CONSTRUCTION OF LYAPUNOV FUNCTIONS FOR NONSTATIONARY SYSTEMS CONTAINING MEMORYLESS NONLINEÁRITIES

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This paper considers the construction of a Lyapunov function for linear, finite-dimensional, nonstationary control systems with an arbitrary finite number of feedback nonlinearities. The sufficient conditions for the construction of the Lyapunov function may be considered as a stability criterion. To establish stability, diagonal matrices have to be determined so that particular functions involving these matrices, the parameters of the linear nonstationary part of the system, and the sector bounds for the nonlinearities are covariances, or, equivalently, that a certain matrix Riccati equation involving these various system parameters have a completely determinate solution; the solution of the Riccati equation enables the Lyapunov function to be constructed. It may be noted that the covariance-criterion reduces to the well-known criterion for the positiveness of the characteristic numbers previously established for stationary systems. This in turn reduces to the Popov [1] criterion for the single nonlinearity case.

Introduction

The aim of the paper is to obtain a stability criterion for linear, finite-dimensional, nonstationary systems with an arbitrary finite number of stationary memoryless nonlinear elements in the feedback circuit. Special attention is paid to the construction of the Lyapunov functions for such systems, since fulfillment of the stability criterion is checked by using these functions. In the series of papers [2-7], a generalized Popov [1] criterion is obtained; Popov's criterion gives sufficient conditions for the stability of stationary systems whose linear part is stable, but contains a memoryless nonlinearity in a feedback circuit. Some of these papers show a special interest in the construction of Lyapunov functions for systems similar to the one considered in this paper, and, in particular, for finite-dimensional stationary systems containing an arbitrary finite number of memoryless nonlinearities in the feedback circuit [4]. However, the generalization of this case to a system whose linear part is nonstationary has not been considered previously. It is shown in this paper that recent results in the theory of optimal control [8, 9] make it possible to extend known results to this case, and an example is given which illustrates how the stability criterion is applied.

Stability Criterion

Since it suffices to investigate stability for zero input signal, we may consider the system S shown in Fig. 1, without loss of generality. The linear, finite-dimensional nonstationary subsystem W is described by the equation

$$\dot{x} = Fx + Gu, \quad y = H'x, \tag{1}$$

where the dimension of the input vector u and the output vector y is n. The matrix of the periodic impulse functions $w(t, \tau)$ of the subsystem W are specified by

$$w(t,\tau) = H'(t)\Phi(t,\tau)G(\tau)1(t-\tau), \tag{2}$$

where

$$\dot{\Phi}(t,\tau) = F\Phi(t,\tau), \quad \Phi(\tau,\tau) = I_n, \tag{3}$$

and $1(t)$ is the unit jump function. We note some conditions used in this paper.

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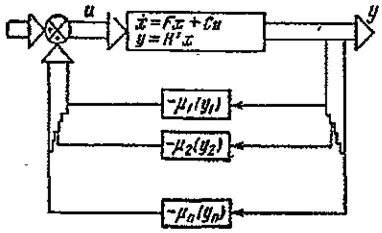

Fig. 1

1. We assume that either the matrix F is asymptotically exponentially stable, or the pair [F, G] is uniformly completely stable [10]. We remark that if F and G are constants, then uniform complete stability reduces to ordinary complete stability. The matrices F, G, and H are bounded, and H is differentiable, $\dot{H}$ bounded.

2. We assume that the memoryless nonlinearities $\mu_i(y_i)$ $(i = 1, 2, \ldots, n)$ in the feedback circuit satisfy

$$\mu'(y)\ y \leqslant y'Ky,$$

where $\mu'(y) = (\mu_1(y_1), \mu_2(y_2), \ldots, \mu_n(y_n))$, and $K = \operatorname{diag}\{k_1, k_2, \ldots, k_n\}$ is a constant, positive definite matrix.

We note that the nonlinearities are constant with respect to time, in the sense that μ_i clearly depends only on y_i, and not on y_i and t.

In order to obtain the stability criterion for the system S described above, we consider the hypothetical system $\mathcal{L}$ which can be characterized by the following matrix of impulsive transition functions:

$$z(t,\tau) = Z(t)K^{-1}\sigma(t-\tau) + A(t)\,w(t,\tau) + B(t)\frac{d}{dt}w(t,\tau), \tag{4}$$

where $\sigma(t)$ is the Dirac delta function, $A(t) = \operatorname{diag}\{a_1(t), a_2(t), \ldots, a_n(t)\}$, $B(t) = \operatorname{diag}\{b_1(t), b_2(t), \ldots, b_n(t)\}$.

3. We will assume that the matrix $B(t)$ is differentiable, and that the matrices $A(t)$, $B(t)$, $-\dot{B}(t)$ are non-negative definite and bounded for all t. Before we formulate a stability theorem, the multidimensional Popov criterion must be considered, for the case where the matrices F, G, H, A, and B are constants [4], and the Laplace transforms $W(s)$, $Z(s)$ exist for the functions $W(t-\tau)$, $Z(t-\tau)$, and

$$Z(s) = AK^{-1} + (A + Bs)W(s). \tag{5}$$

<u>Theorem 1.</u> If the matrices $A = \operatorname{diag}\{a_1, a_2, \ldots, a_n\}$ and $B = \operatorname{diag}\{b_1, b_2, \ldots, b_n\}$ can be expressed so that $a_i > 0$, $b_i > 0$ $(a_i + b_i > 0)$, and if $-a_i/b_i$ is not a pole of the Laplace transform $W(s)$, and if the real part of the Laplace transform is positive, i.e.,

$$\operatorname{Re} Z(s) \geqslant 0, \tag{6a}$$

then the control system depicted in Fig. 1, with a completely controllable pair [F, G] and an asymptotically stable F which satisfies the conditions 2, is Lyapunov stable. The requirement that the real part of Z(s) be positive is equivalent (under a transformation to the time domain) to the condition

$$\int_{T_1}^{T_2}\int_{T_1}^{T_2} q'(t)\,z(t-\tau)\,q(\tau)\,dt\,d\tau \geqslant 0, \tag{6b}$$

where $q(\cdot)$ is an arbitrary real vector function defined on $[T_1, T_2]$ for arbitrary T_1 and T_2 [11].

We now show that the stability criterion corresponding to (6a) and (6b) for the nonstationary, nonlinear system S (Fig. 1) satisfying conditions 1 and 2 is the fulfillment of condition 4.

4. The sum $[z(t,\tau) + z(\tau, t) - 2\eta\delta(t-\tau)]$ is a covariance for some positive η and some $A(t)$ and $B(t)$ satisfying condition 1.

We note that condition 4 is equivalent to the condition

$$\int_{T_1}^{T_2}\int_{T_1}^{T_2} q'(t)\,[z(t,\tau) - \eta I_n\delta(t-\tau)]\,q(\tau)\,dt\,d\tau \geqslant 0, \tag{7}$$

where $q(\cdot)$ is an arbitrary vector function defined on $[T_1, T_2]$ for arbitrary T_1 and T_2.

When we take (2) and (4) into account, we may rewrite the impulsive periodic function matrix of the system Z as

$$z(t,\tau)=\frac{1}{2}R_z(t)\delta(t-\tau)+H_z'(t)\Phi(t,\tau)G(\tau)1(t-\tau), \tag{8}$$

where

$$R_z=2AK^{-1}+BH'G+G'HB, \tag{9a}$$

$$H_z=HA+(F'H+\dot{H})B. \tag{9b}$$

5. We assume that the pair [F, H_z'] is uniformly completely observable [10] under conditions 1, 4, and 5. We also assume that the results obtained in optimal control theory [8, 9] (cf. the Appendix) can be used to determine a matrix function P(t) such that, for all t and for certain positive constants α_1 and α_2, the inequality

$$0<\alpha_1 I_n\leqslant P(t)\leqslant\alpha_2 I_n<\infty,\ \ P=P', \tag{10}$$

holds, and its time derivative is determined by

$$-\dot{P}=P(F-GR_z^{-1}H_z')+(F'-H_zR_z^{-1}G')P+PGR_z^{-1}G'P+H_zR^{-1}H_z' \tag{11}$$

for all t.

We now try to construct a Lyapunov function for the system S (Fig. 1) in the form

$$V(\mathbf{x},t)=\mathbf{x}'P(t)\mathbf{x}+2\sum_i b_i(t)\int_0^{y_i}\mu_i(\rho_i)d\rho_i. \tag{12}$$

When we differentiate (12), we get

$$\dot{V}(\mathbf{x},t)=2\dot{\mathbf{x}}'P(t)\mathbf{x}+\mathbf{x}'\dot{P}(\mathbf{x})+2\sum_i b_i(t)\mu_i\dot{y}_i+2\sum_i\dot{b}_i(t)\int_0^{y_i}\mu_i(\rho_i)d\rho_i. \tag{13}$$

We note that the feedback equation has the form $\mathbf{u}=-\mu(\mathbf{y})$ in (1), and therefore when we substitute (1), (9), and (11) into (13) we get

$$\begin{aligned}\dot{V}(\mathbf{x},t)=&-[\mathbf{x}'(H_z-PG)R_z^{-1}-\mu'(\mathbf{y})]\,R_z\,[R_z^{-1}(H_z'-G'P)\mathbf{x}-\mu(\mathbf{y})]\\&-2\mu'(\mathbf{y})A[\mathbf{y}-K^{-1}\mu(\mathbf{y})]+2\sum_i\dot{b}_i(t)\int_0^{y_i}\mu_i(\rho_i)d\rho_i.\end{aligned} \tag{14}$$

It follows from this that the conditions 1, 3, 4, 5 are sufficient to ensure that the function V(x, t) determined by (12) is a Lyapunov function. Since 1, 4, 5 ensure that (10) holds, and 3 guarantees that the matrix B is non-negative, the right side of (12) has the required properties. The covariance condition 4 guarantees that $R_Z-\eta I_n$ is nonnegative definite and that R_Z^{-1} is bounded, and so the first term in (14) is nonpositive. Condition 3 ensures the nonnegative definiteness of the matrices A and $-\dot{B}$, so that the second and third terms in (14) are non-negative; and so the right side of (14) has the required properties.

The interesting case is when $\dot{B}(t)=0$, and consequently V(x, t) = x' P(t)x is a Lyapunov function even if the nonlinearities are time-dependent, i.e., vary with time. In this case, the matrix K of condition 2 also is time-dependent. Moreover, analysis shows that the nonlinearities $\mu(y)$ do not always have to have the form

$$\mu(\mathbf{y})=[\mu_1(y_1),\ \mu_2(y_2),\ldots,\ \mu_n(y_n)].$$

It turns out that the vector μ may be any vector function satisfying the inequality of condition 2.

It is to be noted that, despite the fact the conditions of positive definiteness are imposed on A(t) + B(t) in Theorem 1, clearly they were not used anywhere in the proof. Actually (9a) and the non-negative definiteness required for $R_2-\eta I_n$ ensure that the conditions imposed on A(t) + B(t) are fulfilled. The results obtained are generalized directly, for the case where there is a degree of stability. For this, we require only that $F-\sigma_0 I_n$ is exponentially asymptotically stable (cf., condition 1) and that F be replaced by $(F-\sigma_0 I_n)$ in (9). Besides, an additional term

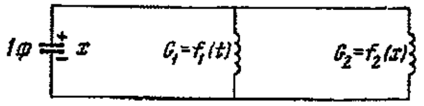

Fig. 2

$2\sigma_0[x'Px + \mu'(y)By]$ has to be considered in (14). The detailed analysis for this case is not done here, but the results are formulated in the following theorem.

Theorem 2. A. The nonlinear nonstationary system S (Fig. 1) with the linear subsystem W of (1)-(3) satisfying condition 1, and with the memoryless feedback elements satisfying condition 2 is Lyapunov stable, if the hypothetical system Z corresponding to S and the diagonal matrices A(t) and B(t) satisfying condition 3 (cf. (4), (8), and (9)) can be chosen such that: a) conditions 4 and 5 are satisfied, or b) the limit

$$P(t) = \lim_{t_1 \to \infty} [-\Pi(t, t_1)] \tag{15}$$

satisfies (10) and (11) for all t (a) implies b)). The function $\Pi(t, t_1)$ is specified by the matrix Riccati equation

$$-\dot{\Pi} = \Pi(F - CR_z^{-1}H_z') + (F' - H_zR_z^{-1}G')\Pi - \Pi GR_z^{-1}G'\Pi - H_zR_z^{-1}H_z', \tag{16a}$$

$$\Pi(t_1, t_1) = 0. \tag{16b}$$

The Lyapunov function for the system S has the form

$$V(\mathbf{x}, t) = \mathbf{x}'P(t)\mathbf{x} + 2\sum_i b_i(t) \int_0^{y_i} \mu_i(\rho_i)\, d\rho_i. \tag{17}$$

B. If the subsystem W of S also satisfies the condition that the matrix $(F - \sigma_0 I_n)$ is exponentially asymptotically stable for some real negative σ_0, then the assertions of Theorem 2A remain valid when F is replaced by $(F - \sigma_0 I_n)$ in (11) and (16). Further, if the nonlinearities satisfy

$$\mu_i(y_i)y_i \geqslant \int_0^{y_i} \mu_i(\rho_i)\, d\rho_i \qquad (i = 1, 2, \ldots, n), \tag{18}$$

then a Lyapunov function V exists for which $\dot{V}/V \leq \sigma_0$, and if the nonlinearities satisfy

$$\mu_i(y_i)y_i \geqslant 2\int_0^{y_i} \mu_i(\rho_i)\, d\rho_i \qquad (i = 1, 2, \ldots, n), \tag{19}$$

then a Lyapunov function V exists for which $\dot{V}/V \leq 2\sigma_0$.

Remark. Equation (18) comes from condition 2 for sufficiently small y_i and for all monotonic nonlinearities. Equation (19) comes from condition 2 for any nonlinearity which is concave up.

Corollary. If $\dot{B}(t)$ is a zero matrix, then Theorem 2 is valid for time-dependent nonlinearities satisfying condition 2.

Example

Consider the stability of a first order system as an example:

$$\dot{x} + [f_1(t) + f_2(x)]x = 0, \tag{20}$$

where $f_2(x)$ is a nonlinearity without any delay. Equation (20) corresponds to the equation describing the electrical circuit of Fig. 2 which has a constant condensor, a time-dependent variable resistance, and a nonlinear resistance.

Write

$$F = -f_1(t), \quad G = 1, \quad H = 1, \quad \mu(x) = f_2(x)x, \tag{21}$$

and from (9) and (21) we get $R_z = 2aK^{-1} + 2b$, $H_z = a - f_1(t)b$ (where a and b are not yet determined), for which we have assumed (cf., condition 2) that

$$0 \leqslant f_2(x) \leqslant k < \infty, \tag{22}$$

where k is arbitrary. This boundedness is equivalent to the requirement that the conductivity of G_2 be nonnegative and bounded.

Further, we assume that the following condition is fulfilled; this condition also has a obvious physical meaning

$$0 < \alpha \leqslant f_1(t) \leqslant \beta < \infty \quad \text{for all } t. \tag{23}$$

The condition guarantees the exponential asymptotic stability of F in (21). By taking b = 0 and using (9), we compute

$$z(t, \tau) = ak^{-1} + a\exp\left[-\int_\tau^t f_1(\lambda)\,d\lambda\right]1(t-\tau). \tag{24}$$

It is now clear that the sufficient conditions for stability are satisfied (conditions 1-5), if a is an arbitrary positive constant and the function

$$r(t, \tau) = a\exp\left[-\int_\tau^t f_1(\lambda)\,d\lambda\right]1(t-\tau) + a\exp\left[-\int_t^\tau f_1(\lambda)\,d\lambda\right]1(t-\tau) \tag{25}$$

is a covariance. This function can be rewritten in the form

$$r(t, \tau) = c(t)\left[\frac{d(\tau)}{c(\tau)}1(t-\tau) + \frac{d(t)}{c(t)}1(\tau-t)\right]c(\tau), \tag{26}$$

where

$$c(t) = a\exp\left[-\int_{t_0}^t f_1(\lambda)\,d\lambda\right], \tag{27a}$$

$$d(t) = \exp\left[\int_{t_0}^t f_1(\lambda)\,d\lambda\right]. \tag{27b}$$

When we apply the theorem presented in [12], we get that $r(t, \tau)$ is a covariance if

$$c(t) \neq 0, \tag{28a}$$

$$\frac{d}{dt}\left[\frac{d(t)}{c(t)}\right] \geqslant 0. \tag{28b}$$

Equation (27a) implies that (28a) holds, and since

$$\frac{d}{dt}\left[\frac{d(t)}{c(t)}\right] = \frac{1}{a}\exp\left[2\int_{t_0}^t f_1(\lambda)\,d\lambda\right]2f_1(t),$$

(28b) is satisfied; so $r(t, \tau)$ is a covariance.

Thus, if (22) and (29) are satisfied, then the stability of the system (20) is ensured.

CONCLUSION

The above theorem is proposed as a means of verifying the stability of nonstationary systems with nonlinearities whose graphs are confined to a sector, and it makes possible the construction of Lyapunov functions for such systems. It is a pity that the application of this procedure may turn out to be significantly more complicated than in the stationary case, even if there is only a single nonlinearity. Finally, the stationarity calculation entails the effective application of methods based on the Fourier transform, but, despite all this, these methods are useless in practice. In [13], the essential property of structural stability of nonstationary linear systems is indicated, i.e., that narrow sectors for the nonlinearity do not affect stability under determinate conditions of boundedness, controllability, etc.

APPENDIX

Suppose that the system described by

$$z(t, \tau) = \frac{1}{2} R_z(t)\delta(t-\tau) + H_z'(t)\Phi(t, \tau)G(\tau)1(t-\tau) \tag{A.1}$$

(where $\Phi(t, \tau)$ is the transition matrix for the system $\dot{x} = Fx$), is such that $z(t, \tau) + z'(\tau, t) - \eta I_n \delta(t-\tau)$ is a covariance, that the pair [F, G] is uniformly completely controllable, that F, G, H_Z, and R_Z are bounded, and that [F, H_Z] is uniformly completely observable. We show that there is a symmetric, positive definite matrix P(t) such that $0 < \alpha_1 I_n \leq P(t) \leq \alpha_2 I_n < \infty$ for certain positive constants α_1 and α_2. The case is considered in [8, 9] where the matrix F is required to be exponentially asymptotically stable instead of the pair [F, G] being uniformly completely controllable. This proof is not given here since it is more complicated than that given in this paper.

Consider the following optimal control problem.

We want to find the control u, which minimizes the functional

$$V(x_0, u, t_0, t_1) = \int_{t_0}^{t_1} (u'R_z u + 2x'H_z u)\, dt \tag{A.2}$$

under the conditions

$$\dot{x} = Fx + Gu, \qquad x(t_0) = x_0. \tag{A.3}$$

We show first, that if the conditions imposed on F, G, etc., are fulfilled, then there is a positive constant K such that

$$V(x_0, u, t_0, t_1) \geqslant -k\|x_0\|^2 \tag{A.4}$$

for all x_0, u, t_0, and t_1.

The uniform complete controllability for all t_0 guarantees the existence of a control u_c defined on $[t_0-T_0, t_0]$, where T_0 does not depend on t_0, where $x(t_0-T_0) = 0$, $x(t_0) = x_0$; $\|u_c(t)\|$ and $\|x(t)\|$ are bounded, by an estimate which depends on x_0, but is independent of t_0 for $t \in [t_0-T_0, t_0]$; this is proved in [8].

From (A.1) and (A.3) there follows

$$\begin{aligned} \int_{t_0-T_0}^{t_0} (u_c'R_z u_c + 2x'H_z u_c)\, dt &= 2\int_{t_0-T_0}^{t_0} u_c'(t) \int_{t_0-T_0}^{t} z(t, \tau)\, u_c(\tau)\, d\tau \\ &= \int_{t_0-T_0}^{t_0}\int_{t_0-T_0}^{t_0} u_c(t)\,[z(t, \tau) + z'(\tau, t)]\, u_c(\tau)\, dt\, d\tau. \end{aligned} \tag{A.5}$$

The covariance condition implies that this quantity is positive and bounded from above by the quantity $k\|x_0\|^2$, where k is a positive constant independent of x_0 and t_0 [8].

Equation (A.4) follows from these. Actually, if u is arbitrary on $[t_0, t_1]$ and is equal to u_c on $[t_0-T_0, t_0]$, then

$$V(x_0, u, t_0, t_1) = V(0, u, t_0-T_0, t_1) - V(0, u, t_0-T_0, t_0). \tag{A.6}$$

The first term on the right is positive, since the covariance condition is fulfilled, and (A.5) holds if t_1 is replaced by t_0. Then

$$V(x_0, u, t_0, t_1) \geqslant -V(0, u, t_0-T_0, t_0) \geqslant -k\|x_0\|^2,$$

q.e.d.

We note that the upper estimate for the minimum over u of the functional of (A.2) corresponds to the control u = 0. The functional is zero for this control. Thus, there are upper and lower estimates of the minimum of the functional under conditions where the minimum exists.

The Hamilton-Jacobi theorem (e.g., [14], [15]) now makes it possible to find the minimum of the functional if there is a priori information about its upper and lower bounds. If procedures analogous to those used to minimize loss functions of the form $u'Ru + x'Qx$ are applied, then we find that the minimum of (A.2) is $x_0'\Pi(t_0, t_1)x_0$, where $\Pi(\cdot, t_1)$ is the symmetric matrix given by

$$-\dot{\Pi} = \Pi(F - GR_z^{-1}H_z') + (F' - HR_z^{-1}G')\Pi - \Pi GR_z^{-1}G'\Pi - HR^{-1}H' \qquad (A.7)$$

under the boundary condition $\Pi(t_1, t_1) = 0$.

Finiteness of the upper and lower estimates of the functional ensures that the solution of (A.7) is bounded for all $t \le t_1$.

Further analysis [9, 14] shows that the limit

$$\lim_{t_1 \to \infty} [-\Pi(t, t_1)] = P(t)$$

exists.

The estimate from below still holds if $t_1 \to \infty$, so $x_0'P(t_0)x_0 \le k\|x_0\|^2$ or, which is the same, there is an α_2 such that $P(t) \le \alpha_2 I$ for all t.

The existence of the lower estimate $\alpha_1 I$ for P(t) follows from the discussions in [9], where the uniform complete observable bounds are used.

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