Note that the slope limits

$$\alpha \leq f'(X \sin \theta) \leq \beta$$

give limits on $c$ as

$$-\beta \leq c \leq -\alpha. \quad (9)$$


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## Comments on "On the Noninferiority of Nash Equilibrium Solutions"

Z. V. REKASIUS AND W. E. SCHMITENDORF

In the above paper,[1] a $k$-player nonzero sum differential game is considered and a sufficient condition for a Nash equilibrium solution to be inferior is presented. Further investigation by the authors has revealed an error in the paper.

In order that the theorem presented be correct, the definition of an admissible vector must be modified to read as follows.

*Definition:* The $k$-vector $h(x,t)$ will be called admissible if for $i = 1, 2, \ldots, k$, $h_i(x,t) < 0$.

Using this definition, the main result, presented as a theorem, and its proof are correct as stated. Corollary 1 is also correct, but Corollary 2 is no longer valid. Corollary 3 is correct if the word "nonzero" is deleted. This corollary may be useful if one is examining a non-Nash solution for inferiority.

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## A Note on Bounds on Solutions of the Riccati Equation

K. L. HITZ, T. E. FORTMANN, AND BRIAN D. O. ANDERSON

It is well known that for $t > t_0 + 2\sigma$ ($\sigma$ will be defined later) the solution of the matrix Riccati differential equation

$$\begin{aligned}\dot{P} &= F(t)P + PF'(t) - PH'(t)H(t)P + G(t)G'(t)\\ P(t_0) &= \Gamma = \Gamma' \geq 0 \end{aligned} \quad (1)$$

has upper and lower bounds independent of the initial value $\Gamma$, provided that $F(\cdot)$, $G(\cdot)$, and $H(\cdot)$ are bounded and that there exist positive scalars $\alpha$, $\beta$, $\sigma$ such that $\beta I > C(t + \sigma, t) > \alpha I$ and $\beta I > W(t, t - \sigma) > \alpha I$ for all $t \geq$ some $t_1$, with $t_1 \geq t_0 + \sigma$, where

$$C(t + \sigma, t) = \int_t^{t+\sigma} \Phi(t + \sigma, s)G(s)G'(s)\Phi'(t + \sigma, s)\, ds$$

$$W(t, t - \sigma) = \int_{t-\sigma}^{t} \Phi'(s, t)H'(s)H(s)\Phi(s, t)\, ds$$

and where $\Phi(t, s)$ is the transition matrix of $F(t)$. (The notation $A > B$ for symmetric matrices $A$ and $B$ means $A - B$ is positive definite.) In [1, lemmas 4 and 5], Bucy has claimed that the bounds on the solution $\Pi(t, \Gamma, t_0)$ of (1) have the particularly simple form

$$\begin{aligned}[C^{-1}(t, t - \Delta) + W(t, t - \Delta)]^{-1} &\leq \Pi(t, \Gamma, t_0)\\ &\leq W^{-1}(t, t - \Delta) + C(t, t - \Delta) \end{aligned} \quad (2)$$

where $t > t_0 + \Delta$ and $\Delta$ can be set equal to $2\sigma$. An entirely analogous result is derived in Jazwinski [2, lemmas 7.1 and 7.2] for the solution of the matrix Riccati difference equation. There the result is used in proving the stability of the discrete Kalman–Bucy filter. Kalman [3] obtains similar formulas in order to conclude exponential asymptotic stability of a class of linear optimal control systems.

We wish to point out that the proofs are incorrect in all three cases. In Bucy an essential step in the proof of [1] (2) is the assertion that the inequality (in Bucy's notation)

$$-K(\sigma)H'HK(\sigma) \geq -\Pi(\sigma, \Gamma_0, t - \Delta)H'H\Pi(\sigma, \Gamma_0, t - \Delta)$$

is implied by $\Pi(\sigma, \Gamma_0, t - \Delta) \geq K(\sigma) \geq 0$. However, $\Pi \geq K \geq 0$ does not imply $\Pi^2 \geq K^2 \geq 0$, as the following example shows:

$$K = \begin{bmatrix} 4.4 & 2.5 \\ 2.5 & 1.5 \end{bmatrix} > 0$$

$$\Pi = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} > K > 0$$

$$\Pi^2 - K^2 = \begin{bmatrix} 8.39 & 6.25 \\ 6.25 & 4.50 \end{bmatrix} \ngeq 0.$$

A similar error occurs in the proof of the discrete-time counterpart of (2) in [2]. There it is asserted that

$$\operatorname{cov}\left\{W_c^{-1} \sum_{i=k-N}^{k} D_i \sum_{j=i}^{k-1} e_j\right\} \leq \operatorname{cov}\left\{W_c^{-1} \sum_{i=k-N}^{k} D_i \sum_{j=k-N}^{k-1} e_j\right\} \quad (3)$$

where, in Jazwinski's notation, $D_i = \Phi^T(i, k)M^T(i)R_i^{-1}M(i)\Phi(i, k)$, $R_i > 0$, and $e_j = \Phi(k, j + 1)\Gamma(j)w_{j+1}$, with $w_j$ a white Gaussian random vector with covariance $Q_j$, and $W_c$ is a quantity analogous to $W(t, t - \sigma)$. Set $Q_j = I$, $k = 3$, and $N = 2$. Then (3) reduces to $D_1^2 + (D_1 + D_2)^2 \leq 2(D_1 + D_2)^2$, which, by the preceding example, is not true for all $D_1, D_2 \geq 0$.

In [3] it is claimed that $B > 0$ and $B \geq A \geq 0$ imply $\lambda_{\max} (B^{-1}A^2B^{-1}) \leq 1$. Identification of $B$ with $\Pi$ and $A$ with $K$ above leads to

$$B^{-1}A^2B^{-1} = \begin{bmatrix} 1.06 & -0.63 \\ -0.63 & 0 \end{bmatrix}$$

for which the maximum eigenvalue is approximately 1.5.

Although certain results in [1]–[3] as they stand are invalidated, it is still possible [4] to obtain bounds on the solutions of the Riccati equations in [1]–[3] of the same qualitative character as obtained in these references. In quantitative terms though, the bounds are not as tight as those claimed in [1]–[3]. This means that the stability results of these references are not invalidated.

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