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Some Remarks on Simplified Stability Criteria for Continuous Linear Systems

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***Abstract*—Relations between reduced-order Markov and Hermite criteria are discussed.**

The purpose of this note is to comment on a relationship between the Liénard–Chipart–Markov criterion [1] and a modified form of the Hermite criterion discussed in [2]. These criteria arise in studying the root distribution of a real polynomial

$$F(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0.$$

The Liénard–Chipart–Markov criterion involves the Markov parameters s_i associated with $F(s)$. Write $F(s) = H(s^2) + sG(s^2)$ and define the sequence $\{s_i\}$ by

$$\frac{G(u)}{H(u)} = s_{-1} + \frac{s_0}{u} - \frac{s_1}{u^2} + \frac{s_2}{u^3} - \cdots.$$

If $a_i > 0$ for all i, then $F(s)$ is stable if and only if either

$$s_0 > 0, \quad \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} > 0, \quad \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} > 0, \cdots,$$

or

$$s_1 > 0, \quad \begin{vmatrix} s_1 & s_2 \\ s_2 & s_3 \end{vmatrix} > 0, \quad \begin{vmatrix} s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \\ s_3 & s_4 & s_5 \end{vmatrix} > 0 \cdots. \tag{1}$$

In (1), determinants up to dimension $m \times m$ are considered, with $m = n/2$ or $\frac{1}{2}(n-1)$, depending on whether n is even or odd.

Reference [2], using the Liénard–Chipart criterion in conjunction with the inners concept [3], shows that, with $a_i > 0$ for all i, $F(s)$ is stable if and only if the matrix $C = (c_{ij})$ is positive definite symmetric, where

$$c_{ij} = \sum_{k=1}^{2i-1} (-1)^{k+2i-1} a_{n-k+1} a_{n-2i-2j+k+2}, \qquad j \geq i. \tag{2}$$

For a sixth-degree polynomial one has, for example,

$$C = \begin{bmatrix} a_6a_5 & a_6a_3 & a_6a_1 \\ a_6a_3 & a_6a_1 - a_5a_2 + a_4a_3 & -a_5a_0 + a_4a_1 \\ a_6a_1 & -a_5a_0 + a_4a_1 & -a_3a_0 + a_2a_1 \end{bmatrix}. \tag{3}$$

The Liénard–Chipart–Markov criterion simplifies the Markov criterion, which states that $F(s)$ is stable if and only if all determinants in (1) are positive, with $s_{-1} > 0$ if $n = 2m + 1$. The criterion associated with (2) simplifies the Hermite criterion for stability, which states that $F(s)$ is stable if and only if the symmetric matrix $D = (d_{ij})$ is positive definite, where

$$d_{ij} = \begin{cases} \displaystyle\sum_{k=1}^{i} (-1)^{k+i} a_{n-k+1} a_{n-i-j+k}, & j \geq i, \quad i + j \text{ even} \\ 0, & i + j \text{ odd}. \end{cases} \tag{4}$$

For a sixth-degree polynomial, one has

$$D = \begin{bmatrix} a_6a_5 & 0 & a_6a_3 & 0 & a_6a_1 & 0 \\ 0 & -a_6a_3 + a_5a_4 & 0 & -a_6a_1 + a_5a_2 & 0 & a_5a_0 \\ a_6a_3 & 0 & a_6a_1 - a_5a_2 + a_4a_3 & 0 & -a_5a_0 + a_4a_1 & 0 \\ 0 & -a_6a_1 + a_5a_2 & 0 & a_5a_0 - a_4a_1 + a_3a_2 & 0 & a_3a_0 \\ a_6a_1 & 0 & -a_5a_0 + a_4a_1 & 0 & -a_3a_0 + a_2a_1 & 0 \\ 0 & a_5a_0 & 0 & a_3a_0 & 0 & a_1a_0 \end{bmatrix}.$$

The fact that the matrix C in (3) may be obtained from D by deleting the even rows and columns in D is a general property of the matrices C and D. For this reason, we apply the name "reduced Hermite criterion" to a stability test[1] based on C.

In this note, we wish first to indicate a close connection between the Liénard–Chipart–Markov criterion and the reduced Hermite criterion. This follows from a formula developed in [4] in relating the Markov criterion to the Hermite criterion. For simplicity, assume that n is even. In [4] the following formula is established (see especially [4, eq. (38)]):

$$\begin{bmatrix} s_0 & 0 & s_1 & 0 & s_2 & \cdots & 0 \\ 0 & s_1 & 0 & s_2 & 0 & \cdots & \cdot \\ s_1 & 0 & s_2 & 0 & s_3 & \cdots & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \\ \cdot & & \cdot & \cdot & \cdot & & s_{n-1} \end{bmatrix} = W'EW \tag{5}$$

where E is the Hermite matrix with rows and columns reversed, i.e., the i,j element of E is the $(n+1-i),(n+1-j)$ element of D and W is a matrix with the important structure

$$VW = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ \cdot & \cdot & & & \cdot & \cdot & x \\ \cdot & & & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & x \\ \cdot & 1 & 0 & x & \cdot & \cdot & \cdot \\ 1 & 0 & x & 0 & x & \cdot & \cdot \end{bmatrix}.$$

The symbol x denotes an element whose value is irrelevant. By setting

$$V = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 1 \\ 0 & 0 & \cdot & \cdot & 1 & 0 \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

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[1] This reduced form can also be utilized for determining the root distribution of a polynomial.

$$W = \begin{bmatrix} 1 & 0 & x & 0 & x & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & x & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & x & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & & & & \cdot \\ \cdot & & & & \cdot & & & \cdot \\ \cdot & & & & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & & \cdot \end{bmatrix},$$

(5) can be made to involve the Hermite matrix D as follows:

$$\begin{bmatrix} s_0 & 0 & s_1 & 0 & s_2 & \cdots & 0 \\ 0 & s_1 & 0 & s_2 & 0 & \cdots & \cdot \\ s_1 & 0 & s_2 & 0 & s_3 & \cdots & \cdot \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & s_{n-1} \end{bmatrix} = V'DV. \tag{6}$$

A rearrangement of rows and columns in (6) can be effected by multiplying on the left by a matrix U' and on the right by a matrix U, where

$$u_{ij} = \begin{cases} 1, & i = 2j+1, \;\; j \le \dfrac{n}{2} \text{ and } i = 2j, \;\; j > \dfrac{n}{2} \\ 0, & \text{otherwise.} \end{cases}$$

The result, obtained using the special structure of V and D, is

$$\left[\begin{array}{cccc|cccc} s_0 & s_1 & s_2 & \cdots & & & & \\ s_1 & s_2 & s_3 & \cdots & & O & & \\ \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & & & & & \\ \hline & & & & s_1 & s_2 & s_3 & \cdots \\ & O & & & s_2 & s_3 & s_4 & \cdots \\ & & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \vdots & \end{array}\right]$$

$$= \left[\begin{array}{c|c} X' & 0 \\ \hline 0 & Y' \end{array}\right] \left[\begin{array}{c|c} C & 0 \\ \hline 0 & \hat{C} \end{array}\right] \left[\begin{array}{c|c} X & 0 \\ \hline 0 & Y \end{array}\right]. \tag{7}$$

Here, X and Y are nonsingular matrices, the precise form of which is unimportant, C is as before, and the symmetric matrix $\hat{C}$ is given by

$$\hat{c}_{ij} = \sum_{k=1}^{2i} (-1)^{k+2i} a_{n-k+1} a_{n-2i-2j+k}, \qquad j \ge i. \tag{8}$$

(The matrix $\hat{C}$ is obtainable from D by deleting the odd rows and columns.)

From (8), connection between the criteria in question is immediate and one criterion can evidently be proved from the other.

In conclusion, we shall comment on a number of other connections between the two criteria discussed and the Liénard–Chipart criterion [5], [6]. In [1] the Liénard–Chipart–Markov criterion is proved by using a formula relating the odd- and even-order Markov determinants to the odd- and even-order Hurwitz determinants. The odd- and even-order Hurwitz determinants are relatable also to the principal minors of the matrices C and $\hat{C}$, respectively. Effectively, the formulas are in [7]. Parks [8] has also commented on the existence of such formulas, which can be derived from formulas connecting principal minors of the Hermite matrix with products of pairs of Hurwitz determinants [9], [10]. A straightforward proof of the Liénard–Chipart–Markov criterion may also be obtained from several properties of the Cauchy index (see [6, vol. 2, ch. 15]). The proof is very similar to a proof in [6] for the Liénard–Chipart criterion, which uses the Cauchy index.

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