The Small-Gain Theorem, the Passivity Theorem and their Equivalence†

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ABSTRACT: *Two small-gain theorems and a passivity theorem yielding stability results for general feedback systems are shown to be equivalent.*

I. Introduction

We are concerned with the system shown in Fig. 1. In this section, we describe the results of succeeding sections in as qualitative a fashion as possible, reserving most mathematical details for later sections.

The figure is a symbolic representation of the equations

$$\left.\begin{aligned} e_1 &= u_1 - G_2 e_2, \\ e_2 &= u_2 + G_1 e_1. \end{aligned}\right\} \tag{1}$$

We are interested in studying the stability of the closed-loop system; and, in particular, in obtaining stability results with the minimum of assumptions on G_1 and G_2. Stability here carries the connotation that, roughly, bounded u_i should lead to bounded e_i, the bound being obtained with some suitable norm.

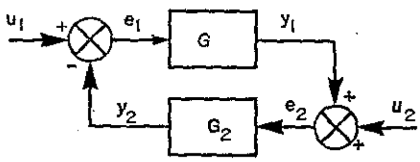

FIG. 1. Basic system whose stability is to be studied; it is also discussed in Ref. (2).

One important result, generally called the *small-gain* theorem, is to the effect that if the loop gain, defined in an appropriate sense, is less than one, then the system is stable (1, 2). A second important result, called the *passivity* theorem, is to the effect that if G_1 and G_2 are passive or positive operators, with at least one of them strictly positive, the closed-loop is stable, (1, 2). This paper exhibits the relations between these theorems.

For the remainder of this section, we discuss the relationship of our results with known results. The paper (1) deals with a variant on the system of Fig. 1; this variant is shown in Fig. 2. A major difference between the two arrangements is that the system of Fig. 2 has in effect a single input x, multiplied by constants a_1 and a_2. The symbols w_1 and w_2 are functions which can be used to account for initial conditions, but in any case are fixed

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functions. Each H_i is a relation (one-to-many mapping) rather than an operator (one-to-one mapping), which permits incorporation of initial conditions into H_i if desired.

In (**1**), it is shown how to assign to each H_i a positive number $\gamma(H_i)$, called the gain of H_i; amongst other properties,

$$\gamma(H_1 H_2) \leqslant \gamma(H_1)\,\gamma(H_2). \tag{2}$$

The small-gain theorem then becomes a statement that stability follows if $\gamma(H_1)\,\gamma(H_2) < 1$. From the small-gain theorem, there is deduced the passivity theorem, which is actually a special case of a theorem predicting stability in case certain conicity results are satisfied.

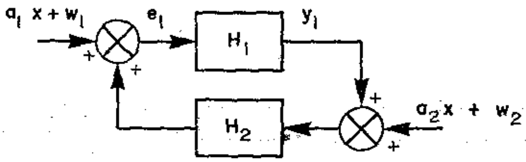

FIG. 2. System whose stability is considered in Ref. (**1**); the H_i are relations, or one-to-many operators.

In (**2**), again it is shown how to assign each G_i a gain $\gamma(G_i)$, actually now an operator norm. The small-gain theorem is concerned with establishing stability under the condition

$$\gamma(G_2 G_1) < 1 \tag{3}$$

which, in view of (2), is a stronger result than that of (**1**); however, in (**2**), stronger continuity conditions are imposed on the operator G_2 than in (**1**). A passivity theorem is also derived in (**2**). Again, it is a special case of a continuity result, but the derivation does not make use of the small-gain theorem.

Our aim here is to show the equivalence of the two versions of the small-gain theorem and the passivity theorem, in the following sense: any one of the three theorems implies the other two theorems. In so doing, we note a network theoretic interpretation via scattering operators of the theorems which, to those familiar with the properties of scattering matrices in the usual network sense, may render the connection between the theorems more transparent.

II. Mathematical Preliminaries

In large measure, we follow the definitions of Ref. (**2**). However, in order to apply the passivity theorem, it is necessary that the signals in the loop of the system in Fig. 1 be functions of time contained not merely in a linear normed space, but in a space with an inner product. Therefore, we shall assume that the signals in the loop of Fig. 1 take values at each time in finite-dimensional Euclidean space, and that for any finite T_1, T_2 contained in the time interval over which the system is defined, the pointwise norm of the signal over the interval $[T_1, T_2]$ is in the space $\mathscr{L}_2[T_1, T_2]$. We also use the symbol $\mathscr{L}_2[T_1, T_2]$ to denote vector functions of time of arbitrary finite dimension whose pointwise norms over $[T_1, T_2]$ lie in the space $\mathscr{L}_2[T_1, T_2]$.

The time interval over which the system is defined, denoted by S, is normally $(-\infty, \infty)$ or $[T_0, \infty)$. All signals will be assumed to lie in the extended space $\mathscr{L}_{2e}$, which is defined with the aid of a projection operator as follows. The operator P_T maps any (vector) function of time $y(\cdot)$ defined for $t \in S$ into another such function according to

$$\begin{aligned} P_T y(t) &= y(t), \quad t \leqslant T, \quad t \in S, \\ &= 0, \qquad t > T, \quad t \in S. \end{aligned}$$

Then $\mathscr{L}_{2e}$ is defined by $x \in \mathscr{L}_{2e}$ if and only if $P_T x \in \mathscr{L}_2$ for all $T \in S$, where $\mathscr{L}_2$ here denotes $\mathscr{L}_2(-\infty, \infty)$ or $\mathscr{L}_2[T_0, \infty)$, depending on the definition of S.

Next, we define the notion of stability. Let us define G as $(G_2 e_2, -G_1 e_1)$, u as (u_1, u_2) and e as (e_1, e_2); this allows the equation describing the system of Fig. 1 to be written as

$$(I+G)\,e = u. \tag{4}$$

If for any given $u \in \mathscr{L}_2$, e exists, the system is said to be stable if:

(i) $e \in \mathscr{L}_2$;

(ii) there exists a positive constant K, independent of u, such that $\|e\|_{\mathscr{L}_2} \leqslant K \|u\|_{\mathscr{L}_2}$.

It proves convenient to be able to assign norms to signals $x \in \mathscr{L}_{2e}$ in the following way

$$\|x\|_{\mathscr{L}_{2e}} = \sup_{T \in S} \|P_T x\|_{\mathscr{L}_2}$$

which implies that if $x \in \mathscr{L}_2$, $\|x\|_{\mathscr{L}_{2e}} = \|x\|_{\mathscr{L}_2}$, while if $x \notin \mathscr{L}_2$, $\|x\|_{\mathscr{L}_{2e}} = \infty$.

Finally, we assume some conditions on G, as follows:

(i) G is causal, i.e. $P_T G = P_T G P_T$;

(ii) G maps $\mathscr{L}_2$ into $\mathscr{L}_2$, $G(0) = 0$ and G is continuous on $\mathscr{L}_2$.

Conditions (i) and (ii) imply that G maps $\mathscr{L}_{2e}$ into $\mathscr{L}_{2e}$; two norms of G can therefore be defined, depending on whether G is regarded as mapping $\mathscr{L}_2$ into $\mathscr{L}_2$ or $\mathscr{L}_{2e}$ into $\mathscr{L}_{2e}$. In the event that either norm is finite, it is possible to show that the two norms are equal (**1**). Henceforth, norms of operators and functions will be denoted without any subscript.

III. The Small-gain Theorems and the Passivity Theorem

In this section, we review the statements of the small-gain theorem and the passivity theorem following (**1**) and (**2**).

Theorem I. (**2**). Consider the system depicted in Fig. 1, and suppose that the operators G_1 and G_2 satisfy conditions as noted in the previous section together with the conditions:

(i) G_1 is bounded and G_2 is Lipschitz continuous;

(ii) $\|G_2 G_1\| < 1$.

Then the system is stable.

One might imagine that the small-gain theorem could equally well be stated with G_1 and G_2 interchanged, i.e. if $\|G_1 G_2\| < 1$, then the system is stable. This is only partly true; observe that the condition on G_2 is stronger than that on G_1, in that G_2 is assumed Lipschitz continuous, whereas G_1 is not. One can however obtain a small-gain theorem by dropping the Lipschitz continuity requirement and demanding that $\|G_2\|\,\|G_1\|<1$; of course, this is a stronger condition than $\|G_2G_1\|<1$.

Theorem II. (**1**). Consider the system depicted in Fig. 1 and suppose that the operators G_1 and G_2 satisfy conditions as noted in the previous section. Suppose also that $\|G_2\|\,\|G_1\|<1$. Then the system of Fig. 1 is stable.

To state the passivity theorem, the notion of a passive or positive operator is required: F, mapping an inner product space X into itself, is said to be a passive or positive operator if

$$\operatorname{Re}<x,\quad Fx>_X\geqslant 0\quad\text{for all}\quad x\in X.$$

Theorem III (**1, 2**). Consider the system depicted in Fig. 1 and suppose that G_1 and G_2 satisfy conditions as noted in the previous section. Suppose also that the dimensions of e_1 and y_1 (and thus of all signals depicted in Fig. 1) are the same. Then the system of Fig. 1 is stable if $G_1-\varepsilon I$ and G_2 are passive for some $\varepsilon>0$ and $\|G_1\|<\infty$, or $G_2-\varepsilon I$ and G_1 are passive for some $\varepsilon>0$, and $\|G_2\|<\infty$.

IV. Passive and Contractive Operators

We have already defined the concept of a passive operator; the reader familiar with network theory will recognize that a passive operator is a generalization of a passive impedance or admittance, these being linear operators describable via impulse responses, with the Laplace transform of the impulse response being positive real (**3**). Let us define a *contractive operator* G as one for which $\|G\|\leqslant 1$ and such that G maps an inner product space into the same space. Then G can be regarded as the generalization of a passive scattering matrix (or function); passive scattering matrices are linear operators arising in network theory and are describable via impulse responses, the Laplace transforms of which are bounded real (**3**).

Network theory in fact suggests two possible ways of describing passive systems; one way is by linear operators which are passive or positive in the sense already defined, and the other way is by linear operators which are contractive. Thus network theory suggests that when appropriate operators are used to describe a passive system, even a general system of the form of Fig. 1, the operators may have the property of having norm bounded by unity. In other words, network theory suggests that to exhibit the equivalence of the small-gain and the passivity theorem, the arrangement of Fig. 1 should be redrawn so that the passivity property can be interpreted as a gain-bounded-by-unity property.

In order to do this, it is necessary to have some scheme for relating arbitrary passive operators to contractive operators, in the same manner that passive impedance or admittance matrices are related to passive scattering matrices†. The required scheme is described in the following theorem.

† Professor C. A. Desoer has pointed out that a partial development along these lines was achieved in Refs. (**4**) and (**5**), which are concerned with monotone, rather than passive, operators. An operator F is monotone if $\operatorname{Re}\langle x_1-x_2,\ F(x_1-x_2)\rangle_X\geqslant 0$ for all $x_1, x_2\in X$.

Theorem IV. Let G be a passive operator, and let operators E and F map the input v^i into the output v^r of the system shown in Figs. 3(a) and 3(b)† respectively. Then E and F are contractive. If G is bounded and $G-\varepsilon I$ is passive for some $G>0$, then $\|E\|<1$ and $\|F\|<1$. Conversely, let E be a contractive operator, and let operators C and D map the input e into the output y of the system shown in Figs. 4(a) and 4(b) respectively. Then C and D are passive. Moreover, if $\|E\|<1$, then C and D are bounded and $C-\varepsilon I$ and $D-\varepsilon I$ are passive for some $\varepsilon>0$.

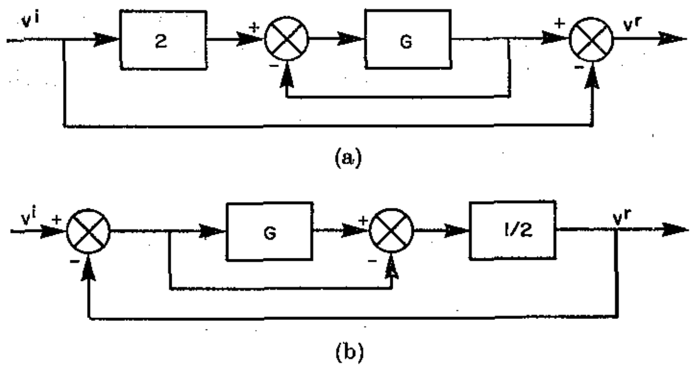

FIG. 3(a). Conversion of a passive operator G to a contractive operator E. (b). Conversion of a passive operator G to a contractive operator F.

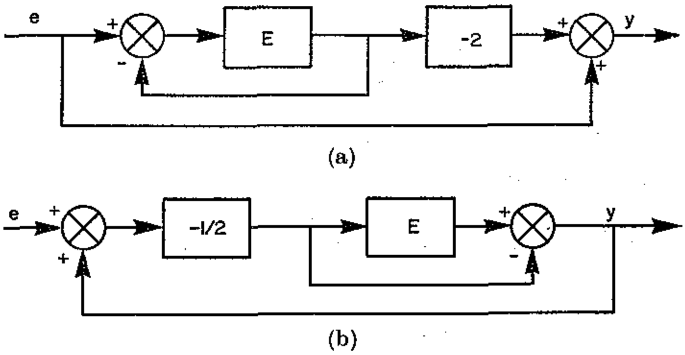

FIG. 4(a). Conversion of a contractive operator E to a passive operator C. (b). Conversion of a contractive operator E to a passive operator D.

† The author is indebted to Professor C. A. Desoer for pointing out that if in Fig. 3(a), the gain block 2 is moved to the immediate left of the right-hand summing point, the operator mapping v^i into v^r is $(G-I)\ (G+I)^{-1}$, while in the case shown, it is $\frac{1}{2}(G-I)\ (G+I)^{-1}$. Both are of course contractive, but the latter is more like a true scattering operator. The same remark holds mutatis mutandis for the arrangement of Fig. 3(b).

Proof: First note that the theorem makes no actual claim about the existence of the operators E and F mapping v^i into v^r in Figs. 3(a) and 3(b), nor the operators C and D mapping e into y in Figs. 4(a) and 4(b). It is now generally recognized that the question of existence of operators is a separate one from the question of boundedness. Generally speaking, existence can only be ensured by demanding some extra set of conditions, typically conditions which require that there be a infinitesimal delay around any loop. We assume here that all operators are well-defined, either as a result of infinitesimal delay in the loop or some other reason. This actually rules out the case of $E = -I$ in Figs. 4(a) and 4(b); inspection shows that y will be infinite for all e. On the other hand, $E = I$ leads to the operators C and D being identically zero. We now turn to the proof proper.

For the schemes of Fig. 3, it is straightforward to check that the input to the G block is $v^i - v^r$ and the output is $v^i + v^r$. Hence,

$$G(v^i - v^r) = v^i + v^r.$$

Therefore

$$\begin{aligned} 0 \leqslant \langle v^i - v^r, G(v^i - v^r)\rangle &= \langle v^i, v^i\rangle - \langle v^r, v^r\rangle \\ &= \| v^i \|^2 - \| v^r \|^2. \end{aligned} \tag{5}$$

From this equation it follows that E and F are contractive. If also $G - \varepsilon I$ is passive for $\varepsilon > 0$, then (5) can be strengthened to

$$\| v^i \|^2 - \| v^r \|^2 \geqslant \varepsilon \| v^i - v^r \|^2. \tag{6}$$

Further, if G is bounded,

$$\| v^i + v^r \| \leqslant \| G \| \, \| v^i - v^r \|$$

or

$$\| v^i - v^r \| + \| v^i + v^r \| \leqslant (1 + \| G \|) \| v^i - v^r \|$$

or

$$\| v^i \| \leqslant \tfrac{1}{2}(1 + \| G \|) \| v^i - v^r \|. \tag{7}$$

Hence, combining (6) and (7):

$$\| v^i \|^2 - \| v^r \|^2 \geqslant \frac{4\varepsilon}{(1 + \| G \|)^2} \| v^i \|^2$$

or

$$\frac{\| v^r \|^2}{\| v^i \|^2} \leqslant 1 - \frac{4\varepsilon}{(1 + \| G \|)^2} \tag{8}$$

from which the bounds $\| E \| < 1$ and $\| F \| < 1$ follow.

Next, we prove the converse. It is not hard to check that the input to block E in Fig. 4(a) is $\frac{1}{2}(e + y)$ and the output is $\frac{1}{2}(e - y)$, i.e.

$$E(\tfrac{1}{2}\overline{e + y}) = \tfrac{1}{2}\overline{e - y}.$$

Inspection of Fig. 4(b) will reveal that

$$E(-\tfrac{1}{2}\overline{e + y}) = -\tfrac{1}{2}\overline{e - y}.$$

In either case, the contractive property of E forces

$$\tfrac{1}{4}\|e+y\|^2-\tfrac{1}{4}\|e-y\|^2 \geqslant 0 \tag{9}$$

or

$$\langle y, e\rangle \geqslant 0. \tag{10}$$

This establishes the passivity property for C and D. If now $\|E\|<1$, (9) can be straightened to

$$\tfrac{1}{4}\|e+y\|^2-\tfrac{1}{4}\|e-y\|^2 \geqslant \varepsilon\|e+y\|^2$$

for some ε with $0<\varepsilon<\frac{1}{4}$. This inequality implies

$$\langle y, e\rangle \geqslant \varepsilon\|y\|^2+2\varepsilon\langle y, e\rangle+\varepsilon\|e\|^2$$

which in turn implies

$$\|y\|\;\|e\| \geqslant \varepsilon\|y\|^2 \tag{11}$$

and

$$\langle y, e\rangle-\varepsilon\|e\|^2 \geqslant 0. \tag{12}$$

Equation (11), being $\|y\| \leqslant \varepsilon^{-1}\|e\|$, establishes boundedness of C and D, while (13) establishes the passivity of $C-\varepsilon I$ and $D-\varepsilon I$.

If in the above theorem, all operators are linear, the connection with network theory becomes very clear. In the case of the schemes of Figs. 3(a) and 3(b),

$$E=F=(G-I)(G+I)^{-1}. \tag{13}$$

If G is an impedance, $E=F$ is the associated scattering matrix, while if G is an admittance, $E=F$ is the negative of the usually defined associated scattering matrix, but nevertheless meets all the conditions requiring it to be itself a scattering matrix (**3**). In the case of the schemes of Figs. 4(a) and 4(b),

$$C=D=(I-E)(I+E)^{-1}, \tag{14}$$

C and D are both the admittances associated with the scattering matrix E; and C and D are the impedances associated with $-E$ (**3**).

V. Equivalence of the Small-gain Theorems and the Passivity Theorem

In this section, we show the equivalence of the two versions of the small-gain theorem and the passivity theorem. We first argue that the small-gain Theorem II implies the passivity Theorem III. The main tools for this are the theorem of the previous section and the following result:

Lemma 1. The systems of Figs. 1 and 5 are the same in the sense that if functions u_1 and u_2 are the same for both systems, then the functions e_1, e_2, y_1 and y_2 are the same for both systems. The proof of this lemma follows by simple block diagram manipulation and will be omitted.

Using this lemma and Theorem IV we establish the following:

Theorem V. If Theorem II is true, then Theorem III is true.

Proof: Our task is as follows: We assume that G_1 and G_2 are given, with $G_1 - \varepsilon I$ and G_2 passive and $\|G_1\| < \infty$. Using the small-gain Theorem II, we establish stability. The case when the conditions on G_1 and G_2 are interchanged proceeds the same way, and will not be done.

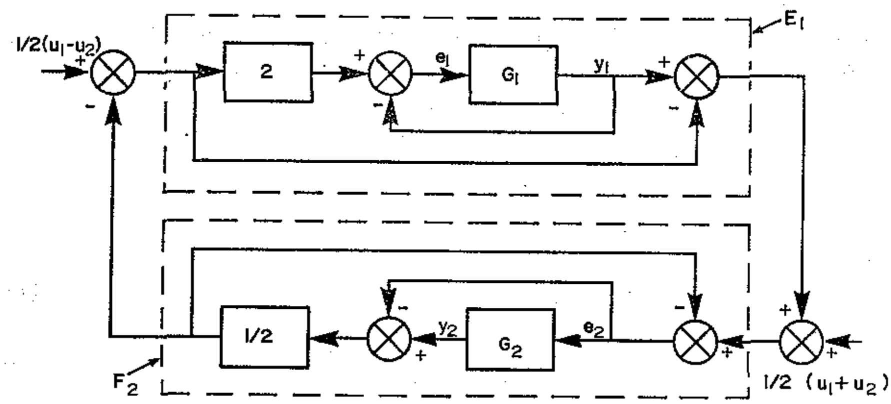

FIG. 5. Passivity properties of G_1 and G_2 in the system of Fig. 1 are converted to contractive properties of E_1 and F_2 in this equivalent system.

By Theorem IV, the operators E_1 and F_2 depicted in Fig. 5 are contractive, with $\|E_1\| < 1$. Hence

$$\|F_2\| \, \|E_1\| < 1. \tag{15}$$

By Theorem II, arbitrary signals $\frac{1}{2}(u_1 - u_2)$ and $\frac{1}{2}(u_1 + u_2)$ in $\mathscr{L}_2$ will lead to signals including e_i and y_i in Fig. 5 being in $\mathscr{L}_2$. Since the system of Fig. 5 is equivalent to that of Fig. 1 and since permitting $\frac{1}{2}(u_1 - u_2)$ and $\frac{1}{2}(u_1 + u_2)$ to be arbitrary is equivalent to permitting u_1 and u_2 to be arbitrary, the system of Fig. 1 is stable.

Theorem VI. If Theorem III is true, then Theorem I is true.

Proof: Our task is as follows: We assume that in Fig. 1 G_1 and G_2 are given, with $\|G_2 G_1\| < 1$, G_1 bounded and G_2 Lipschitz continuous. Using the passivity theorem, we shall establish stability. Note that we do not assume that the dimensions of e_1 and y_1 are the same.

It is easy to check that the scheme of Fig. 6 is equivalent to that of Fig. 1, under the identification

$$\hat{u}_2 = G_2(u_2 + G_1 e_1) - G_2(G_1 e_1). \tag{16}$$

By the Lipschitz continuity of G_2,

$$\|\hat{u}_2\| \leqslant K \|u_2\| \tag{17}$$

with K some positive constant, so that if $u_2 \in \mathscr{L}_2$, then $\hat{u}_2 \in \mathscr{L}_2$. Next, it is straightforward to check that the scheme of Fig. 7(a) is equivalent to that of Fig. 6. The scheme of Fig. 7(a) is redrawn in Fig. 7(b), to define operators C_1 and D_2; by Theorem IV, $C_1 - \varepsilon I$ is passive for some ε, with $\|C_1\| < \infty$, while D_2 is the zero operator. The passivity Theorem III guarantees stability of the scheme of Fig. 7 for all u_1 and $\hat{u}_2$, and in particular those $\hat{u}_2$ resulting from a u_1 and u_2 according to (16). A simple argument then yields the stability of the system of Fig. 1.

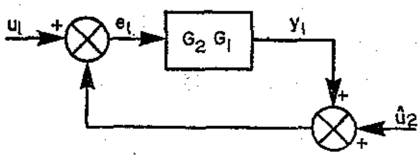

FIG. 6. A system equivalent to that of Fig. 1: Lipschitz continuity of G_2 is used in defining $\hat{u}_2$ and $G_2\,G_1$ is contractive.

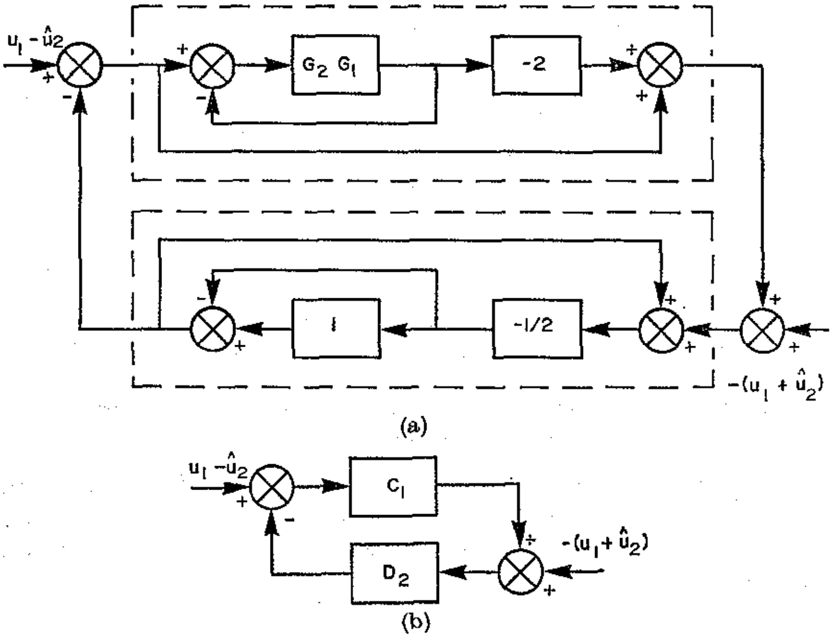

FIG. 7(a). A system equivalent to that of Fig. 6. (b). Fig. 7(a) redrawn to define contractive operators C_1 and D_2, with D_2 the zero operator.

Our final task is to show that the version of the small-gain theorem contained in Theorem I implies the version contained in Theorem II. Taken with the results of Theorem V and VI, this implies that the two versions of the small-gain theorem and the passivity theorem are equivalent results.

Theorem VII. If Theorem I is true, then Theorem II is true.

Proof: Our task is as follows: We assume that in Fig. 1, operators G_1 and G_2 are prescribed with $\|G_1\|\,\|G_2\| < 1$. Our aim is to prove stability, using Theorem I. Let α be a positive constant. From the definition of $\|G_1\|$ as

$$\|G_1\| = \sup_{x\epsilon L_2} \frac{\|G_1 x\|}{\|x\|} \tag{18}$$

it easily follows that

$$\|\alpha I G_1\| = \|G_1 \alpha I\| = \alpha \|G_1\|. \tag{19}$$

In other words, G_1 followed or preceded by a gain of α yields a composite operator with norm $\alpha\|G_1\|$. Now evidently, the scheme of Fig. 8(a) is equivalent in an obvious fashion to that of Fig. 1, and if $\|G_1\|\,\|G_2\| < 1$, there exists a value of α such that with

$$G_1^* = \alpha I G_1, \quad G_2^* = G_2 \alpha^{-1} I \tag{20}$$

then

$$\|G_1^*\| < 1, \quad \|G_2^*\| < 1. \tag{21}$$

We now seek to establish the stability of the system of Fig. 8(a), or equivalently Fig. 8(b); evidently this is as good as establishing stability for the

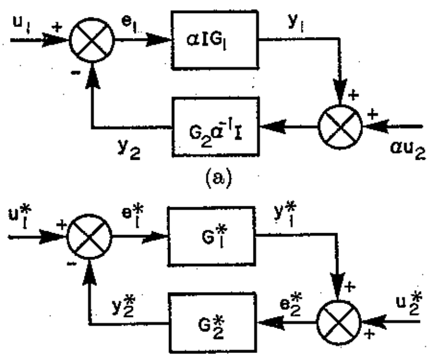

FIG. 8. (a) A system obviously equivalent to the system of Fig. 1. (b) Figure 8(a) redrawn to define G_1^* and G_2^*, which are contractive.

system of Fig. 1. Suppose u_1, e_1 and y_2 have dimension p, and that u_2, e_2 and y_1 have dimension q. Consider the scheme of Fig. 9. All signals have dimension $p+q$, and it is not hard to check that the signals of Fig. 9 are related to those of Fig. 8 by

$$\bar{e}_1 = \begin{bmatrix} e_1^* \\ e_2^* \end{bmatrix}, \qquad \bar{y}_1 = \begin{bmatrix} y_1^* \\ y_2^* \end{bmatrix},$$

$$\bar{e}_2 = \begin{bmatrix} y_1^* + u_2^* \\ y_2^* \end{bmatrix} = \begin{bmatrix} e_2^* \\ y_2^* \end{bmatrix}, \quad \bar{y}_2 = \begin{bmatrix} y_2^* \\ -e_2^* \end{bmatrix}. \tag{22}$$

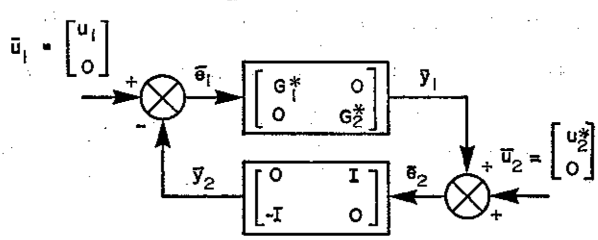

FIG. 9. System equivalent to that of Fig. 8(b) from which stability can be concluded.

Consequently, the system of Fig. 8 is stable if the system of Fig. 9 is stable. Stability of Fig. 9, however, is a consequence of Theorem I identifying

$$G_1 \quad \text{with} \quad \begin{bmatrix} G_1^* & 0 \\ 0 & G_2^* \end{bmatrix},$$

$$G_2 \quad \text{with} \quad \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

We see that G_2 is clearly Lipschitz continuous, while the fact that $\|G_2 G_1\| < 1$ follows easily from $\|G_1^*\| < 1$, $\|G_2^*\| < 1$. Hence Theorem I guarantees stability of the scheme of Fig. 9.

VI. Conclusions

The significance of the equivalence results established is really as follows. In order to establish stability for a certain system, the small-gain theorems and the passivity theorem are equally powerful, so that nothing can be proved with one theorem that could not be proved, after transformation, with another. It is therefore interesting to speculate as to the existence of nontrivial stability results which are more powerful than the theorems discussed.

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