OUTLINE DESIGNS FOR STABLE CONTINUOUS-TIME FIXED-LAG SMOOTHERS

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Three basic methods for constructing suboptimal continuous-time linear fixed-lag smoothers are described. In contrast to the optimal smoother, all the suboptimal smoothers are stable. The degree of complexity allowed in the suboptimal smoothers determines how closely they perform to the limit obtainable from the theoretical optimal linear fixed-lag smoother.

We consider the filtering and smoothing problems for a linear finite-dimensional system of the form

$$\left.\begin{aligned} \dot{x} &= Fx+Gw \\ z &= H'x+v \end{aligned}\right\} \qquad (1)$$

where v and w are noise processes and all the usual assumptions apply to guarantee that the associated Kalman–Bucy filter is exponentially asymptotically stable.[1] It is then true[2] that a fixed-lag smoother for eqn. 1 is unstable, in the sense that any physical realisation proposed is unstable. This is somewhat surprising, in view of the existence of a relation between the fixed-lag smoothed estimate $\hat{x}(\cdot-\Delta/\cdot)$, filtered estimate $\hat{x}(\cdot/\cdot)$ and innovations process $\nu(\cdot)$ of the form[3]

$$\hat{x}(t-\Delta/t) = \hat{x}(t-\Delta/t-\Delta) + \int_{t-\Delta}^{t} K(t,s)\,\nu(s)\,ds \qquad (2)$$

This shows that the impulse response mapping $\nu(\cdot)$ and $\hat{x}(\cdot/\cdot)$ into $\hat{x}(\cdot-\Delta/\cdot)$ is bounded-input bounded-output, and one would therefore expect that a stable physical realisation exists as in the discrete-time case.[4]

Here, we propose three basic methods, with a number of variations, for the design of suboptimal, exponentially stable fixed-lag smoothers; it is hoped that they will, like the optimal, practically unrealisable fixed-lag smoother, offer lower error variance than the filter. We hope to present further details and performance analyses for our suboptimal smoothers in a future paper.

Method 1. Sampling and reconstruction: The measurement process $z(\cdot)$ is sampled, and a stable discrete-time smoother is built, the outputs of which provide approximately $\hat{x}(t-\Delta/t)$ for values of t equal to the sampling instants. An approximation to $\hat{x}(t-\Delta/t)$ for values of t between the sampling instants is easily obtained. As the sampling interval goes to zero, the performance approaches that of the optimal smoother, except that stability is retained.

Method 2. Use of exact filters for approximate models:

(*a*) The model defined by eqn. 1 is replaced by that shown in Fig. 1*a*; $z(\cdot)$ remains as the measurement process. One builds an exponentially stable Kalman–Bucy filter for the new model, and recovers from it the exact filtered estimate of an approximation to $x(t-\Delta)$, i.e. one obtains an approximation to $\hat{x}(t-\Delta/t)$.

(*b*) One may instead adopt the arrangement of Fig. 1*b*. If w has dimension less than that of x, there may be a saving in complexity of the model, and therefore the filter. On the other hand, a better approximation is required in Fig. 1*b* than Fig. 1*a*, because $w(\cdot)$ is of wider bandwidth than $x(\cdot)$.

(*c*) If case-signal estimation or estimation of $H'x$, rather than state estimation, is required, the signal $H'x$ rather than x may be passed through the block S. If $H'x$ has lower dimension than x, as is usual, there is reduction in complexity.

(*d*) Because a Kalman–Bucy filter looks like a copy of the original system driven by an error signal, the block S appears in the filter for each of the models of Fig. 1. It may be replaced by an exact time-delay in the filter.

Of course, the higher the dimension of the finite-dimensional block S is (and hence the higher the filter dimension), the closer the performance can be to optimal.

Method 3. Use of approximate smoothers for exact models: From the practical point of view, this method is essentially limited to time-invariant systems with stationary noise.

(*a*) One may adopt the brute-force approach of writing down the smoother impulse response, which maps $z(\cdot)$ into $\hat{x}(\cdot-\Delta/\cdot)$, and then obtain an approximation with a rational Laplace transform possessing no left-halfplane poles. This procedure takes no advantage of the structure of the problem.

(*b*) A variation of method 3*a* is to approximate (again with an impulse response of stable Laplace transforms) the impulse response $K(t-s)\,l(t-s)\,l(s-t+\Delta)$ in eqn. 2. [Note

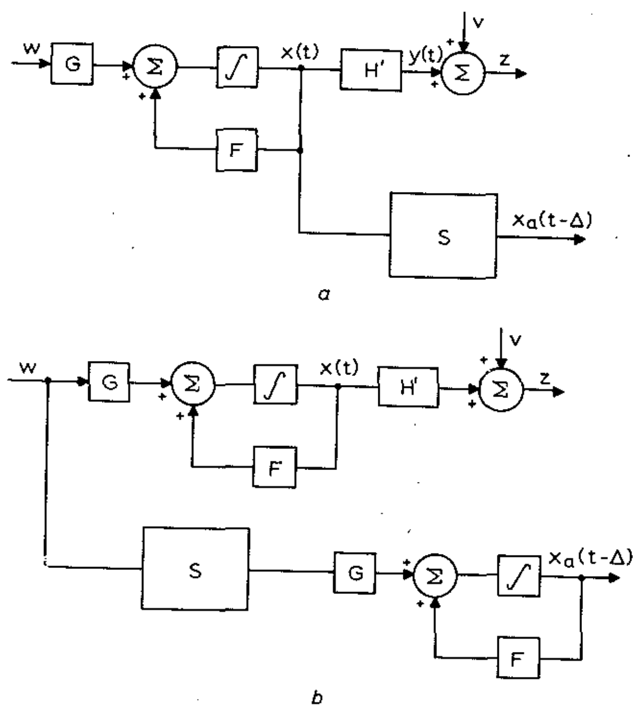

Fig. 1 *Approaches to suboptimal smoothing by constructing approximate models*

S denotes a finite-dimensional system approximating a time delay Δ
The output at time t, $x_a(t-\Delta)$, is an approximation to $x(t-\Delta)$

that the time-invariance stationarity assumption forces $K(t,s) = K(t-s)$].

(*c*) Yet a further variation relies on use of a formula of the form[5]

$$\hat{x}(t-\Delta/t) = M(\Delta)\,\hat{x}(t/t) + \int_{t-\Delta}^{t} L(t-s)\,\hat{x}(s/s)\,ds \qquad (3)$$

Then one approximates $L(t-s)\,l(t-s)\,l(s-t+\Delta)$ in the same way. This is preferable to method 3*b* in that, with $\hat{x}(\cdot/\cdot)$ band-limited and $v(\cdot)$ not band-limited, errors of approximation of $L(\cdot)$ may be less troublesome than errors in $K(\cdot)$. But method 3*c* may not be preferable in that $\hat{x}(\cdot/\cdot)$ generally has greater dimension than $v(\cdot)$.

(*d*) One can check that the impulse response

$$K(t-s)\,l(t-s)\,l(s-t+\Delta)$$

in eqn. 3 has a Laplace transform of the form

$$(sI-A)^{-1}\,[e^{-A\Delta} - e^{-s\Delta}I]\,B \qquad \ldots\ldots (4)$$

for constant matrices A and B, with $\mathrm{Re}\,\lambda_i(A) > 0$. [Note that each zero of $(sI-A)$ is a zero of $\det(e^{-A\Delta} - e^{-s\Delta}I)$, so that unstable poles are cancelled.] By diagonalising A, the problem of providing a stable rational approximation to eqn. 4 is made equivalent to the problem of providing stable rational approximations for

$$\frac{e^{-a\Delta} - e^{-s\Delta}}{s-a}$$

and

$$\begin{bmatrix} s+a & b \\ -b & s+a \end{bmatrix}^{-1} \begin{bmatrix} e^{-a\Delta}\cos b\Delta - e^{-s\Delta} & e^{-a\Delta}\sin b\Delta \\ -e^{-a\Delta}\sin b\Delta & e^{-a\Delta}\cos b\Delta - e^{-s\Delta} \end{bmatrix} \qquad \ldots (5)$$

where a, b and Δ are positive constants. The approximation problem for eqn. 5 can be solved once and for all for a variety of a, b and Δ; one thereby has a permanent approximation to eqn. 4, assuming the ability to diagonalise A.

In conclusion, we note that the approximation problem is a standard one of network synthesis, and many techniques are available for dealing with it. Frequently, if one fixes the pole positions *a priori*, optimum zero positions follow by linear algebra.

Performance of all suboptimal smoothers depends on how close they are to the optimum smoother. If the approximation is very coarse, a worse performance than filtering may result. However, the suboptimal smoothers of methods 3*b*, 3*c* and 3*d* always offer superior performance to the filter by itself.

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