined by

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{2i-1} (-1)^{k+2i-1} a_{n-k+1} a_{n-2i-2j+k+2}, & j \geq i \\ &= c_{ji}, & i < j \end{aligned} \tag{10}$$

and

$$\begin{aligned} d_{ij} &= \sum_{k=1}^{2i} (-1)^{k+2i} a_{n-k+1} a_{n-2i-2j+k}, & j \geq i \\ &= d_{ji}, & i < j. \end{aligned} \tag{11}$$

In the summations, one takes $a_l = 0$ if l is not in the range $0,1,\cdots,n$.

Both C and D are submatrices of the Hermite matrix P, obtainable by eliminating certain rows and columns. Accordingly, as n is even or odd, C is $n/2 \times n/2$ or $(n + 1)/2 \times (n + 1)/2$ and is obtained by deleting even-numbered rows and columns from P; the matrix D is $n/2 \times n/2$ or $(n - 1)/2 \times (n - 1)/2$ and is obtained by deleting odd-numbered rows and columns from P. If $F(s)$ is a fifth-degree polynomial, one has, for example,

$$C = \begin{bmatrix} a_5a_4 & a_5a_2 & a_5a_0 \\ a_5a_2 & a_5a_0 - a_4a_1 + a_3a_2 & a_3a_0 \\ a_5a_0 & a_3a_0 & a_1a_0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} -a_5a_2 + a_4a_3 & -a_5a_0 + a_4a_1 \\ -a_5a_0 + a_4a_1 & -a_3a_0 + a_2a_1 \end{bmatrix}.$$

If $n = 6$, then

$$C = \begin{bmatrix} a_6a_5 & a_6a_3 & a_6a_1 \\ a_6a_3 & a_6a_1 - a_5a_2 + a_4a_3 & -a_5a_0 + a_4a_1 \\ a_6a_1 & -a_5a_0 + a_4a_1 & -a_3a_0 + a_2a_1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} -a_6a_3 + a_5a_4 & -a_6a_1 + a_5a_2 & a_5a_0 \\ -a_6a_1 + a_5a_2 & a_5a_0 - a_4a_1 + a_3a_2 & a_3a_0 \\ a_5a_0 & a_3a_0 & a_1a_0 \end{bmatrix}.$$

Theorem (Reduced-Order Hermite Criterion): With the previous definitions $F(s)$ is Hurwitz if and only if any one of the following four sets of inequalities holds:

$$C > 0,\ a_0 > 0,\ a_2 > 0,\ a_4 > 0,\ \cdots \tag{12}$$

$$C > 0,\ a_0 > 0,\ a_1 > 0,\ a_3 > 0,\ \cdots \tag{13}$$

$$D > 0,\ a_0 > 0,\ a_2 > 0,\ a_4 > 0,\ \cdots \tag{14}$$

$$D > 0,\ a_0 > 0,\ a_1 > 0,\ a_3 > 0,\ \cdots. \tag{15}$$

This theorem will be proved in the next section. Meanwhile, we comment on another property of the matrices C and D. The polynomial $s^nF(s^{-1})$ clearly has roots which are the reciprocals of those of $F(s)$; denote the associated C and D matrices by C_r and D_r. Then one can check that with n even, C_r is D with reversal of rows and columns, while D_r is C with reversal of rows and columns. If n is odd, then C_r is C with reversal of rows and columns, and D_r is D with reversal of rows and columns. This point will be used in proving the theorem.

Of course, just as the Liénard–Chipart criterion essentially halves the computational complexity of the Hurwitz criterion, so the reduced Hermite criterion halves the computational complexity of the Hermite criterion. This is sharply pointed up by noting that, with reordering of rows and columns, the matrix P becomes the direct sum of C and D. The Hermite criterion therefore requires that both C and D be checked for positive definiteness; the reduced criterion requires that only one be checked, together with the signs of coefficients of $F(s)$.

III. Proof of Reduced Hermite Criterion

The necessity of conditions (12)–(15) is straightforward to establish and will not be proved here. In order to prove sufficiency, it is enough to show it in the case of (12) or (13), holding that $D > 0$, and in the case of (14) or (15), holding $C > 0$. (This is because the Hermite matrix with rearranged rows and columns is, as we noted, the direct sum of C and D; its positive definiteness implies $F(s)$ is Hurwitz).

As in proofs of the Liénard–Chipart criterion in [3], it proves convenient to distinguish between the case of n even and n odd. With the possibility of any one of (12)–(15) holding, this makes eight cases to consider. The principal argument is that given for Case 1 below; the remainder of the arguments constitute minor variations or extensions.

Case 1: Assume (12) and that $n = 2m$. Define the matrix

$$A_1 = \begin{bmatrix} -\frac{a_{n-2}}{a_n} & -\frac{a_{n-4}}{a_n} & -\frac{a_{n-6}}{a_n} & \cdots & -\frac{a_0}{a_n} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & & \cdot & \\ & & & \cdot & \\ & & & \cdot & \end{bmatrix}. \tag{16}$$

It is easy to verify by direct calculation that the $i - j$ element of CA_1 is

$$-c_{i1}\frac{a_{n-2j}}{a_n}+c_{i,j+1}$$

$$= -a_n a_{n-2i+1}\frac{a_{n-2j}}{a_n} + \sum_{k=1}^{2i-1}(-1)^{k+2i-1}a_{n-k+1}a_{n-2i-2j+k}, \qquad j \geq i-1$$

$$= \sum_{k=1}^{2i}(-1)^{k+2i-1}a_{n-k+1}a_{n-2i-2j+k}, \qquad j \geq i-1$$

$$= -d_{ij}, \qquad j \geq i-1.$$

The first equality follows from (10), and the last from (11). It is similarly checked that this relation holds for $j < i-1$, and so, using the symmetry of D,

$$CA_1 = A_1'C = -D. \tag{17}$$

Since C is positive definite by assumption, it has a nonsingular square root $C^{\frac{1}{2}}$ [15], so that

$$C^{\frac{1}{2}}A_1C^{-\frac{1}{2}} = -C^{-\frac{1}{2}}DC^{-\frac{1}{2}}. \tag{18}$$

Equation (18) implies that A_1 is similar to a symmetric matrix and has all real eigenvalues. These eigenvalues are the zeros of the polynomial

$$s^{n/2}+\frac{a_{n-2}}{a_n}s^{(n/2)-1}+\cdots+\frac{a_0}{a_n}.$$

By (12), the polynomial must be positive for all nonnegative s, and so all eigenvalues of A_1 are negative real. From (18), $D > 0$ is immediate.

Case 2: Assume (14) and that $n = 2m$. $C > 0$ follows either by using (12) and the fact that the matrices C_r and D_r associated with $s^nF(s^{-1})$ are C and D with their rows reversed, or that with

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 1 \\ -\frac{a_n}{a_0} & -\frac{a_{n-2}}{a_0} & -\frac{a_{n-4}}{a_0} & \cdot & \cdot & -\frac{a_2}{a_0} \end{bmatrix} \tag{19}$$

one has $DA_2 = A_2'D = -C$, and then one argues as in Case 1.

Case 3: Assume (13) and $n = 2m+1$. In this case, set

$$A_3 = \begin{bmatrix} -\frac{a_{n-2}}{a_n} & -\frac{a_{n-4}}{a_n} & \cdot & \cdot & \cdot & -\frac{a_1}{a_n} & 0 \\ 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \end{bmatrix} \tag{20}$$

and then

$$CA_3 = A_3'C = -\left[\begin{array}{c|c} D & \begin{matrix}0\\ \cdot\\ \cdot\\ \cdot\\ 0\end{matrix} \\ \hline 0\cdots 0 & 0 \end{array}\right]. \tag{21}$$

A minor variation on earlier arguments establishes the result.

Case 4: Assume (12) and $n = 2m+1$. We define

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \\ 0 & -\frac{a_{n-1}}{a_0} & -\frac{a_{n-3}}{a_0} & & & & -\frac{a_2}{a_0} \end{bmatrix} \tag{22}$$

and then

$$CA_4 = A_4'C = -\left[\begin{array}{c|c} 0 & 0\cdots 0 \\ \hline \begin{matrix}0\\ \cdot\\ \cdot\\ \cdot\\ 0\end{matrix} & D \end{array}\right]. \tag{23}$$

The rest is easy. Alternatively, one can use Case 3 and the fact that the matrices C_r and D_r associated with $s^nF(s^{-1})$ are C and D with their rows and columns reversed.

Case 5: Assume (14) and $n = 2m+1$. Let C_ϵ and D_ϵ be the matrices associated with

$$(s+\epsilon)F(s) = a_ns^{n+1} + (a_{n-1}+\epsilon a_n)s^n + \cdots + (a_0+\epsilon a_1)s + \epsilon a_0. \tag{24}$$

It is straightforward to check that, to first order in ϵ,

$$C_\epsilon = C + \epsilon\bar{C} \tag{25}$$

and

$$D_\epsilon = \begin{bmatrix} D+\epsilon\bar{D}_{11} & \epsilon\bar{D}_{12} \\ \epsilon\bar{D}_{12}' & \epsilon a_0^2 \end{bmatrix} \tag{26}$$

where $\bar{C}$ and $\bar{D}_{ij}$ are independent of ϵ. The inequalities (14) imply for small enough ϵ that $D_\epsilon > 0$. Now using Case 2, it follows that $C_\epsilon > 0$ and then, for suitably small ϵ, that $C > 0$.

Case 6: Assume (15) and $n = 2m+1$. The desired result follows by applying Case 5 to the polynomial $s^nF(s^{-1})$.

Case 7: Assume (13) and $n = 2m$. Let C_ϵ and D_ϵ be the matrices associated with $(s+\epsilon)F(s)$. Then one finds that, to first order in ϵ,

$$C_\epsilon = \begin{bmatrix} C+\epsilon\bar{C}_{11} & \epsilon\bar{C}_{12} \\ \epsilon\bar{C}_{12}' & \epsilon a_0^2 \end{bmatrix} \tag{27}$$

and

$$D_\epsilon = D + \epsilon\bar{D}. \tag{28}$$

With suitably small ϵ, (13) implies $C_\epsilon > 0$ and then, via Case 1, $D_\epsilon > 0$. Then $D > 0$.

Case 8: Assume (15) and $n = 2m$. Apply Case 7 to the polynomial $s^nF(s^{-1})$ to conclude that $C > 0$.

IV. Connection with the Liénard–Chipart Criterion

As noted earlier, there are connections between the minors of the Hurwitz and Hermite matrices. Liénard and Chipart [4] suggest they were aware of the existence of such connections, but the first formally stated connection appears to be that of Fujiwara [16]. Formulas relating the minors appear also in [2], [7], and [17].

These extensions show that the odd-order principal minors of the Hurwitz matrix H, i.e., $\Delta_1, \Delta_3, \cdots$, are the same as the principal minors of the matrix C, while the even-order Hurwitz determinants $\Delta_2, \Delta_4, \cdots$ are the same as the principal minors of D. Accordingly, the Liénard–Chipart inequalities (3)–(6) are the same as the reduced Hermite criterion inequalities (12)–(15).

V. Conclusion

We have presented a Hermite test for checking the Hurwitz nature of a polynomial which is simpler to apply than the usual one, requiring approximately half the computational effort. In addition, the test also proves to be equivalent to the Liénard–Chipart test.

From the practical point of view, application of the reduced Hermite test requires checking the positive definiteness of a matrix of dimension $n/2 \times n/2$ or $(n \pm 1)/2 \times (n \pm 1)/2$, depending on whether n, the degree of the polynomial being examined, is even or odd. Checking of the signs of the coefficients of the polynomial is immediate, and if any are nonpositive, the polynomial is not Hurwitz. In case n is odd, it is evidently best to examine the matrix D, which has smaller dimensions than the matrix C.

Finally, we comment that the statement and proof of the main theorem could be simplified by replacing the positivity condition on half the a_i by a positivity condition on all the a_i. From the practical point of view, such a simplification is probably welcome. However, the theorem statement as given is of interest in that the number of inequalities presented are minimal.

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