New Linear Smoothing Formulas

BRIAN D. O. ANDERSON AND S. CHIRARATTANANON

Abstract—**Formulas are given expressing the smoothed estimate of the state of a noisy linear system in terms of filtered estimates of the state, for both continuous and discrete time.**

We begin by defining the various quantities of interest. We are given the system

$$\frac{dx(t)}{dt} = F(t)x(t) + G(t)u(t) \quad (1)$$

$$y(t) = H'(t)x(t) + v(t) \quad (2)$$

with $u(\cdot)$ and $v(\cdot)$ zero-mean white Gaussian processes such that

$$E[u(t)u'(\tau)] = Q(t)\delta(t-\tau)$$

$$E[v(t)v'(\tau)] = R(t)\delta(t-\tau)$$

and with $R(t)$ nonsingular for all t. It is assumed that (1) and (2) apply for $t \geq t_0$, that $x(t_0)$ is Gaussian with mean $\bar{x}_0$ and covariance P_0, and that $x(t_0)$, $u(\cdot)$, and $v(\cdot)$ are mutually independent.

As is well known, the quantity $\hat{x}(t/t) = E\{x(t)/y(\tau), t_0 \leq \tau < t\}$ may be computed as follows. Define $P(t)$ by

$$\dot{P} = PF' + FP - PHR^{-1}H'P + GQG', \qquad P(t_0) = P_0 \quad (3)$$

and $K(t)$ by

$$K = PHR^{-1}. \quad (4)$$

Then

$$\dot{\hat{x}} = F\hat{x} - K(H'\hat{x} - y)$$

$$= (F - KH')\hat{x} + Ky, \qquad \hat{x}(t_0/t_0) = \bar{x}_0. \quad (5)$$

Suppose that $b > t$. Then in [1, eq. (28)] it is shown that the quantity $\hat{x}(t/b) = E\{x(t)/y(\tau), t_0 \leq \tau < b\}$ is given by

$$\hat{x}(t/b) = \hat{x}(t/t) + P(t)\int_t^b \Phi_K'(\tau,t)H(\tau)R^{-1}(\tau)\nu(\tau)\,d\tau \quad (6)$$

where $\Phi_K(\cdot,\cdot)$ is the transition matrix associated with $F - KH'$, and the innovation process $\nu(t)$ is defined by

$$\nu(t) = y(t) - H'(t)\hat{x}(t/t). \quad (7)$$

Equations (6) and (7) imply that a smoothed estimate $\hat{x}(t/b)$ can be regarded as a linear functional of the measurements $y(\tau)$ for $t \leq \tau < b$ and the filtered estimate $\hat{x}(\tau/\tau)$ for $t \leq \tau < b$. The content of our main result is that $\hat{x}(t/b)$ can be regarded as a linear functional of only the filtered estimate $\hat{x}(\tau/\tau)$ for $t \leq \tau \leq b$.

Theorem: With all quantities as defined previously, suppose that $P(\tau)$ is nonsingular for $t \leq \tau \leq b$. Then the following formula expresses $\hat{x}(t/b)$ as a linear functional of $\hat{x}(\tau/\tau)$, $t \leq \tau \leq b$:

$$\hat{x}(t/b) = P(t)\Phi_K'(b,t)P^{-1}(b)\hat{x}(b/b)$$
$$+ P(t)\int_t^b \Phi_K'(\tau,t)P^{-1}(\tau)G(\tau)Q(\tau)G'(\tau)P^{-1}(\tau)\hat{x}(\tau/\tau)\,d\tau. \quad (8)$$

Proof: From the filter equation (5) and the definition (7) of $\nu(\cdot)$ we see that

$$\dot{\hat{x}}(t/t) - F(t)\hat{x}(t/t) = K(t)\nu(t) = P(t)H(t)R^{-1}(t)\nu(t).$$

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The authors are with the Department of Electrical Engineering, University of Newcastle, Newcastle, N.S.W. 2308, Australia.

Equation (6) then becomes

$$\hat{x}(t/b) = \hat{x}(t/t) + P(t)\int_t^b \Phi_K'(\tau,t)P^{-1}(\tau)[\dot{\hat{x}}(\tau/\tau) - F(\tau)\hat{x}(\tau/\tau)]\,d\tau. \quad (9)$$

Now it follows through integration by parts that

$$P(t)\int_t^b \Phi_K'(\tau,t)P^{-1}(\tau)\dot{\hat{x}}(\tau/\tau)\,d\tau$$
$$= P(t)[\Phi_K'(\tau,t)P^{-1}(\tau)\hat{x}(\tau/\tau)]_t^b - P(t)\int_t^b \Phi_K'(\tau,t)$$
$$\cdot[F'(\tau) - H(\tau)K'(\tau)]P^{-1}(\tau)\hat{x}(\tau/\tau)\,d\tau - P(t)\int_t^b \Phi_K'(\tau,t)$$
$$\cdot[-P^{-1}(\tau)F(\tau) - F'(\tau)P^{-1}(\tau) + H(\tau)R^{-1}(\tau)H'(\tau)$$
$$- P^{-1}(\tau)G(\tau)Q(\tau)G'(\tau)P^{-1}(\tau)]\hat{x}(\tau/\tau)\,d\tau$$
$$= -\hat{x}(t/t) + P(t)\Phi_K'(b,t)P^{-1}(b)\hat{x}(b/b)$$
$$+ P(t)\int_t^b \Phi_K'(\tau,t)[P^{-1}(\tau)F(\tau)$$
$$+ P^{-1}(\tau)G(\tau)Q(\tau)G'(\tau)P^{-1}(\tau)]\hat{x}(\tau/\tau)\,d\tau.$$

Substituting this in (9), we obtain the desired result.

Remark 1: The formula (8) may be more attractive than (6) for the purposes of computing a smoothed estimate. On the other hand, for theoretical work (6) is probably more attractive; for example, computation of the error covariance matrix associated with $\hat{x}(t/b)$ is straightforward using (6), but would be difficult using (8).

Remark 2: The integration by parts used in the proof is possible because the integrals are at worst Wiener integrals, so that the special techniques usually required for dealing with Ito integrals are not needed here.

Remark 3: The condition that $P(t)$ be nonsingular is not a severe one, and in fact, invertibility of $P(t)$ is a standard assumption made often without comment in linear smoothing work (see, e.g., [1]). Actually, as shown in [2], $P(t)$ is nonsingular if and only if

$$W(t,t_0) = \Phi_0(t,t_0)P_0\Phi_0'(t,t_0) + \int_{t_0}^t \Phi_0(t,\tau)G(\tau)Q(\tau)G'(\tau)\Phi_0'(t,\tau)\,d\tau$$

is nonsingular. (Here, $\Phi_0(\cdot,\cdot)$ is the transition matrix associated with F.) Nonsingularity of $W(\cdot,\cdot)$ is easily guaranteed by, for example, nonsingularity of P_0 or complete controllability of $[F, GQ^{1/2}]$. For more extended remarks, see [2].

Remark 4: In the event that the input noise $u(\cdot)$ and output noise $v(\cdot)$ fail to be independent, (8) fails and there appears to be no simple way of modifying it to cope with the dependence.

For the discrete-time case, the basic system is

$$x(k+1) = \phi_k x(k) + G_k w(k)$$
$$y(k+1) = H_{k+1}'x(k+1) + v(k+1). \quad (10)$$

We assume $w(\cdot)$ and $v(\cdot)$ are zero-mean Gaussian sequences, with $E[w(k)w'(l)] = Q_k\delta(k-l)$ and $E[v(k)v(l)] = R_k\delta(k-l)$. Equation (10) is defined for integer $k \geq k_0$, with $x(k_0)$ a Gaussian random variable of mean $\bar{x}_0$ and covariance P_0. Further, $v(\cdot)$, $w(\cdot)$, and $x(k_0)$ are mutually independent.

The estimate $\hat{x}(k/k)$ may be obtained as follows. Define a recursion commenced with $P_{k_0} = P_0$ by

$$\bar{P}_k = \phi_k P_k \phi_k' + G_k Q_k G_k' \quad (11)$$

$$K_{k+1} = \bar{P}_k H_{k+1}(H_{k+1}'\bar{P}_k H_{k+1} + R_{k+1})^{-1} \quad (12)$$

$$P_{k+1} = (I - K_{k+1}H_{k+1}')\bar{P}_k = \bar{P}_k(I - H_{k+1}K_{k+1}'). \quad (13)$$

Then

$$\hat{x}(k+1/k+1) = \phi_k \hat{x}(k/k) + K_{k+1}[y(k+1) - H_{k+1}'\phi_k \hat{x}(k/k)], \qquad \hat{x}(k_0/k_0) = \bar{x}_0. \quad (14)$$

With $b > k$, one equation for the smoothed estimate is readily shown to be

$$\hat{x}(k/b) = \hat{x}(k/k) + \sum_{k+1}^{b} E[x(k)\nu'(l)]\{E[\nu(l)\nu'(l)]\}^{-1}\nu(l) \quad (15)$$

where

$$\nu(l) = y(l) - H_l'\phi_l \hat{x}(l-1/l-1). \quad (16)$$

A sequence of manipulations will lead to the following expression:

$$\hat{x}(k/b) = \sum_{l=k}^{b-1} \left[\prod_{m=k}^{l} D_m^{-1}\right](D_l - \phi_l)\hat{x}(l/l) + \left(\prod_{m=k}^{b-1} D_m^{-1}\right)\hat{x}(b/b) \quad (17)$$

where

$$D_m = \bar{P}_m(P_m\phi_k')^{-1}.$$

Remarks 1 and 3 made for the continuous-time case apply, *mutatis mutandis*, to the discrete-time case.

References

[1] T. Kailath and P. Frost, "An innovations approach to least-squares estimation: Part II—Linear smoothing in additive white noise," *IEEE Trans. Automat. Contr.*, vol. AC-13, pp. 655–660, Dec. 1968.
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