Iterative method of computing the limiting solution of the matrix Riccati differential equation

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ABSTRACT

The paper describes an iterative algorithm for computing the limiting, or steady-state, solution of the matrix Riccati differential equation associated with quadratic minimisation problems in linear systems. It is shown that the positive-definite solution of the algebraic equation $PF + F'P - PGR^{-1}G'P + S = 0$, provided that it exists and is unique, can be obtained as the limiting solution of a quadratic matrix difference equation that converges from any nonnegative definite initial condition. The algorithm is simple, and, at least for moderate dimensions of the solution matrix, competitive in computational effort with other current techniques for obtaining the limiting solution of the Riccati equation.

1 INTRODUCTION

The bilinear transformation of complex-function theory has long been a useful tool in the analysis of linear time-invariant systems, particularly in relating stability criteria for continuous time and discrete time systems. Steiglitz[1] has shown that the sequence of operations, Laplace transformation → bilinear transformation → inverse Z transformation, establishes an explicit isomorphism between the spaces of square integrable functions and square summable sequences. In this paper, it will be shown that, by means of an application of the bilinear transformation to the state-space realisation of a transfer-function matrix, a continuous-time quadratic minimisation problem over an infinite time interval can be transformed into a discrete-time problem. More precisely, an algebraic matrix equation of the form

$$\Phi F + F'\Phi - \Phi GR^{-1}G'\Phi + S = 0 \tag{1}$$

can be transformed into the equation

$$A'\Phi A - \Phi - A'\Phi B(U + B'\Phi B)^{-1}B'\Phi A + Q = 0 \tag{2}$$

where U is positive-definite and Q positive-semidefinite if R and S are positive-definite and positive-semidefinite, respectively. It will be shown that the conditions that guarantee eqn. 1 to have a unique positive-definite solution are just those that allow this solution to be obtained as the limiting solution of the Riccati difference equation

$$\Phi(i+1) = A'[\Phi(i) - \Phi(i)B\{U + B'\Phi(i)B\}^{-1}B'\Phi(i)]A + Q$$

$$\Phi(0) = V \tag{3}$$

starting at any positive-semidefinite initial condition V.

This yields a new algorithm for computing the limiting solutions of Riccati differential equations that is simple, numerically stable and competitive in computational effort with other known methods.

Before developing the details of the method, it will be convenient in Section 2 of the paper to summarise the relevant results on the Riccati difference equation having a positive-semidefinite limiting solution, and to develop a necessary extension. The algorithm itself will be described in Section 3, followed in Section 4 by a discussion of its computational efficiency.

2 RICCATI DIFFERENCE EQUATION

It will be assumed that the reader is familiar with the discrete-time and continuous-time versions of the state-regulator problem and the Kalman filtering theory, which are the two main areas in which the Riccati difference equation, eqn. 3, arises. Detailed expositions may be found in References 2-4. In the sequel, the coefficient matrices A, B, U and Q in eqn. 3 will be assumed to be real constant matrices of dimension $n \times n$, $n \times m$, $m \times m$ and $n \times n$, respectively; U will be assumed to be positive-definite symmetric, and Q will be assumed to be positive-semidefinite symmetric. (The term 'positive semidefinite' will be taken to include 'positive definite' throughout the paper.) In contrast with most developments in the literature, the matrix A will not be assumed nonsingular. This means that the usual assumption of complete controllability of the pair [A, B] will be replaced by assuming [A, B] to be completely reachable: an integer $k_c \leq n$ exists such that the matrix

$$W_R = \sum_{i=0}^{k_c-1} (A)^i BU^{-1}B'(A')^i$$

is nonsingular. It will also be assumed that the pair $[A, Q^{1/2}]$ is completely observable, which means that, for some integer $k_0 \leq n$, the matrix

$$W_0 = \sum_{i=0}^{k_0-1} (A')^i QA^i$$

is nonsingular.

Let $\Phi_0(\cdot)$ denote the solution of eqn. 3 corresponding to $\Phi(0) = 0$. The following results on the existence and properties of a limiting solution of eqn. 3 are well known, and will be stated without proof.

Theorem 1: If the pair [A, B] in eqn. 3 is completely reachable, $\lim_{i\to\infty} \Phi_0(i) = \bar{\Phi}$ exists. If, in addition, the pair $[A, Q^{1/2}]$ is completely observable, $\bar{\Phi}$ is the unique solution Φ, which is positive-definite, of eqn. 2, with the property that the free system $x(i+1) = \{I - B(U + B'\Phi B)^{-1}B'\Phi\}Ax(i)$ is asymptotically stable.

The following theorem on the asymptotic behaviour of eqn. 3 appears to have been proved so far only for the special case when the matrix A is nonsingular.[4] This restriction, however, can be removed.

Theorem 2: If [A, B] is completely reachable, and $[A, Q^{1/2}]$ is completely observable, the solution of eqn. 3 converges to $\bar{\Phi}$, as defined in theorem 1, for all positive-semidefinite initial conditions V.

Theorem 2 can be proved by observing that $\Delta_i \triangleq \bar{\Phi} - \Phi(i)$ satisfies the equation $\Delta_{i+1} = A_i'\Delta_i A_c$, where $A_c = (I - BM^{-1}B'\bar{\Phi})A$ and $A_i = \{I - BN_i^{-1}B'\Phi(i)\}A$ with $M \triangleq U + B'\bar{\Phi}B$ and $N_i \triangleq U + B'\Phi(i)B$.

Therefore

$$\Delta_k = \psi(K, 0)\Delta_0 A_c^i \tag{4}$$

where $\psi(k, 0)$ is the transition matrix of the system

$$y(i+1) = A_i'y(i) \tag{5}$$

It can now be shown that, with $\Phi(0)$ positive-semidefinite, the scalar function $y'(i)\Phi^{-1}(i)y(i)$, $i \geq \max\{k_0, k_c\}$, is a Lyapunov function establishing the stability of the system in eqn. 5

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even when $\mathbf{A}$ is singular, so that there exists some positive constant β such that $\|\psi(K,0)\| \le \beta < \infty$ for all integers $K \ge 0$. This, together with the fact that $\|A_c\| < 1$ by theorem 1, shows, by eqn. 4, that $\lim_{K\to\infty} \|\Delta_K\| = 0$.

We point out, in passing, that, by means of theorems 1 and 2, it is not hard to show that, under the hypotheses of theorem 2, the system of eqn. 5 is, in fact, asymptotically stable, whether or not $\mathbf{A}$ is singular.

3 MAIN RESULT

Let $\mathbf{F}, \mathbf{G}, \mathbf{R}$ and $\mathbf{S}$ be a set of constant real matrices of dimensions $n \times n, n \times m, m \times m$ and $n \times n$, respectively, $\mathbf{R}$ being positive-definite symmetric and $\mathbf{S}$ being positive-semidefinite symmetric. The limiting, or steady-state solution

$$\bar{\Phi} = \lim_{t\to-\infty} P(t)$$

of the Riccati differential equation

$$-\frac{d\mathbf{P}}{dt} = \mathbf{F}'\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F} - \mathbf{P}(t)\mathbf{G}\mathbf{R}^{-1}\mathbf{G}'\mathbf{P}(t) + \mathbf{S}$$

$$\mathbf{P}(0) = \mathbf{P}'(0) \ge 0$$

is a positive-definite solution of the algebraic equation eqn. 1, whose existence and uniqueness is guaranteed whenever the pair $[\mathbf{F}, \mathbf{G}]$ is completely controllable and $[\mathbf{F}, \mathbf{S}^{1/2}]$ completely observable.[3] The proposed algorithm for the computation of $\bar{\Phi}$ is summarised in the following theorem.

Theorem 3: Let $[\mathbf{F}, \mathbf{G}]$ be completely controllable and $[\mathbf{F}, \mathbf{S}^{1/2}]$ be completely observable. Let $\bar{\Phi}$ denote the unique positive-definite solution of eqn. 1. Then, for any real $\alpha > 0$ not equal to a real positive eigenvalue of $\mathbf{F}$,

$$\bar{\Phi} = \lim_{i\to\infty} \Phi(i)$$

where $\Phi(\cdot)$ is the solution of the difference equation

$$\Phi(i+1) = \mathbf{A}'\Phi(i)\mathbf{A} - \{\mathbf{A}'\Phi(i)\mathbf{B} + \mathbf{H}\}\{\mathbf{U} + \mathbf{B}'\Phi(i)\mathbf{B}\}^{-1}\{\mathbf{B}'\Phi(i)\mathbf{A} + \mathbf{H}'\} + \mathbf{Q} \quad (6)$$

whose coefficients are related to those of eqn. 1 by

$$\mathbf{A} = (\alpha\mathbf{I} - \mathbf{F})^{-1}(\alpha\mathbf{I} + \mathbf{F}) \quad (7)$$

$$\mathbf{B} = \alpha\sqrt{2}(\alpha\mathbf{I} - \mathbf{F})^{-1}\mathbf{G} \quad (8)$$

$$\mathbf{Q} = 2\alpha(\alpha\mathbf{I} - \mathbf{F}')^{-1}\mathbf{S}(\alpha\mathbf{I} - \mathbf{F})^{-1} \quad (9)$$

$$\mathbf{H} = \tfrac{1}{2}\sqrt{2}\mathbf{Q}\mathbf{G} \quad (10)$$

$$\mathbf{U} = \alpha\mathbf{R} + \tfrac{1}{2}\mathbf{G}'\mathbf{Q}\mathbf{G} \quad (11)$$

The solution of eqn. 6 converges to $\bar{\Phi}$ from any positive-semidefinite symmetric initial condition $\Phi(0)$.

Proof: The proof proceeds in several steps. In the first, eqn. 1 will be transformed into the form of eqn. 2. It will then be shown that the conditions that guarantee the existence of a unique positive-definite solution of eqn. 1 imply that the conditions of theorems 1 and 2 hold for the transformed algebraic equation, so that its positive-definite solution can be obtained from a difference equation of the form of eqn. 3. Finally, this difference equation will be shown to be identical with eqn. 6.

By the restriction placed on α, the matrix $\mathbf{A}$ defined by eqn. 7 exists. A simple calculation shows that

$$\mathbf{F} = \alpha(\mathbf{A} - \mathbf{I})(\mathbf{A} + \mathbf{I})^{-1} \quad (12)$$

$$\mathbf{S} = 2\alpha(\mathbf{A}' + \mathbf{I})^{-1}\mathbf{Q}(\mathbf{A} + \mathbf{I})^{-1} \quad (13)$$

$$\mathbf{G} = \sqrt{2}(\mathbf{A} + \mathbf{I})^{-1}\mathbf{B} \quad (14)$$

$$\mathbf{H} = \mathbf{Q}(\mathbf{A} + \mathbf{I})^{-1}\mathbf{B} \quad (15)$$

Substituting eqns. 12, 13 and 14 into eqn. 1, defining

$$\mathbf{R}_1 = \alpha\mathbf{R} \quad (16)$$

and simplification yields

$$\mathbf{A}'\Phi\mathbf{A} + \mathbf{Q} - \Phi = (\mathbf{A}' + \mathbf{I})\Phi(\mathbf{A} + \mathbf{I})^{-1}\mathbf{B}\mathbf{R}_1^{-1}\mathbf{B}'(\mathbf{A}' + \mathbf{I})^{-1}\Phi(\mathbf{A} + \mathbf{I}) \quad (17)$$

Define

$$\mathbf{L} \triangleq (\mathbf{A}' + \mathbf{I})\Phi(\mathbf{A} + \mathbf{I})^{-1}\mathbf{B}\mathbf{R}_1^{-1/2} \quad (18)$$

and

$$\mathbf{W} \triangleq \mathbf{R}_1^{1/2} + \mathbf{L}'(\mathbf{A} + \mathbf{I})^{-1}\mathbf{B} \quad (19)$$

Then, from eqn. 17,

$$\mathbf{L}\mathbf{L}' = \mathbf{A}'\Phi\mathbf{A} - \Phi + \mathbf{Q} \quad (20)$$

and, from eqn. 18,

$$\mathbf{L}\mathbf{R}_1^{1/2} = (\mathbf{A}' + \mathbf{I})\Phi(\mathbf{A} + \mathbf{I})^{-1}\mathbf{B} \quad (21)$$

From eqn. 19,

$$\mathbf{W}'\mathbf{W} = \mathbf{R}_1 + \mathbf{B}'(\mathbf{A}' + \mathbf{I})^{-1}\mathbf{L}\mathbf{R}_1^{1/2} + \mathbf{R}_1^{1/2}\mathbf{L}'(\mathbf{A} + \mathbf{I})^{-1}\mathbf{B} + \mathbf{B}'(\mathbf{A}' + \mathbf{I})^{-1}\mathbf{L}\mathbf{L}'(\mathbf{A} + \mathbf{I})^{-1}\mathbf{B}$$

Using eqns. 20 and 21,

$$\begin{aligned}\mathbf{W}'\mathbf{W} &= \mathbf{R}_1 + \mathbf{B}'\Phi(\mathbf{A} + \mathbf{I})^{-1}\mathbf{B} + \mathbf{B}'(\mathbf{A}' + \mathbf{I})^{-1}\Phi\mathbf{B} \\ &\quad + \mathbf{B}'(\mathbf{A}' + \mathbf{I})^{-1}(\mathbf{A}'\Phi\mathbf{A} - \Phi + \mathbf{Q})(\mathbf{A} + \mathbf{I})^{-1}\mathbf{B} \\ &= \mathbf{U} + \mathbf{B}'\Phi\mathbf{B}\end{aligned} \quad (22)$$

by eqns. 11, 14 and 16.

Clearly, $\mathbf{U}$ is positive-definite if Φ is any positive-semidefinite solution of eqn. 1. Consequently, $\mathbf{W}$ may, by eqn. 22, be taken to be nonsingular. From eqn. 19,

$$\begin{aligned}\mathbf{L}\mathbf{W} &= \mathbf{L}\mathbf{R}_1^{1/2} + \mathbf{L}\mathbf{L}'(\mathbf{A} + \mathbf{I})^{-1}\mathbf{B} \\ &= \mathbf{A}'\Phi\mathbf{B} + \mathbf{H}\end{aligned}$$

by eqns. 20, 21 and 15. Hence,

$$\mathbf{L} = (\mathbf{A}'\Phi\mathbf{B} + \mathbf{H})\mathbf{W}^{-1}$$

and

$$\mathbf{L}\mathbf{L}' = (\mathbf{A}'\Phi\mathbf{B} + \mathbf{H})(\mathbf{W}'\mathbf{W})^{-1}(\mathbf{B}'\Phi\mathbf{A} + \mathbf{H}')$$

Equating this to eqn. 20 and using eqn. 22 gives

$$\Phi = \mathbf{A}'\Phi\mathbf{A} - (\mathbf{A}'\Phi\mathbf{B} + \mathbf{H})(\mathbf{U} + \mathbf{B}'\Phi\mathbf{B})^{-1}(\mathbf{B}'\Phi\mathbf{A} + \mathbf{H}') + \mathbf{Q} \quad (23)$$

Now, define

$$\mathbf{Q}_1 \triangleq \mathbf{Q} - \mathbf{H}\mathbf{U}^{-1}\mathbf{H}' \quad (24)$$

Substituting into eqn. 23, and expanding, gives

$$\begin{aligned}\Phi &= \mathbf{A}'\Phi\mathbf{A} + \mathbf{Q}_1 - \mathbf{A}'\Phi\mathbf{B}(\mathbf{U} + \mathbf{B}'\Phi\mathbf{B})^{-1}\mathbf{B}'\Phi\mathbf{A} \\ &\quad - \mathbf{H}(\mathbf{U} + \mathbf{B}'\Phi\mathbf{B})^{-1}\mathbf{B}'\Phi\mathbf{A} \\ &\quad - \mathbf{H}(\mathbf{U} + \mathbf{B}'\Phi\mathbf{B})^{-1}\mathbf{H}' \\ &\quad - \mathbf{A}'\Phi\mathbf{B}(\mathbf{U} + \mathbf{B}'\Phi\mathbf{B})^{-1}\mathbf{H}' + \mathbf{H}\mathbf{U}^{-1}\mathbf{H}' \\ &= \mathbf{A}'\Phi\mathbf{A} + \mathbf{Q}_1 - \mathbf{A}'\Phi\mathbf{B}(\mathbf{U} + \mathbf{B}'\Phi\mathbf{B})^{-1}\mathbf{B}'\Phi\mathbf{A} \\ &\quad - \mathbf{H}\mathbf{U}^{-1}\{\mathbf{I} - \mathbf{B}'\Phi\mathbf{B}(\mathbf{U} + \mathbf{B}'\Phi\mathbf{B})^{-1}\}\mathbf{B}'\Phi\mathbf{A} \\ &\quad - \mathbf{A}'\Phi\mathbf{B}\{\mathbf{I} - (\mathbf{U} + \mathbf{B}'\Phi\mathbf{B})^{-1}\mathbf{B}'\Phi\mathbf{B}\}\mathbf{U}^{-1}\mathbf{H}' \\ &\quad + \mathbf{H}\mathbf{U}^{-1}\{\mathbf{B}'\Phi\mathbf{B} - \mathbf{B}'\Phi\mathbf{B}(\mathbf{U} + \mathbf{B}'\Phi\mathbf{B})^{-1}\mathbf{B}'\Phi\mathbf{B}\}\mathbf{U}^{-1}\mathbf{H}'\end{aligned}$$

or

$$\Phi = \mathbf{A}_1'\Phi\mathbf{A}_1 - \mathbf{A}_1\Phi\mathbf{B}(\mathbf{U} + \mathbf{B}'\Phi\mathbf{B})^{-1}\mathbf{B}'\Phi\mathbf{A}_1 + \mathbf{Q}_1 \quad (25)$$

where

$$A_1 = A - BU^{-1}H' \tag{26}$$

This shows that any positive-semidefinite solution of eqn. 1 is also a solution of eqn. 25, which is of the form of eqn. 2. In particular, the positive-definite solution $\bar{\Phi}$ of eqn. 1, known to be the unique positive-definite solution of eqn. 1, is a solution of eqn. 25. However, before theorems 1 and 2 can be applied to establish that this solution is also the limiting solution of a Riccati difference equation starting at arbitrary initial conditions $\Phi(0) \geq 0$, it is necessary to show that Q_1 is positive-semidefinite, that $[A_1, B]$ is completely reachable and that $[A_1, Q_1^{1/2}]$ is completely observable.

It is well known that the complete controllability of the pair $[F, G]$ and the complete observability of $[F, S^{1/2}]$ are equivalent to the triple $[F, G, S^{1/2}]$ being a minimal realisation of the transfer-function matrix

$$W(s) = S^{1/2}(sI - F)^{-1}G \tag{27}$$

It is not difficult to show that the bilinear transformation $z = (\alpha - s)^{-1}(\alpha + s)$ transforms W(s) into

$$V(z) = S^{1/2}(\alpha I - F)^{-1}G + \sqrt{2}S^{1/2}(\alpha I - F)^{-1}(zI - A)^{-1}\alpha\sqrt{2}(\alpha I - F)^{-1}G$$
$$= \frac{1}{\sqrt{\alpha}}\left\{\frac{1}{\sqrt{2}}K'G + K'(zI - A)^{-1}B\right\} \tag{28}$$

where A and B are given by eqns. 7 and 8, respectively, and where $K' = \sqrt{(2\alpha)}S^{1/2}(\alpha I - F)^{-1}$, so that, by eqn. 9, $KK' = Q$. Now, assume that eqn. 28 does not define a minimal realisation of V(z). It is then possible to find matrices T_1, K_1, B_1 and A_1 such that

$$V(z) = T_1 + K_1'(zI - A_1)^{-1}B_1 \tag{29}$$

where A_1 is of smaller dimension than A. Applying the inverse of the bilinear transformation to eqn. 29 gives

$$W(s) = T_2 + S_1^{1/2}(sI - F_1)^{-1}G_1$$

where T_2, S_1 and G_1 are some constant matrices and where $F_1 = \alpha(A_1 - I)(A_1 + I)^{-1}$ has a smaller dimension than F, contradicting the minimality of $[F, G, S^{1/2}]$.

Therefore $[A, B]$ is completely reachable and $[A, Q^{1/2}]$ completely observable. Since $A_1 = A - BU^{-1}H'$, $[A_1, B]$ is immediately completely reachable; also, because $H = \frac{1}{2}\sqrt{2}QG$, it follows, by duality, that $[A_1, Q^{1/2}]$ is completely observable. Finally, this implies that $[A_1, Q_1^{1/2}]$ is completely observable. To see this, note that, from eqns. 24, 11 and 10,

$$x'Q_1x = x'\{Q - \tfrac{1}{2}QG(\alpha R + \tfrac{1}{2}G'QG)^{-1}G'Q\}x$$
$$= \{x' \mid -x'QG(\alpha R + G'QG)^{-1}\}\begin{bmatrix} Q & \frac{1}{\sqrt{2}}QG \\ \frac{1}{\sqrt{2}}G'Q & \alpha R + \frac{1}{2}G'QG \end{bmatrix}\begin{bmatrix} x \\ -(\alpha R + \frac{1}{2}G'QG)^{-1}G'Qx \end{bmatrix}$$
$$= \{x' \mid -x'QG(\alpha R + \tfrac{1}{2}G'QG)^{-1}\}\begin{bmatrix} I \\ \frac{1}{\sqrt{2}}G' \end{bmatrix} Q(I \; \frac{1}{\sqrt{2}}G)\begin{bmatrix} x \\ -(\alpha R + \frac{1}{2}G'QG)^{-1}G'Qx \end{bmatrix}$$
$$+ x'QG(\alpha R + \tfrac{1}{2}G'QG)^{-1}\alpha R(\alpha R + \tfrac{1}{2}G'QG)^{-1}G'Qx \tag{30}$$

Both terms on the right-hand side of eqn. 30 are evidently nonnegative. Therefore, if $Q_1x = 0$, both terms on the right-hand side are zero, which implies that $x'Qx = 0$, or $Qx = 0$. Clearly, $Qx = 0$ implies that $Q_1x = 0$. Therefore, the null spaces of Q and Q_1, and therefore of $Q^{1/2}$, and $Q_1^{1/2}$, are the same, and accordingly, $[A_1, Q_1^{1/2}]$ is completely observable, because $[A_1Q^{1/2}]$ is. Moreover, it has been shown that Q_1 is positive-semidefinite.

By theorem 1, the difference equation

$$\Phi(i+1) = A_1'[\Phi(i) - \Phi(i)B\{U + B'\Phi(i)B\}^{-1}B'\Phi(i)] \tag{31}$$
$$A_1 + Q_1$$
$$\Phi(0) = 0$$

converges to a matrix that is the unique positive-definite solution of eqn. 25, known to be $\bar{\Phi}$. Theorem 2 then guarantees that eqn. 31 converges from any positive-semidefinite initial condition.

Finally, the proof of the theorem is completed by noting that the right-hand sides of eqns. 23 and 25 were shown to be identical, with their equality not depending on the fact that each is equal to Φ. Consequently, the difference equations eqns. 6 and 31 must have the same solutions for the same initial conditions.

4 COMPUTATIONAL ASPECTS

The ease of implementation of many of the results of linear optimal-control and filtering theory depends largely on efficient algorithms for the solution of eqn. 1, and the problem has received much attention in the literature. It appears to be accepted now that a very efficient way to solve eqn. 1 is to compute the eigenvalues and half the set of eigenvectors of the system matrix M of the Hamiltonian differential system

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} F & -GR^{-1}G' \\ -S & -F' \end{bmatrix}\begin{bmatrix} x \\ p \end{bmatrix} \triangleq M\begin{bmatrix} x \\ p \end{bmatrix}$$

which is the result of a variational treatment of the optimisation problem associated with eqn. 1. If M has linearly independent eigenvectors, a nonsingular matrix T can be formed from them such that

$$T^{-1}MT = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}^{-1}\begin{bmatrix} F & -GR^{-1}G' \\ -S & -F' \end{bmatrix}\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix}$$

where Λ is a diagonal matrix whose entries have nonnegative real parts. It can then be shown that the positive-definite solution $\Phi = \bar{\Phi}$ of eqn. 1 is given by $\bar{\Phi} = T_{21}T_{11}^{-1}$. A refined version of this technique that avoids complex arithmetic is described by Fath,[7] who reports a solution time of 4 s on an IBM 360-44 computer for an example in which $n = 6$ (as before, n and m are the dimensions of F and R, respectively, in eqn. 1).

It is obviously desirable to check the effectiveness of a new algorithm by comparing the solution times for a wide range of problems with those of current methods, such as Fath's, obtained on the same computer. This we are unable to do as yet. However, our initial experience with the algorithm has been quite encouraging. A number of equations were solved on an IBM 1130 computer, and the results are summarised in Table 1. In most of the examples, S and R were identity matrices of appropriate dimension, and F and G were in the 'control canonical' form of Anderson and Luenberger,[8] in which F has $p \leq m$ blocks of companion matrix form centred on the main diagonal, the blocks being coupled by some nonzero entries below the diagonal. The eigenvalues of F were within a radius of 8 in all cases. In Table 1, k is the

TABLE 1

SOLUTION TIMES FOR THE ITERATIVE ALGORITHM ALGORITHM OF THEOREM 3

Number of equations	n	m	k	Execution time
				s
25	6	1	10-62, 27 av.	65 average
1	8	2	39	210
1	8	2	29	160
1	12	3	24	280
1	12	4	32	460
1	20	4	44	44 × 60

number of iterations of the difference equation, eqn. 6, required for the norm of the solution matrix to converge to an accuracy of five significant figures, the solution starting from an initial value $\Phi(0) = 0$. The norm was taken as the sum of the absolute values of the elements so that the examples with larger n are somewhat penalised by this convergence criterion. The execution times quoted were obtained on an IBM 1130 without hardware floating-point arithmetic and quite a slow printer. Bearing in mind that the speeds of the IBM 360-44 and the IBM 1130 differ by a factor of the order of 25 for problems with a high content of floating-point arithmetic with subscripted variables, the average execution time for our 6th-order examples should be about 2 s on the faster machine, and this compares favourably with the 4 s quoted by Fath for his example.

The problems of Table 1 were all solved with a value of unity for the scalar α in theorem 3, and no attempt was made to influence the convergence rate of the difference equation by a suitable choice of α. However, the convergence rate does depend on α, and some care is necessary in its choice. The dependence is illustrated in Fig. 1 for some typical examples. The main feature of the curves is the increase, quite sharp in some cases, in the number of iterations required as α becomes either very large or very small. A consideration of eqns. 6-11 readily shows why this occurs. Eqn. 7 shows that the eigenvalues of A are related to those of F by $\lambda_i(A) = \{\alpha - \lambda_i(F)\}^{-1}\{\alpha + \lambda_i(F)\}$, so that, as α becomes either very large or very small, all the moduli of the eigenvalues of A tend to unity, and, consequently, the difference in the magnitude of the elements of $A'\Phi A$ and Φ becomes quite small. As α increases, it is clear that the entries of Q and H become very small and those of U very large, while B tends to a constant matrix $\sqrt{2}G$. Assuming, for the moment, that F has no eigenvalue near zero, a decrease in α to very small values again results in small elements in Q and H, and the negative term in eqn. 6 then tends to $2\alpha A'\Phi(i)L\{R + L'SL + 2\alpha L'\Phi(i)L\}^{-1}L'\Phi(i)A$, where $L = (\alpha I - F)^{-1}G$, the entries of which decrease linearly with α. Therefore, for either very large or very small values of α, the solution of eqn. 6 evolves initially at a nearly linear rate with very small increments, so that a large number of iterations are required before the negative term grows sufficiently to level off further changes. It seems difficult to obtain a criterion for the optimal value of α by analytical methods, but the curves of Fig. 1 suggest that an α equal to the average of the absolute values of the eigenvalues of F will be quite close to the optimum. A complication arises if F has a real positive eigenvalue, such as curve e on Fig. 1. A choice of α close to this eigenvalue will result in some very large entries in $(\alpha I - F)^{-1}$, and, hence, in all the coefficients of eqn. 6, so that, after a few iterations, the solution of eqn. 6 is computed as the difference of matrices with very large (and so 'nearly equal') entries, with a serious loss in the number of significant figures obtainable in the solution. In example (e) on Fig. 1, where F has an eigenvalue of +2, single-precision arithmetic, and a choice of 1·9 and 2·1, results in the convergence of the norm of the solution matrix to two significant figures in nine iterations, and, from there, no further improvement is possible. It is obvious therefore that a value of α near a positive eigenvalue of F should be avoided.

Apart from α, an important factor in the convergence rate of eqn. 6 appears to be what could be called the 'degree of controllability' of the matrices F and G, the extent to which the states x of the system $\dot{x} = Fx + Gu$ are directly, rather than indirectly through other states, affected by the inputs u. In our examples, this degree of controllability is directly related to m/n, where m is the number of columns of G, and a comparison of curves (d) and (e) in Fig. 1, and of curves (b) and (c) with the 6th-order problems of Table 1 shows that considerably faster convergence rates of eqn. 6 are obtained for higher values of m/n. This suggests that the reduction in the degree of controllability resulting from an increase in n, the dimension of the solution matrix, without a corresponding increase in the number of linearly independent columns of G, will increase the minimum number of iterations required for eqn. 6 to converge. It therefore seems unlikely that the algorithm will be computationally attractive for problems of large dimension (n greater than 20, say). A count of the number of multiplications required gives the expression $(4{\cdot}5 + 1{\cdot}5k)n^3 + 2m(1 + 2k)n^2$. An analysis of the algorithm of Fath, using the estimates given by Wilkinson[9] for the number of multiplications required in the various steps of Fath's method, leads to the expression $\beta_1 n^3 + \beta_2 n^2$, where β_1 and β_2 are of the order of 20 and 300, respectively, but depend somewhat on the convergence rates in the iterative computation of eigenvalues and eigenvectors. These estimates support the

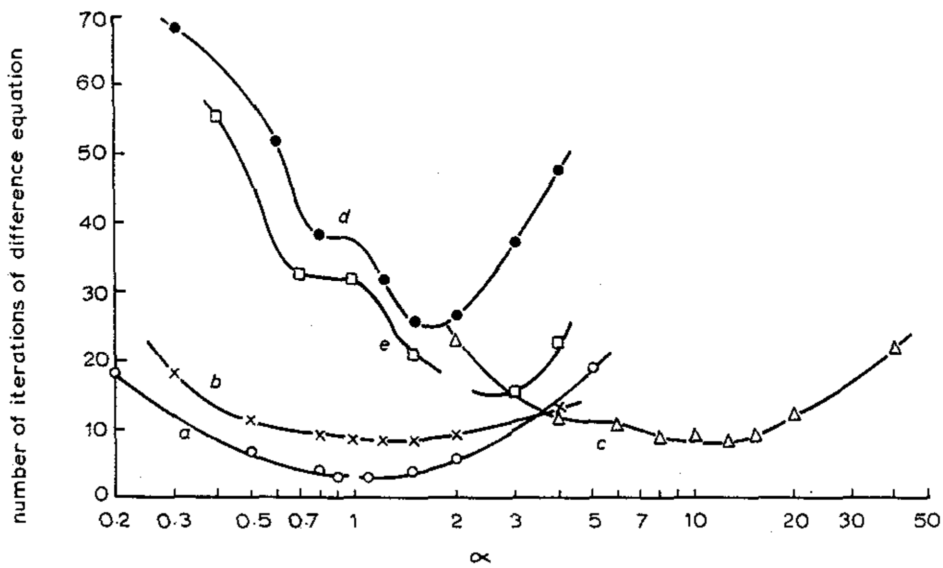

Fig. 1

Dependence of convergence rate of eqn. 6 on parameter α

	n	m	$\frac{1}{n}\sum\lvert\lambda_i(F)\rvert$	
(a)	2	1	1·0	*
(b)	6	3	0·91	†
(c)	6	3	9·1	†
(d)	8	2	1·96	
(e)	12	4	1·85	‡

* F_1 *has eigenvalue at +1*
† $\lambda_i(F_3) = 10\lambda_i(F_2)$
‡ F_5 *has eigenvalue at +2*

view that the algorithm of theorem 3 is not likely to be suitable for large problems.

For the range of problem sizes we have tested, however, the algorithm does give good results, provided that α is chosen to be not too far from the average of the magnitudes of the eigenvalues of F and not too close to a positive eigenvalue. Since, in most problems requiring a solution of eqn. 1, some prior knowledge of the dynamics of the system $\dot{\mathbf{x}} = \mathbf{F}\mathbf{x}$ is available, this does not appear to be a serious restriction on the applicability of the method.

5 CONCLUSIONS

In this paper, we have presented an iterative method for the calculation of the limiting solution of a stationary Riccati equation. We have found that the method compares favourably with another commonly used technique relying on the determination of the eigenvectors of a matrix.

As a possible future extension of the ideas of this paper, we envisage the application of the bilinear transformation to problems of singular optimal control. In discrete time, one can expect the singular problem to be much more tractable.

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