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Continuously Equivalent State Variable Realizations

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Abstract—Suppose one is given two minimal realizations of the same transfer function matrix. The question is asked: When does there exist a family of coordinate transformations defined by a set of nonsingular matrices $T(\lambda)$, continuously dependent on λ, with $T(0) = I$ and with $T(1)$ mapping the state vector associated with one minimal realization into the state vector associated with the other? The quesion is answered, and a procedure is given for constructing the family when it exists.

The theory of continuously equivalent networks [1]–[7], and especially the noncompleteness result given in [5], suggests a related problem in the theory of continuously equivalent state realizations. The solution of this problem, given below following a formal statement of the problem, may shed light on the continuously equivalent network problem through a description of the latter in state-space terms.

As usual, a minimal state realization $\{A, B, C\}$ of a real rational transfer function matrix $W(s)$ with $W(\infty)=0$ is a triple of real matrices satisfying

$$W(s) = C'(sI - A)^{-1}B \tag{1}$$

with A being of minimum dimension.

Continuously Equivalent State Realization Problem

Given two minimal state realizations $\{A_0, B_0, C_0\}$ and $\{A_1, B_1, C_1\}$ of the same transfer function matrix, does there exist a continuously variable family of real nonsingular matrices $T(x)$, $0 \leq x \leq 1$, such that $T(0)=I$ and $T(1)=T_1$, where T_1 is the unique matrix such that $A_1 = T_1A_0T_1^{-1}$, $B_1 = T_1B_0$, and $C_1' = C_0'T_1^{-1}$. (Existence of T_1 is established in [8], for example, which also describes a procedure for computing T_1.) If there is such a family, how may it be found?

Remark: Such a family $T(x)$ would yield a continuously varying family of state realizations $\{A(x), B(x), C(x)\}$, with the end members of the family comprising $\{A_0, B_0, C_0\}$ and $\{A_1, B_1, C_1\}$.

Theorem

A family $T(x)$ of nonsingular real matrices, continuously dependent on x in the interval $[0, 1]$ and with $T(0)=I$ and $T(1)=T_1$, exists if and only if $\det T_1 > 0$.

Proof: First, suppose to the contrary that a family exists with $\det T_1 < 0$. Now continuity of the family implies that $\det T(x)$ varies continuously, and nonsingularity of the family implies that $\det T(x)$ is never zero. This contradicts the property $\det T(0)=1$ and the assumption $\det T(1)<0$. We have thus shown that existence of the family implies that $\det T_1 > 0$.

Now suppose $\det T_1 > 0$. Write T_1 as

$$T_1 = UH \tag{2}$$

where U is orthogonal, and H is positive definite symmetric. Evidently, $\det H > 0$; because also $\det T_1 > 0$, it follows that $\det U > 0$. Therefore, for some orthogonal matrix V, U can be written as

$$U = V'\Lambda V \tag{3}$$

with

$$\Lambda = \operatorname{diag}\left\{1, \cdots, 1, -1, \cdots, -1, \begin{bmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{bmatrix}, \cdots, \begin{bmatrix}\cos\theta_r & -\sin\theta_r\\ \sin\theta_r & \cos\theta_r\end{bmatrix}\right\} \tag{4}$$

and with the number of -1 entries being even [9, p. 244]. Recognize that

$$\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

with $\theta = -\pi$. This means that Λ is of the form

$$\Lambda = \operatorname{diag}\left\{1, \cdots, 1, \begin{bmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{bmatrix}, \cdots, \begin{bmatrix}\cos\theta_s & -\sin\theta_s\\ \sin\theta_s & \cos\theta_s\end{bmatrix}\right\}. \tag{5}$$

Now set

$$T(x) = V'\Lambda(x)V[xH + (1-x)I] \tag{6}$$

with

$$\Lambda(x) = \operatorname{diag}\left\{1, \cdots, 1, \begin{bmatrix}\cos x\theta_1 & -\sin x\theta_1\\ \sin x\theta_1 & \cos x\theta_1\end{bmatrix}, \cdots, \begin{bmatrix}\cos x\theta_s & -\sin x\theta_s\\ \sin x\theta_s & \cos x\theta_s\end{bmatrix}\right\}. \tag{7}$$

It is easily checked that $T(0)=I$ and $T(1)=UH=T_1$. Obviously, $T(x)$ is continuously dependent on x, and is nonsingular, being of the form $U(x)H(x)$ for orthogonal $U(x)$ and positive definite $H(x)$. This proves the theorem and, at the same time, we have indicated how the family $T(x)$ may be found.

Remark: It follows easily from the theorem that the set of all minimal state realizations of a transfer function matrix falls into two disjoint subsets, with all members of the one subset being continuously equivalent. It is not possible for three or more realizations to be such that no pair is continuously equivalent.

Remark: Define the matrices W_0 and W_1 by

$$W_i = [B_i\ A_iB_i \cdots A_i^{n-1}B_i], \qquad i = 0, 1 \tag{8}$$

where A_i is $n \times n$. The W_i have rank n by the minimality of $\{A_i, B_i, C_i\}$. The matrix T_1 is uniquely determined by $T_1W_0 = W_1$ or $T_1W_0W_0' = W_1W_0'$; see [8]. Since W_0W_0' evidently has positive determinant, the condition $\det T_1 > 0$ is equivalent to $\det W_1W_0' > 0$.

Remark: The result extends easily to the time-varying case. Let $\{A_i(t), B_i(t), C_i(t)\}$ be two realizations of the same weighting function, related by $A_1 = T_1(t)A_0T_1^{-1}(t) + \dot{T}_1(t)T_1^{-1}(t)$, $B_1 = T_1(t)B_0$, $C_1' = C_0'T_1^{-1}(t)$; see [8]. It is implicitly assumed that $T_1(t)$ is differentiable and nonsingular for all t. For fixed t, a family $T(x, t)$, $0 \leq x \leq 1$, exists taking A_0 to A_1 if and only if $\det T_1(t) > 0$. Also, if $\det T_1(t)$ is positive for one particular value of t, it is positive for all t because $T_1(\cdot)$ is continuous and always nonsingular. Accordingly, the family $T(x, t)$ exists for all t.

There is no question that there remains a gap between the statement of this theorem and its use in the design problem of varying the elements of a network so as to preserve the terminal behavior but achieve a more satisfactory internal configuration. The bridging of the gap will undoubtedly require use of the state-space approach to network synthesis; see [10], for example.

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