Hurwitz Polynomials, *LC* and *RC* Positive Real Functions, and the Hermite Matrix

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***Abstract*—Classical results connecting Hurwitz polynomials, *LC* positive real functions, and *RC* positive real functions are all viewed as being equivalent to the positive definiteness of a symmetric matrix.**

Introduction

Let $f_1(s)$ be a real polynomial defined by

$$f_1(s) = s^n + a_1 s^{n-1} + \cdots + a_n. \tag{1}$$

For convenience, we shall assume n is even. Consider also the transfer functions

$$f_2(s) = \frac{a_1 s^{n-1} + a_3 s^{n-3} + \cdots}{s^n + a_2 s^{n-2} + \cdots} \tag{2}$$

$$f_3(s) = \frac{a_1 s^{(n/2)-1} + a_3 s^{(n/2)-2} + \cdots}{s^{n/2} + a_2 s^{(n/2)-1} + \cdots}. \tag{3}$$

It is established in a number of network theory texts, see e.g., [1], that the following three properties are equivalent.

Property 1: $f_1(s)$ is Hurwitz (i.e., its zeros lie in Re $[s]<0$).

Property 2: $f_2(s)$ is positive real and the impedance of an *LC* network.

Property 3: $f_3(s)$ is positive real and the impedance of an *RC* network, and $a_n \neq 0$.

Our aim here is to show that these three properties are all reflections of a basic matrix identity.

Algebraic Identity

Let us define a symmetric matrix P by

$$P = \begin{bmatrix} \vdots & \vdots & & & \vdots \\ \cdots & -a_9 + a_1a_8 - a_2a_7 + a_3a_6 & 0 & -a_7 + a_1a_6 & 0 \\ & 0 & a_7 - a_1a_6 + a_2a_5 & 0 & a_5 \\ & -a_7 + a_1a_6 - a_2a_5 + a_3a_4 & 0 & -a_5 + a_1a_4 & 0 \\ & 0 & a_5 - a_1a_4 + a_2a_3 & 0 & a_3 \\ & -a_5 + a_1a_4 & 0 & -a_3 + a_1a_2 & 0 \\ \cdots & 0 & a_3 & 0 & a_1 \end{bmatrix}. \tag{4}$$

(The matrix P is actually the Hermite matrix associated with the polynomial $f_1(s)$. See, e.g., [2]). Also define

$$F_2 = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & -a_4 & 0 & -a_2 & 0 \end{bmatrix} \quad g_2 = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \quad h_2 = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ 0 \\ a_2 \\ 0 \\ a_1 \end{bmatrix}. \tag{5}$$

Then it is easily checked that

$$Pg_2 = h_2 \qquad PF_2 = -F_2'P. \tag{6}$$

Equations (5) and (6) will be linked shortly with (2) and Property 2.

Further Algebraic Consequences

Let us define

$$F_1 = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix}. \tag{7}$$

We see at once that $F_1 = F_2 - g_2h_2'$, and so (6) implies

$$PF_1 + F_1'P = -2h_2h_2'. \tag{8}$$

Equation (8) will be linked shortly with (1) and Property 1.

Second, let us reorder the rows and columns of P and F_2 so that the first, third, fifth, $\cdots$, rows and columns become the first, second, third, $\cdots$, $(n/2)$th rows and columns, and the second, fourth, sixth, $\cdots$, rows and columns become the $[(n/2)+1]$th, $[(n/2)+2]$th, $\cdots$, nth. Then P and F_2 after this rearrangement have the structure

$$\begin{bmatrix} P_3^{(1)} & 0 \\ 0 & P_3^{(2)} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & I \\ F_3 & 0 \end{bmatrix}$$

where each submatrix is of dimension $(n/2)\times(n/2)$ and

$$F_3 = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & \cdot \\ \cdot & & & & & & 1 \\ \cdot & \cdot & \cdot & \cdot & -a_6 & -a_4 & -a_2 \end{bmatrix}. \tag{9}$$

Symmetry of P is translated into symmetry of $P_3^{(1)}$ and $P_3^{(2)}$.

Manuscript received August 26, 1971; revised March 6, 1972. This work was supported in part by the Australian Research Grants Committee.
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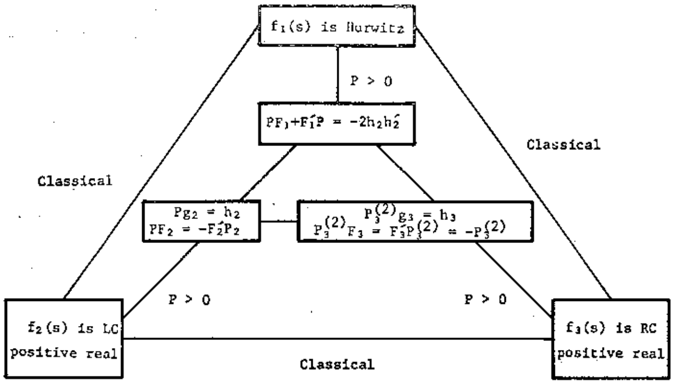

Fig. 1.

Define also the $(1/2)n$-dimensional vectors:

$$g_3 = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \qquad h_3 = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ a_5 \\ a_3 \\ a_1 \end{bmatrix}. \tag{10}$$

Then (6) becomes

$$\begin{bmatrix} P_3^{(1)} & 0 \\ 0 & P_3^{(2)} \end{bmatrix} \begin{bmatrix} 0 \\ g_3 \end{bmatrix} = \begin{bmatrix} 0 \\ h_3 \end{bmatrix}$$

$$\begin{bmatrix} P_3^{(1)} & 0 \\ 0 & P_3^{(2)} \end{bmatrix} \begin{bmatrix} 0 & I \\ F_3 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & F_3' \\ I & 0 \end{bmatrix} \begin{bmatrix} P_3^{(1)} & 0 \\ 0 & P_3^{(2)} \end{bmatrix}$$

or

$$P_3^{(2)} g_3 = h_3 \qquad P_3^{(2)} F_3 = F_3' P_3^{(2)} = -P_3^{(1)}. \tag{11}$$

Equations (9)–(11) will be linked with (3) and Property 3.

Connection of the Algebraic Identities to the Hurwitz and Positive Real Properties

To illustrate the connection between the algebraic identities and the positive real properties, we make the following claim.

Theorem: Properties 1–3 are equivalent to the following property.

Property 4: The matrix P is positive definite.

We shall prove this theorem by showing the equivalence of each of Properties 1–3 to Property 4. Fig. 1 helps illustrate the demonstrations.

Property 1⇔Property 4: This has been proved by Parks [2]. The idea is that the matrix F_1 is the companion matrix associated with the polynomial $f_1(s)$, and $x'Px$ is a Lyapunov function establishing stability of $\dot{x} = F_1 x$. By (8), the Lyapunov function has a nonpositive derivative.

Property 2⇔Property 4: This has been pointed out in [3]. One observes that $h_2'(sI - F_2)^{-1} g_2 = f_2(s)$, and then one uses the lossless positive real lemma [4], [5] which states that $f_2(s)$ is lossless positive real if and only if there exists a unique positive definite P satisfying (6).

Property 4⇒Property 3: Because of the way $P_3^{(1)}$ and $P_3^{(2)}$ are defined, each of these matrices is positive definite when P is. The equations

$$P_3^{(2)} F_3 + F_3' P_3^{(2)} = -2P_3^{(1)} \qquad P_3^{(2)} g_3 = h_3 \tag{12}$$

imply by the positive real lemma [4], [5] that $h_3'(sI - F_3)^{-1} g_3 = f_3(s)$ is positive real. The equations

$$P_3^{(2)} F_3 = F_3' P_3^{(2)} \qquad P_3^{(2)} g_3 = h_3 \tag{13}$$

imply that $f_3(s)$ is RC positive real [5]–[8]. That $a_n \neq 0$ is trivial.

Property 3⇒Property 4: The RC property and (13) imply by [5]–[8] that $P_3^{(2)}$ is positive definite. Therefore, $P_3^{(2)}$ possesses a square root and, by (11),

$$[P_3^{(2)}]^{1/2} F_3 [P_3^{(2)}]^{-1/2} = -[P_3^{(2)}]^{-1/2} P_3^{(1)} [P_3^{(2)}]^{-1/2}. \tag{14}$$

Since the right side is symmetric, we see that F_3 possesses all real eigenvalues. Positive realness and the fact that $a_n \neq 0$ imply that eigenvalues of F_3 have negative real parts, and so the eigenvalues of F_3 are negative real. By (14), the eigenvalues of $P_3^{(1)}$ are positive and so $P_3^{(1)}$, as well as $P_3^{(2)}$, is positive definite. Hence P is positive definite.

A Further Algebraic Identity

Property 4, requiring that the $n \times n$ matrix P is positive definite' can be replaced by the simpler Property 5 of the following theorem.

Theorem: Property 4 is equivalent to the following property.

Property 5: $a_i > 0$ for $i = 1, 2, \cdots, n$ and $P_3^{(2)}$ is positive definite.

Proof: Property 4 implies Property 1, which implies $a_i > 0$, and directly implies $P_3^{(2)}$ is positive definite. So, Property 4 implies Property 5. Conversely, positivity of the a_i forces $|sI - F_3|$ to have all positive coefficients; hence, F_3 has no real nonnegative eigenvalues. Following earlier remarks, F_3 has all negative real eigenvalues, whence $P_3^{(1)}$ is positive definite. Then P is positive definite, i.e., Property 5 implies Property 4.

We note that the equivalence of Properties 5 and 3 may be found in [9] and of Properties 5 and 1 in [10].

Note also that the positivity of the a_i for odd i is not actually used above. Finally, note that almost the same proof shows that Property 4 is equivalent to Property 5 with $P_3^{(2)}$ replaced by $P_3^{(1)}$.

Conclusions

The main theorem illustrates that three properties of Hurwitz polynomials and positive real functions, normally related via analysis, may be viewed as flowing from a set of algebraic relations and the positive definite nature of a certain matrix.

Variations of the results to deal with, for example, odd degree polynomials $f_1(s)$ and RL positive real functions, are very straightforward.

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Reprinted from IEEE TRANSACTIONS ON *CIRCUIT THEORY*
Volume CT-19, Number 5, September, 1972
pp. 495-496