Reprinted from IEEE TRANSACTIONS ON AUTOMATIC CONTROL
Vol. AC-17, No. 1, February 1972, pp. 107-111

External and Internal Stability of Linear Systems—A New Connection

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***Abstract*—Finite-dimensional linear systems are considered with the properties of uniform complete controllability and observability. It is shown that exponential asymptotic stability of the homogeneous part of the system is equivalent to the system mapping inputs with finite $\mathcal{L}_2$ norm into outputs with finite $\mathcal{L}_2$ norm.**

I. Introduction

A linear finite-dimensional system is studied, described by

$$\frac{dx(t)}{dt} = F(t)x(t) + G(t)u(t) \tag{1}$$

$$y(t) = H'(t)x(t). \tag{2}$$

Here, $x(\cdot)$ is an n-dimensional real vector function of time, $u(t)$ is a p-dimensional real vector function of time, and y is an m-dimensional real vector function of time. The matrices $F(\cdot)$, $G(\cdot)$, and $H(\cdot)$ of real functions have appropriate dimensions and are assumed to be such that conditions guaranteeing existence and uniqueness of solutions of (1) and (2) are satisfied. In particular it is assumed that a transition matrix $\Phi(\cdot, \cdot)$ exists, associated with the homogeneous equation

$$\frac{dx(t)}{dt} = F(t)x(t). \tag{3}$$

However, *it is not assumed that the entries of* $F(\cdot)$, $G(\cdot)$, *and* $H(\cdot)$ *are bounded.*

The internal stability properties of the system (1) and (2) are determined by the homogeneous equation (3), while the external or input–output stability properties are determined by the impulse response

$$W(t, \tau) = H'(t)\Phi(t, \tau)G(\tau). \tag{4}$$

The impulse response provides, for arbitrary t_0, a straightforward description of the map taking input functions $u(\cdot)$ defined on $[t_0, \infty)$ into output functions $y(\cdot)$ defined on $[t_0, \infty)$, under the constraint $x(t_0) = 0$:

$$y(t) = \int_{t_0}^{t} W(t, \tau)u(\tau)\, d\tau \tag{5}$$

In this paper it is shown that, when $F(\cdot)$, $G(\cdot)$, and $H(\cdot)$ satisfy certain conditions, then it is possible to relate an internal stability property with an external stability property. Leaving the precise definitions to a later section, we can in fact say that, when (1) is uniformly completely controllable and observable, the internal property of exponential asymptotic stability implies and is implied by the external property that $W(\cdot, \cdot)$ maps p-vector functions $u(\cdot)$ square integrable on $[t_0, \infty)$ into m-vector functions $y(\cdot)$ square integrable on $[t_0, \infty)$, with the operator norm associated with $W(\cdot, \cdot)$ finite and bounded independently of t_0.

Several other connections between the external stability and internal stability of (1) and (2) are known [1]–[5]. For a discussion of the connections in [1]–[4], see [4], which also establishes the result closest to that obtained in this paper. The result of [4] shows that exponential asymptotic stability implies and is implied by a rather complicated external stability property: the impulse response $W(\cdot, \cdot)$ maps input functions $u(\cdot)$ defined on $[t_0, \infty)$ into output functions $y(\cdot)$ defined on $[t_0, \infty)$ such that, if the integral of the square of the norm of the input function over any interval of length of some fixed number σ is bounded, then the integral of the square of the norm of the output function over any interval of length σ is also bounded. Roughly speaking, input functions of bounded power are mapped into output functions of bounded power.

The result of this paper is closest to that of [4] for several reasons. First, [4] establishes the equivalence between the two kinds of stability under the uniform complete controllability and observability constraints. Second, [4] does not assume boundedness of the entries of $F(\cdot)$, $G(\cdot)$, and $H(\cdot)$, in contrast to [1]–[3]. In fact, the same hypotheses are assumed in [4] as here on $F(\cdot)$, $G(\cdot)$, and $H(\cdot)$. Third, the techniques of this paper for deriving the main result are similar to those of [4].

Reference [5] is concerned only with the problem of deducing internal stability, given the presence of external stability. While, as [5] points out, the converse problem is usually easily solved, this is not the case when $F(\cdot)$, $G(\cdot)$, and $H(\cdot)$ are unbounded. Indeed, as the later arguments of this paper show, it seems just as hard to deduce external stability, given internal stability, as it is to deduce internal stability, given external stability. Reference [5] is, however, far more general than this paper in that no restriction is made to linear systems. The basic stability result obtained states that for

Manuscript received December 23, 1970; revised September 9, 1971. Paper recommended by J. C. Willems, Chairman of the IEEE S-CS Stability, Nonlinear Systems, Distributed Systems Committee. This work was supported by the Australian Research Grants Committee.
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any dynamic system with certain controllability and observability properties—these properties differing somewhat from those of this paper—external stability of the sort considered in this paper implies global asymptotic stability of the autonomous system.

II. Definition and Preliminary Material

In this section we shall quote definitions and simple results from [4]. We say that the pair $[F(t), G(t)]$ appearing in (1) is uniformly completely controllable if, following [6], the following conditions hold for some $\delta_c > 0$:

$$\alpha_1 I \le M(s-\delta_c, s) \le \alpha_2 I, \qquad \forall\ s \tag{6}$$

$$\|\Phi(t,\tau)\| \le \alpha_3(|t-\tau|), \qquad \forall\ t \text{ and } \tau \tag{7}$$

where

$$M(s-\delta_c, s) = \int_{s-\delta_c}^{s} \Phi(s-\delta_c, t)G(t)G'(t)\Phi'(s-\delta_c, t)\, dt. \tag{8}$$

Here, α_1 and α_2 are positive constants, while $\alpha_3(\cdot)$ maps the reals into the reals and is bounded on bounded intervals. For symmetric matrices X and Y, the notation $X \ge Y (X > Y)$ means $X - Y$ is nonnegative (positive) definite. For a q-dimensional vector w, $\|w\|$ is $\left(\sum_{i=1}^{q} w_i^2\right)^{1/2}$, and the usual induced matrix norm applies.

As is pointed out in [4], relations such as the right-hand inequality in (6) can be replaced by inequalities between scalar quantities, e.g.,

$$\int_{s-\delta_c}^{s} \|\Phi(s-\delta_c, t)G(t)\|^2\, dt \le n\alpha_2. \tag{9}$$

It should also be noted that, if (6) and (7) hold for some δ_c, they also hold for any $\delta > \delta_c$, perhaps with redefinition of α_2.

The final fact concerning uniform complete controllability which we shall need is the following, again established in [4]. Let s and $x(s)$ be an arbitrary time and an arbitrary state vector, respectively. Then (6) implies and is implied by the existence of a minimal energy input $u_1(\cdot)$ taking (1) from the zero state at time $s - \delta_c$ to the state $x(s)$ at time s, together with positive constants α_4 and α_5 such that

$$\alpha_4\|x(s)\|^2 \le \int_{s-\delta_c}^{s} u_1'(t)u_1(t)\, dt \le \alpha_5\|x(s)\|^2. \tag{10}$$

The energy associated with $u_1(\cdot)$ over $(s-\delta_c, s)$ is the value of the integral appearing in (10), and $u_1(\cdot)$ is a minimal energy input if no other input taking the zero state at time $s-\delta_c$ to the state $x(s)$ at time s has an associated smaller energy. [Actually, the right-hand inequality of (6) is equivalent to the left-hand inequality of (10) independent of whether the left-hand inequality of (6) holds, or the right-hand inequality of (10) holds. In the same way, the left-hand inequality of (6) is equivalent to the right-hand inequality of (10).]

Uniform complete observability is defined for the matrix pair $[F(t), H(t)]$ appearing in (1) and (2), with $[F(t), H(t)]$ termed uniformly completely observable if the following conditions hold:

$$\alpha_6 I \le N(s, s+\delta_0) \le \alpha_7 I, \qquad \forall\ s \tag{11}$$

$$\|\Phi(t,\tau)\| \le \alpha_3(|t-\tau|), \qquad \forall\ t \text{ and } \tau \tag{7}$$

where

$$N(s, s+\delta_0) = \int_{s}^{s+\delta_0} \Phi'(t,s)H(t)H'(t)\Phi(t,s)\, dt. \tag{12}$$

in (11), α_6 and α_7 are positive constants.

If (11) and (7) hold for any one δ_0, then they will hold for any $\delta > \delta_0$, perhaps with redefinition of α_7. Therefore, if (1) and (2) define a system with both the uniform complete controllability and observability properties, one can, without loss of generality, assume that $\delta = \delta_0 = \delta_c$ in the definitions of uniform complete controllability and observability. This follows by replacing δ_0 and δ_c by $\delta = \max(\delta_0, \delta_c)$.

Finally, we define the term exponential asymptotic stability. As noted earlier, the term applies to the homogeneous equation (3), which is said to be exponentially asymptotically stable if there exist positive constants α_8 and α_9 such that

$$\|\Phi(t,\tau)\| \le \alpha_8 \exp[-\alpha_9(t-\tau)] \tag{13}$$

for all t, τ with $t \ge \tau$. Exponential asymptotic stability is equivalent to the property that solutions of (3) decay at least as fast as a decaying exponential.

III. Main Result

In this section we state the main result. A proof will be found in Sections IV and V.

Theorem: Consider the system

$$\frac{dx(t)}{dt} = F(t)x(t) + G(t)u(t) \tag{1}$$

$$y(t) = H'(t)x(t) \tag{2}$$

where $x(\cdot)$, $u(\cdot)$, and $y(\cdot)$ are real vector functions of time of dimension n, p, and m. The matrices $F(\cdot)$, $G(\cdot)$, and $H(\cdot)$ have appropriate dimension and are such as to guarantee existence and uniqueness of solutions of (1) and (2) for appropriate boundary conditions and $u(\cdot)$. Let $\Phi(\cdot,\cdot)$ denote the transition matrix associated with $F(\cdot)$. Suppose further that (1) and (2) are uniformly completely observable and controllable, i.e., there exist positive constants δ, α_1, α_2, α_6, and α_7 and a function $\alpha_3(\cdot)$ mapping the reals into the reals and bounded on bounded intervals, such that for all s,

$$\alpha_1 I \le \int_{s-\delta}^{s} \Phi(s-\delta, t)G(t)G'(t)\Phi'(s-\delta, t)\, dt \le \alpha_2 I \tag{14}$$

$$\alpha_6 I \le \int_{s}^{s+\delta} \Phi'(t,s)H(t)H'(t)\Phi(t,s)\, dt \le \alpha_7 I \tag{15}$$

$$\|\Phi(t,\tau)\| \le \alpha_3(|t-\tau|). \tag{7}$$

Part 1: If (3) is exponentially asymptotically stable, if t_0 is arbitrary, if $x(t_0) = 0$, and if $u(\cdot)$ is such that

$$\int_{t_0}^{\infty} u'(t)u(t)\, dt = 1, \tag{16}$$

then there exists a positive constant α_{10} independent of t_0 and $u(\cdot)$ such that

$$\int_{t_0}^{\infty} y'(t)y(t)\, dt \le \alpha_{10}. \tag{17}$$

Part 2: Conversely, if with arbitrary t_0, with $x(t_0) = 0$, and $u(\cdot)$ such that

$$\int_{t_0}^{\infty} u'(t)u(t)\, dt = 1, \tag{16}$$

it follows that

$$\int_{t_0}^{\infty} y'(t)y(t)\, dt \le \alpha_{10} \tag{17}$$

for some positive constant α_{10}, independent of t_0 and $u(\cdot)$, then (3) is exponentially asymptotically stable.

Several points should be noted. First, the external stability described in the above theorems requires $W(t,\tau)$ to be a bounded mapping of $\mathfrak{L}_2[t_0, \infty)$ functions into $\mathfrak{L}_2[t_0, \infty)$ functions, with the $\mathfrak{L}_2$ norm of $W(\cdot,\cdot)$ being independent of t_0. Second, it is possible to allow $x(t_0) \ne 0$, in which case α_{10} becomes dependent on $\|x(t_0)\|$. The extension is straightforward and will be omitted. Third, we reiterate that $F(\cdot)$, $G(\cdot)$, and $H(\cdot)$ are not assumed to contain bounded entries. If $F(\cdot)$, $G(\cdot)$, and $H(\cdot)$ are assumed bounded, Part 1 of the theorem is trivial. It is then also true (see [3]) that bounded-input bounded-output stability implies and is implied by exponential asymptotic stability. Fourth, the difference between this theorem and the main result of [4] is simply that (16) and (17) are replaced in [4] by

$$\int_s^{s+\delta} u'(t)u(t)\,dt \le 1, \qquad \forall\ s \tag{18}$$

and

$$\int_s^{s+\delta} y'(t)y(t)\,dt \le \alpha_{10}, \qquad \forall\ s. \tag{19}$$

IV. Proof of Part 1

Suppose that $x(t_0) = 0$ for some arbitrary t_0 and that

$$\int_{t_0}^{\infty} u'(t)u(t)\,dt = 1. \tag{16}$$

For $t_0 + r\delta \le t \le t_0 + \overline{r+1}\,\delta$ where r is a positive integer, we have

$$y(t) = H'(t)\Phi(t, t_0 + r\delta) \cdot \left[x(t_0 + r\delta) + \int_{t_0+r\delta}^{t} \Phi(t_0 + r\delta, \tau)G(\tau)u(\tau)\,d\tau\right]$$

by a well-known formula. Let us set

$$z_r(t) = \int_{t_0+r\delta}^{t} \Phi(t_0 + r\delta, \tau)G(\tau)u(\tau)\,d\tau$$

and then

$$\begin{aligned}\int_{t_0+r\delta}^{t_0+\overline{r+1}\delta} y'(t)y(t)\,dt &= \int_{t_0+r\delta}^{t_0+\overline{r+1}\delta} [x'(t_0 + r\delta) + z_r'(t)] \\ &\quad \cdot [\Phi'(t, t_0 + r\delta)H(t)H'(t)\Phi(t, t_0 + r\delta)] \\ &\quad \cdot [x(t_0 + r\delta) + z_r(t)]\,dt \\ &\le \sup_{t_0+r\delta \le t \le t_0+\overline{r+1}\delta} [\|x(t_0 + r\delta) + z_r(t)\|]^2 \\ &\quad \cdot \int_{t_0+r\delta}^{t_0+\overline{r+1}\delta} \|\Phi'(t, t_0 + r\delta)H(t)\|^2\,dt \\ &\le \sup_{t_0+r\delta \le t \le t_0+\overline{r+1}\delta} [\|x(t_0 + r\delta) + z_r(t)\|]^2 n\alpha_7\end{aligned}$$

on using the right-hand inequality of (15).

With the definition

$$w(r) = \sup_{t_0+r\delta \le t \le t_0+\overline{r+1}\delta} \|z_r(t)\|,$$

this inequality may be replaced by

$$\int_{t_0+r\delta}^{t_0+\overline{r+1}\delta} y'(t)y(t)\,dt \le 4[\|x(t_0 + r\delta)\|^2 + w^2(r)]n\alpha_7. \tag{20}$$

Observe also that

$$\begin{aligned}\|z_r(t)\| &= \left\|\int_{t_0+r\delta}^{t} \Phi(t_0 + r\delta, \tau)G(\tau)u(\tau)\,d\tau\right\| \\ &\le \left[\int_{t_0+r\delta_0}^{t} \|\Phi(t_0 + r\delta, \tau)G(\tau)\|^2\,d\tau\right]^{1/2} \cdot \left[\int_{t_0+r\delta}^{t} u'(\tau)u(\tau)\,d\tau\right]^{1/2} \\ &\le \sqrt{n\alpha_2}\left[\int_{t_0+r\delta}^{t_0+\overline{r+1}\delta} u'(\tau)u(\tau)\,d\tau\right]^{1/2}\end{aligned}$$

where we have made use of (9), or effectively, the right-hand inequality of (6). Using the definition of $w(r)$, we then have

$$w^2(r) \le n\alpha_2 \int_{t_0+r\delta}^{t_0+\overline{r+1}\delta} u'(\tau)u(\tau)\,d\tau,$$

which, in turn, implies that

$$\sum_{r=0}^{\infty} w^2(r) \le n\alpha_2. \tag{21}$$

Now consider (20) again. In the light of (21), it is clear that we will have the correct bound on

$$\int_{t_0}^{\infty} y'(t)y(t)\,dt$$

if, with

$$v(r) = x(t_0 + r\delta_0),$$

we can show that $\sum_{r=0}^{\infty} \|v(r)\|^2$ is bounded independent of t_0 and $u(\cdot)$. This we shall now do. Evidently,

$$\begin{aligned}v(r) &= \Phi(t_0 + r\delta, t_0)\int_{t_0}^{t_0+\delta} \Phi(t_0, \tau)G(\tau)u(\tau)\,d\tau \\ &\quad + \Phi(t_0 + r\delta, t_0 + \delta)\int_{t_0+\delta}^{t_0+2\delta} \Phi(t_0 + \delta, \tau)G(\tau)u(\tau)\,d\tau \\ &\quad + \cdots \\ &\quad + \Phi(t_0 + r\delta, t_0 + \overline{r-1}\delta)\int_{t_0+\overline{r-1}\delta}^{t_0+r\delta} \Phi(t_0 + \overline{r-1}\delta, \tau) \\ &\quad \cdot G(\tau)u(\tau)\,d\tau.\end{aligned} \tag{22}$$

Also,

$$\begin{aligned}&\left\|\int_{t_0+\overline{k-1}\delta}^{t_0+k\delta} \Phi(t_0 + \overline{k-1}\delta, \tau)G(\tau)u(\tau)\,d\tau\right\| \\ &\le \left[\int_{t_0+\overline{k-1}\delta}^{t_0+k\delta} \|\Phi(t_0 + \overline{k-1}\delta, \tau)G(\tau)\|^2\,d\tau\right]^{1/2} \cdot \left[\int_{t_0+\overline{k-1}\delta}^{t_0+k\delta} u'(\tau)u(\tau)\,d\tau\right]^{1/2} \\ &\le \sqrt{n\alpha_2}\,w(k-1)\end{aligned}$$

where the uniform complete controllability definition has been used—in particular, the right-hand inequality of (6). Equation (22) now yields

$$\begin{aligned}\|v(r)\| &\le \|\Phi(t_0 + r\delta, t_0)\|\sqrt{n\alpha_2}w(0) \\ &\quad + \|\Phi(t_0 + r\delta, t_0 + \delta)\|\sqrt{n\alpha_2}w(1) \\ &\quad + \cdots + \|\Phi(t_0 + r\delta, t_0 + \overline{r-1}\delta)\|\sqrt{n\alpha_2}w(r-1)\end{aligned}$$

or, on using the exponential asymptotic stability property,

$$\begin{aligned}\|v(r)\| &\le \sqrt{n\alpha_2}\alpha_8\{\exp(-\alpha_9 r\delta)w(0) \\ &\quad + \exp(-\alpha_9\overline{r-1}\delta)w(1) + \cdots + \exp(-\alpha_9\delta)w(r-1)\} \\ &= \sqrt{n\alpha_2}\alpha_8\left\{\exp\left(\frac{-\alpha_9 r\delta}{2}\right)\left[\exp\left(\frac{-\alpha_9 r\delta}{2}\right)w(0)\right]\right. \\ &\quad + \exp\left(\frac{-\alpha_9\overline{r-1}\delta}{2}\right)\left[\exp\left(\frac{-\alpha_9\overline{r-1}\delta}{2}\right)w(1)\right] \\ &\quad \left.+ \cdots + \exp\left(\frac{-\alpha_9\delta}{2}\right)\left[\exp\left(\frac{-\alpha_9\delta}{2}\right)w(r-1)\right]\right\}.\end{aligned}$$

Now use the Cauchy–Schwarz inequality. Thus,

$$\begin{aligned}\|v(r)\| &\le \sqrt{n\alpha_2}\alpha_8\{\exp(-\alpha_9 r\delta) + \exp(-\alpha_9\overline{r-1}\delta) + \cdots \\ &\quad + \exp(-\alpha_9\delta)\}^{1/2}\{\exp(-\alpha_9 r\delta)w^2(0) \\ &\quad + \exp(-\alpha_9\overline{r-1}\delta)w^2(1) + \cdots \\ &\quad + \exp(-\alpha_9\delta)w^2(r-1)\}^{1/2} \\ &\le \frac{\sqrt{n\alpha_2}\alpha_8 e^{-\alpha_9\delta}}{1 - e^{-\alpha_9\delta}}\{\exp(-\alpha_9 r\delta)w^2(0) + \exp(-\alpha_9\overline{r-1}\delta) \\ &\quad \cdot w^2(1) + \cdots + \exp(-\alpha_9\delta)w^2(r-1)\}^{1/2}.\end{aligned}$$

Introducing the positive constant α_{11} defined by

$$\alpha_{11}^{1/2} = \frac{\sqrt{n\alpha_2}\alpha_8 e^{-\alpha_9\delta}}{1 - e^{-\alpha_9\delta}},$$

we then have

$$\begin{aligned}\sum_{r=0}^{\infty} \|v(r)\|^2 &\leq \alpha_{11} \sum_{r=0}^{\infty} \left[\sum_{s=0}^{r-1} \exp(-\alpha_9\overline{r-s}\delta) w^2(s)\right] \\ &= \alpha_{11} \sum_{s=0}^{\infty} \left\{\sum_{r=s+1}^{\infty} [\exp(-\alpha_9 r\delta)\exp(\alpha_9 s\delta) w^2(s)]\right\} \\ &= \alpha_{11} \sum_{s=0}^{\infty} \frac{\exp(-\alpha_9\overline{s+1}\delta)}{1-\exp(-\alpha_9\delta)} \exp(\alpha_9 s\delta) w^2(s) \\ &= \alpha_{11} \frac{e^{-\alpha_9\delta}}{1-e^{-\alpha_9\delta}} \sum_{s=0}^{\infty} w^2(s) \\ &\leq \alpha_{12} \end{aligned} \tag{23}$$

for some positive constant α_{12}, independent of t_0 and $u(\cdot)$.

When (21) and (23) are used in (20), we obtain

$$\int_{t_0}^{\infty} y'(t)y(t)\, dt \leq 4(\alpha_{12} + n\alpha_2)n\alpha_7,$$

which yields the bound α_{10} as required in the theorem statement.

V. Proof of Part 2

As a first step, we shall show that there exists a positive constant α_{13} such that

$$\left\|\int_{t_0}^{\infty} \|\Phi'(t, t_0)\Phi(t, t_0)\, dt\right\| \leq \alpha_{13} \tag{24}$$

for all t_0. Let t_0 be arbitrary, and $x(t_0)$ be arbitrary save that $\|x(t_0)\| = \alpha_5^{-1/2}$. By uniform complete controllability, there exists a minimal energy control $u(\cdot)$ zero outside of $[t_0 - \delta, t_0]$ taking the zero state at time $t_0 - \delta$ to the state $x(t_0)$ at time t_0. Further,

$$\int_{t_0-\delta}^{t} u'(t)u(t)\, dt \leq 1 \tag{25}$$

by (10). [Note that, as remarked earlier, this inequality follows from the first inequality of (6), whereas the proof of Part 1 used the second inequality of (6).] Since $u(\cdot)$ is zero outside of $[t_0 - \delta, t_0]$, the upper limit on the integral in (25) may be replaced by infinity; the theorem statement then implies that

$$\int_{t_0-\delta}^{\infty} y'(t)y(t)\, dt \leq \alpha_{10}$$

and, *a fortiori*,

$$\int_{t_0}^{\infty} y'(t)y(t)\, dt \leq \alpha_{10}. \tag{17}$$

Now for $t \geq t_0$, $u(t)$ is zero and so (17) becomes

$$x'(t_0) \int_{t_0}^{\infty} \Phi'(t, t_0)H(t)H'(t)\Phi(t, t_0)\, dt\, x(t_0) \leq \alpha_{10}$$

or

$$\int_{t_0}^{\infty} \Phi'(t, t_0)H(t)H'(t)\Phi(t, t_0)\, dt \leq \alpha_5\alpha_{10}I. \tag{26}$$

Now observe that

$$\begin{aligned}\int_{t_0}^{\infty} &\Phi'(t, t_0)H(t)H'(t)\Phi(t, t_0)\, dt \\ &= \sum_{r=0}^{\infty} \int_{t_0+r\delta}^{t_0+\overline{r+1}\delta} \Phi'(t, t_0)H(t)H'(t)\Phi(t, t_0)\, dt \\ &= \sum_{r=0}^{\infty} \Phi'(t_0 + r\delta, t_0) \int_{t_0+r\delta}^{t_0+\overline{r+1}\delta} \Phi'(t, t_0 + r\delta)H(t)H'(t) \\ &\qquad \cdot \Phi(t, t_0 + r\delta)\, dt \cdot \Phi(t_0 + r\delta, t_0) \\ &\geq \alpha_6 \sum_{r=0}^{\infty} \Phi'(t_0 + r\delta, t_0)\Phi(t_0 + r\delta, t_0). \end{aligned} \tag{27}$$

Notice that we are using here the left-hand inequality of (11). The right-hand inequality was used in the proof of Part 1.

Equations (26) and (27) lead us to

$$\sum_{r=0}^{\infty} \Phi'(t_0 + r\delta, t_0)\Phi(t_0 + r\delta, t_0) \leq \alpha_5\alpha_6^{-1}\alpha_{10}I. \tag{28}$$

Now we also have

$$\begin{aligned}\int_{t_0+r\delta}^{t_0+\overline{r+1}\delta} &\Phi'(t, t_0)\Phi(t, t_0)\, dt \\ &= \Phi'(t_0 + r\delta, t_0) \int_{t_0+r\delta}^{\overline{t_0+r+1}\delta} \Phi'(t, t_0 + r\delta)\Phi(t, t_0 + r\delta) \\ &\qquad \cdot dt\, \Phi(t_0 + r\delta, t_0) \\ &\leq \{\max_{0 \leq x \leq \delta} \alpha_3^2(x)\}\delta\Phi'(t_0 + r\delta, t_0)\Phi(t_0 + r\delta, t_0) \\ &= \alpha_{14}\Phi'(t_0 + r\delta, t_0)\Phi(t_0 + r\delta, t_0) \end{aligned} \tag{29}$$

for some positive constant α_{14}. Combining this equation with (28) yields (24).

Next, we shall prove the existence of a constant α_{15} such that

$$\|\Phi(t, t_0)\| \leq \alpha_{15}, \qquad \forall\, t, t_0 \text{ with } t \geq t_0. \tag{30}$$

Equation (28) implies that

$$\|\Phi(t_0 + r\delta, t_0)\| \leq \sqrt{\alpha_5\alpha_6^{-1}\alpha_{10}}$$

for all t_0 and positive integer r. Now let t be arbitrary with $t \geq t_0$. Let r be a positive integer such that $t_0 + r\delta \leq t \leq t_0 + \overline{r+1}\delta$. Then

$$\begin{aligned}\|\Phi(t, t_0)\| &\leq \|\Phi(t, t_0 + r\delta)\|\, \|\Phi(t_0 + r\delta, t_0)\| \\ &\leq \max_{0 \leq x \leq \delta} \alpha_3(x)\sqrt{\alpha_5\alpha_6^{-1}\alpha_{10}} \\ &= \alpha_{15}\end{aligned}$$

for some positive α_{15}, as required.

We can now tie (24) and (30) together to conclude exponential asymptotic stability. For arbitrary t_0 and T with $T \geq t_0$,

$$\begin{aligned}\int_{t_0}^{T} \|\Phi(T, t_0)\|^2\, dt &\leq \int_{t_0}^{T} \|\Phi(T, t)\|^2\|\Phi(t, t_0)\|^2\, dt \\ &\leq \alpha_{15}^2 \int_{t_0}^{\infty} \|\Phi(t, t_0)\|^2\, dt \\ &\leq n\alpha_{13}\alpha_{15}^2.\end{aligned}$$

But clearly

$$\int_{t_0}^{T} \|\Phi(T, t_0)\|^2\, dt = (T - t_0)\|\Phi(T, t_0)\|^2$$

so that

$$\|\Phi(T, t_0)\| \leq \frac{\sqrt{n\alpha_{13}\alpha_{15}^2}}{(T - t_0)^{1/2}}.$$

Uniform asymptotic stability is immediate from this; exponential asymptotic stability follows using arguments as in [7, see p. 381].

VI. Conclusion

The main result of this paper has related exponential asymptotic stability with a particular form of input–output stability, involving $\mathcal{L}_2$ norms. One might then ask whether results can be obtained when the input–output stability is described by, say, $\mathcal{L}_p$ norms for arbitrary $p \geq 1$, [8], [9]. To a certain extent, this is an open question, although some comments can be made. The case $p = \infty$ has been considered in detail (see, for example, [1]–[3]), and it is clear from counter examples that more conditions than simply uniform complete controllability and observability are required to achieve a connection between input and output stability. The natural assumption that tends to be made to obtain the connection is that the entries of $F(\cdot)$, $G(\cdot)$, and $H(\cdot)$ are bounded, though this assumption is perhaps too strong.

One can of course attempt to examine input–output stability for other values of p with the boundedness assumption, and it is then easy to show that exponential asymptotic stability implies that $\mathcal{L}_p$ inputs are mapped into $\mathcal{L}_p$ outputs; this result follows without the need to make any controllability or observability assumptions. The converse (requiring one to conclude exponential asymptotic stability from input-output stability), however, does appear to require controllability and observability assumptions, with at least the observability assumptions differing in detail from those of this paper. Exactly what assumptions are necessary to be able to equate the two kinds of stability when the entries of $F(\cdot)$, $G(\cdot)$, and $H(\cdot)$ are unbounded is not clear. The uniform complete controllability and observability assumptions of this paper are clearly matched to considering stability with $\mathcal{L}_2$ norms, but equally clearly require variation for $p \neq 2$.

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