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On the Existence of H Matrices

An earlier correspondence by one of the authors [1] proposed the thesis that every passive time-invariant linear n-port has at least one "H Matrix." The problem is further considered here and a slight modification necessary for completeness of the result is pointed out.

If the port voltage $[V]$ and current $[I]$ vectors of an n-port network N are partitioned according to some partitioning of the ports by $[\tilde{V}] = [\tilde{V}_1, \tilde{V}_2]$, and $[\tilde{I}] = [\tilde{I}_1, \tilde{I}_2]$ (where the tilde denotes transposition), then an H Matrix is an $n \times n$ matrix relating $[\tilde{V}_1, \tilde{I}_2]$ to $[\tilde{I}_1, \tilde{V}_2]$ through an equation of the form

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = [H] \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}. \tag{1}$$

In the earlier discussion it was assumed that the network N could be described by equations of the form

$$[A][V] = [B][I] \tag{2}$$

where $[A]$ and $[B]$ were (implicitly) taken to be $n \times n$ matrices, in general functions of the complex frequency s.

In this note we consider the case where $[A]$ and $[B]$ need not be square; in addition we examine some consequences of the general description (2) when $[A]$ and $[B]$ can be taken square, in particular showing that an H matrix exists for a *passive*, time-invariant, linear n-port if and only if a scattering matrix exists; finally we make some remarks pertinent to active networks.

As a motivating preliminary, consider two uncoupled nullators [2] which may be described by the following equation:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{3}$$

We note that the two uncoupled nullators constitute a 2-port which is passive, time-invariant, and linear (see Newcomb [2], pp. 8–9), but which possesses no H matrix.

More generally, let us suppose that $[A]$ and $[B]$ are $m \times n$ matrices defined for s in some region (in this passive case we will assume for simplicity that this region is Re $s > 0$, and we call values of s in the region *general values* of s). Let us then consider separately the cases $m = n$, $m < n$, and $m > n$. When $m = n$ the results of [1] are immediately applicable and det $([A] + [B]) \neq 0$ for general values of s in the passive case, implying that an H matrix exists. If $m < n$, we can adjoin $n - m$ rows of zeros to $[A]$ and $[B]$ in (2) to force $[A]$ and $[B]$ to be square. Moreover, for these new $[A]$ and $[B]$, det $([A] + [B]) = 0$ for all s, and, as pointed out in [1], this violates the passivity of N. When $m > n$, det $([A] + [B])$ is not defined, since $[A]$ and $[B]$ are not square; however, if rank $[A \mid B] \leq n$ for general values of s, then we can modify (2) such that $m = n$. If rank $[A \mid B] > n$ for at least one general value of s, then this modification is not possible (as, for example, in the case of the two uncoupled nullators above).

In summary, we see that det $([A] + [B])$ will be defined if $[A]$ and $[B]$ are square, while if $[A]$ and $[B]$ are not square but $m > n$ and rank $[A \mid B] \leq n$, then new square $[A]$ and $[B]$ can be found which still describe the network, and for which det $([A] + [B])$ is then defined. In other cases the determinant is not defined or the network cannot be passive.

The following theorem may then be stated.

Theorem

Let N be a passive, time-invariant, linear n-port. Then the following three conditions, valid in Re $s > 0$, are equivalent:

1) if N is described by $[A][V] = [B][I]$ with $[A]$ and $[B]$ square, or in the nonsquare case, being able to be replaced by square matrices and being so replaced, then det $([A] + [B]) \neq 0$
2) N possesses a hybrid matrix $[H]$
3) N possesses a scattering matrix $[S]$.

Several comments should be made here. First, passivity requires det $([A] + [B]) \neq 0$ in Re $s > 0$, but forces no *a priori* restriction on det $([A] + [B])$ in Re $s < 0$. Second, in general, det $([A] + [B]) \neq 0$ in Re $s > 0$; however, it is possible for the determinant to have singularities on Re $s = 0$, an infinitely long R-C transmission-line being one example. For simplicity, therefore, we avoid consideration of the $j\omega$-axis. Third, we are implicitly assuming the network scattering matrix $[S]$ is the particular scattering matrix which is normalized to 1-ohm terminations. Finally, we note that in the light of the theorem we can state [2]: Every passive, time-invariant, linear and *solvable* network possesses a hybrid matrix $[H]$.

Proof: We observe first that in [1] it was established that condition 1 implies condition 2. It is pointed out by Newcomb [2], in eq. (19c), that a necessary condition for the existence of $[S] = ([B] + [A])^{-1}([B] - [A])$ is precisely condition 1 in the foregoing; that is, condition 3 implies condition 1. Alternatively, we have that

$$[V] - [I] = [S]\{[V] + [I]\} \tag{4}$$

from which, with $[1_n]$ the $n \times n$ identity,

$$\{[1_n] - [S]\}[V] = \{[1_n] + [S]\}[I] \tag{5}$$

exhibiting $[A]$ and $[B]$ to be square matrices, simply related to $[S]$. Further, det $([A] + [B]) \neq 0$.

It therefore remains to be shown that condition 2 implies condition 3.

Let us assume that $[H]$ is partitioned in the same manner as the port variables, so that (1) becomes

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}. \tag{6}$$

Here $[H_{11}]$ is a $p \times p$ matrix for some $p \leq n$, $[H_{22}]$ a $q \times q$ matrix where $q = n - p$, etc.

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Defining incident voltage vectors $[V_1^i] = \frac{1}{2}\{[V_1] + [I_1]\}$, $[V_2^i]$ similarly and reflected voltage vectors $[V_1^r] = \frac{1}{2}\{[V_1] - [I_1]\}$, $[V_2^r]$ similarly, we obtain

$$\begin{bmatrix} -(1_p + H_{11}) & H_{12} \\ -H_{21} & 1_q + H_{22} \end{bmatrix} \begin{bmatrix} V_1^r \\ V_2^r \end{bmatrix} = \begin{bmatrix} 1_p - H_{11} & H_{12} \\ -H_{21} & 1_q - H_{22} \end{bmatrix} \begin{bmatrix} V_1^i \\ V_2^i \end{bmatrix}. \quad (7)$$

A scattering matrix then exists if

$$\det \begin{bmatrix} -(1_p + H_{11}) & H_{12} \\ -H_{21} & 1_q + H_{22} \end{bmatrix} \neq 0 \quad \text{for} \quad \mathrm{Re}\, s > 0. \quad (8)$$

Suppose unit resistors are connected across each port, requiring $[V_1] = [-I_1]$, $[V_2] = [-I_2]$. Then (6) becomes

$$\begin{bmatrix} 1_p + H_{11} & -H_{12} \\ H_{21} & -(1_q + H_{22}) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0. \quad (9)$$

Passivity requires that in the right half plane

$$\det \begin{bmatrix} 1_p + H_{11} & -H_{12} \\ H_{21} & -(1_q + H_{22}) \end{bmatrix} \neq 0 \quad (10)$$

which is a restatement of (8). This proves the result.

The importance of passivity in the original proof [1] and the foregoing theorem should be clear. It is apparent, though, that not much can be said about the equivalences of the theorem in the general active case. For example, a -1-ohm resistor possesses an $[A]\,[V] = [B]\,[I]$ description, possesses an $[H]$ matrix, but does not have a scattering matrix (with respect to 1-ohm terminations), and $\det([A] + [B]) = 0$ for all s. Still, after inserting minor details, Youla has shown ([3], p. 196) that $[H]$ exists as well as a scattering matrix $[S_R]$ with respect to some set of positive uncoupled resistive terminations (of diagonal impedance matrix $[R]$) if and only if $n \times n$ $[A]$ and $[B]$ exist with $[A \mid B]$ possessing at least one nonsingular "special" submatrix of order n. Here an $n \times n$ submatrix $[D]$ of a matrix $[C] = [a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n]$, with the a_i's and b_i's representing the n columns of $[A]$ and $[B]$, is called "special" if $[D] = [a_{k_1}, a_{k_2}, \cdots, a_{k_r}, b_{i_1}, b_{i_2}, \cdots, b_{i_{n-r}}]$ has $k_q \neq i_m$ for $q = 1, 2, \cdots, r$ and $m = 1, 2, \cdots, n - r$. We also mention that it is rather difficult to be specific on the s-plane regions of definition of $[A]$ and $[B]$ since one can premultiply (2) by almost any nonsingular matrix while still preserving the network constraints contained in (2).

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B. D. Anderson
R. W. Newcomb
Stanford Electronics Labs.
Stanford, Calif.
J. K. Zuidweg
Royal Naval Academy
Den Helder, The Netherlands

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