Automatica, Vol. 7, pp. 657–669. Pergamon Press, 1971. Printed in Great Britain.

The Fixed-Lag Smoother as a Stable Finite-Dimensional Linear System*

Le lisseur à retard fixe comme un systeme linéaire stable aux dimensions finies

Glättungseinrichtung mit fester Verzögerung als stabiles lineares System endlicher Dimension

Сглаживатель с постоянным запаздыванием как устойчивая линейная система с конечными размерами

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By rewriting optimal, fixed-lag smoothing algorithms in the form of discrete-time state-space equations, instability of the algorithms can be eliminated, and smoothers can be characterized by their input–output performance.

Summary—The optimal fixed-lag smoother is studied, and viewed as a finite-dimensional linear system. This allows construction of a stable smoother algorithm, though hitherto only unstable algorithms have been published. Further, the complexity of the smoother can be quantitatively examined, and design criteria suggested for the choice of lag interval. Practical implementation of the smoother is discussed.

INTRODUCTION

THE KALMAN–BUCY filter [1] is now enjoying great popularity as a tool for on-line estimation of a linear system. In some situations however, notably when a fixed time lag is tolerable in producing a state estimate, the use of a fixed-lag smoother would appear preferable. As is known, a fixed-lag smoother offers an estimation scheme which produces an estimate of lower error variance than the Kalman–Bucy filter [2]. This advantage of fixed-lag smoothing is however countered by two alleged disadvantages, over and above the potential disadvantages associated with the delay in the production of an estimate:

(1) The fixed-lag smoothing algorithms formulated in the literature are numerically unstable, and physical realizations of them can therefore be expected not to work [3].

(2) In contrast to the Kalman–Bucy filter, the fixed-lag smoother has never really been viewed as a finite-dimensional linear dynamical system, in the technical sense of linear system theory, with input comprising the noisy measurements, and output comprising the desired fixed-lag smoothed estimate. Rather, it has generally been portrayed with an unfamiliar structure, making problems of physical realization unclear. In addition, it would probably not be unfair to claim that, as a consequence of describing smoothers in an unfamiliar way, the instability problem was not recognized for some time.

The aim of this paper is to resolve the above two disadvantages for the case of discrete-time fixed-lag smoothers. The first disadvantage will be resolved by stating a smoothing algorithm which does not possess the instability property, but rather enjoys the same stability properties as the Kalman–Bucy filter. The second disadvantage will be resolved by presenting standard-form state-space equations for the smoother; this presentation of state-space equations offers the advantages of

(1) making clear the physical arrangement required to implement a smoother, and particularly, enabling assessment of the complexity of the smoother, as measured by the dimension of its state vector.

(2) enabling easy statement, in the stationary, time-invariant case, of the transfer function matrix of the smoother.

A natural question which may be asked is whether the material of the paper is applicable to continuous time fixed-lag smoothing, and the answer is no. The basic reason is that the continuous-time optimal fixed-lag smoother inevitably contains time-delay elements, as examination of the

* Received 8 March 1971; revised 24 May 1971. The original version of this paper was not presented at any IFAC meeting. It was recommended for publication in revised form by Associate Editor A. Sage. Work supported by the Australian Research Grants Committee.

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standard algorithms will show, see e.g. [4], and accordingly cannot be represented as a finite-dimensional linear system. Then, for a mathematical rather than physical reason, it proves impossible to eliminate the instability of the smoothing algorithm, at least via procedures to be used here. However, it should be noted that this does not exclude the possibility of there being excellent *sub-optimal* continuous time smoothers which are both finite-dimensional and stable.

Of course, the Kalman–Bucy filter in both discrete and continuous-time is finite-dimensional. The nonparallel for the smoother arises for the reason that although a time-delay element destroys finite-dimensionality in a continuous-time system, it is a standard block in a finite-dimensional discrete-time system—at least if the delay is an integer multiple of the sampling period.

An outline of the paper is as follows. In section 2, we review the fixed-lag smoothing equations, and then in section 3, we compute from a known algorithm the transfer function matrix of a stationary fixed-lag smoother. At first appearance, this transfer function matrix would appear to have elements with unstable poles, but close examination reveals a cancellation of unstable poles with zeros, thereby yielding a transfer function matrix with every element possessing only stable poles.

In section 4, we write down a set of state-space equations for the optimal smoother, starting from the same algorithm as used in section 3 to obtain a transfer function matrix. As linear system theory suggests, the cancellation observed in section 3 between the poles and zeros of elements of the transfer function matrix corresponds to the presence of uncontrollable or unobservable states in the state-space equation set of section 4. Actually, uncontrollable states are found for both stationary and nonstationary smoothers, and upon elimination of the uncontrollable states in the usual way, a stable finite-dimensional state-space equation set is found which describes the smoother.

In section 5, we relate the smoother description of section 4 to results of [5], and we indicate a recursive formula for the error covariance.

Section 6 gives a simple illustrative example and section 7 contains a summary and conclusions. For easy reference, the definitions of estimates, filter and smoother are gathered in the summary. The conclusions include a discussion of the complexity of the smoother and the improvement in performance through use of smoothing.

2. REVIEW OF FILTERING AND FIXED-LAG SMOOTHING

We consider systems of the form

$$x(k+1)=\phi_k x(k)+G_k w(k) \tag{1a}$$

$$y(k+1)=H'_{k+1}x(k+1)+v(k+1). \tag{1b}$$

Here $x(\cdot)$ is a state vector, $y(\cdot)$ a measurement vector, $w(\cdot)$ is a zero mean, gaussian white noise sequence with $E[w(k)w'k)]=Q_k$, $v(\cdot)$ is a zero mean, gaussian white noise sequence with $E[v(k)v'(k)]=R_k$, $x(k_0)$ is a gaussian random variable of mean $\bar{x}_{k_0}$ and covariance P_{k_0} and $w(\cdot)$, $v(\cdot)$ and $x(k_0)$ are all independent. For convenience, we shall assume that the matrices ϕ_k and R_k are invertible.

If ϕ_k, G_k, H_k, Q_k and R_k are independent of k, if $k_0=-\infty$, and if (1a) is asymptotically stable, then (1) is a time-invariant system and $y(\cdot)$ is a stationary sequence.

We define the filtered estimate $\hat{x}_f(k/k)$ of $x(k)$ and the smoothed estimate $\hat{x}_s(k-N/k)$ of $x(k-N)$ by

$$\hat{x}_f(k/k)=E\{x(k)/y(k),\, y(k-1),\, \ldots,\, y(k_0+1)\} \tag{2}$$

and

$$\hat{x}_s(k-N/k)=E\{x(k-N)/y(k),\, y(k-1),\, \ldots,\, y(k_0+1)\}. \tag{3}$$

Here, N is a fixed positive integer denoting the lag used in the smoothing operation. Note that the measurements required to compute $\hat{x}_f(k/k)$ are a proper subset of those required to compute $\hat{x}_s(k/k+N)$ for $N>0$, and so the variance of $x(k)-\hat{x}_f(k/k)$ will be greater than that of $x(k)-\hat{x}_s(k/k+N)$. Herein lies the prime motivation for using fixed-lag smoothing, as opposed to simple filtering.

To assist in formulating smoother equations, we shall first recall the equations of the Kalman–Bucy filter [1, 4]. To do this, we make the following definitions:

$$P_k=E\{[x(k)-\hat{x}_f(k/k)][x(k)-\hat{x}_f(k/k)]'\} \tag{4}$$

= error covariance matrix* for the filtered estimate at time k

$$\bar{P}_k=E\{[x(k+1)-\phi_k\hat{x}_f(k/k)][x(k+1)-\phi_k\hat{x}_f(k/k)]'\} \tag{5}$$

= error covariance matrix for the predicted estimate of $x(k+1)$, using measurements up to time k

$$K_{k+1}=\bar{P}_k H_{k+1}(H'_{k+1}\bar{P}_k H_{k+1}+R_{k+1})^{-1}. \tag{6}$$

The relation between P_k and $\bar{P}_k$ is

$$\bar{P}_k=\phi_k P_k \phi'_k+G_k Q_k G'_k \tag{7}$$

while P_{k+1} can be obtained from $\bar{P}_k$ and K_{k+1} by

$$P_{k+1}=(I-K_{k+1}H'_{k+1})\bar{P}_k=\bar{P}_k(I-H_{k+1}K'_{k+1})'. \tag{8}$$

* Note that the notation is consistent with our earlier definition of P_{k0}.

Notice that the use in order of the equations (7), (6) and (8) followed by (7), (6) and (8) again, etc. allows recursive computation forward in time of P_k, $\bar{P}_k$ and K_{k+1}. The process is initialized by P_{k_0}, assumed known. Notice also that if (1) is time-invariant and $y(\cdot)$ is stationary, then P_k, $\bar{P}_k$ and K_k are independent of k, [1, 4].

The Kalman–Bucy filter is defined by

$$\hat{x}_f(k+1/k+1)=\phi_k\hat{x}_f(k/k)+K_{k+1}[y(k+1) - H'_{k+1}\phi_k\hat{x}_f(k/k)] \qquad (9)$$

$$=F_k\hat{x}_f(k/k)+K_{k+1}y(k+1) \qquad (10)$$

where

$$F_k=[I-K_{k+1}H'_{k+1}]\phi_k. \qquad (11)$$

The initial condition is taken as $\hat{x}_f(k_0/k_0)=\bar{x}_{k_0}$. Equation (10) represents the filter as a finite-dimensional linear system, the system input being $y(\cdot)$, and the system output, identical with the system state, being $\hat{x}_f(\cdot/\cdot)$. The stability properties are determined by the matrix F_k, and, as is well-known, physically reasonable conditions guarantee stability, and, actually, boundedness of the filter error covariance matrix. For the precise form of these conditions, see [6] and [7]. *Henceforth in this paper, we shall assume these conditions are fulfilled.*

Now to define the fixed-lag smoother, we introduce the quantities*

$$D_k=\phi_k+G_kQ_kG'_k(P_k\phi'_k)^{-1}=\bar{P}_k(P_k\phi'_k)^{-1}. \qquad (12)$$

Reference [4] quotes the following fixed-lag smoothing algorithm:

$$\hat{x}_s(k-N+1/k+1)=D_{k-N}\hat{x}_s(k-N/k) -\left(\prod_{i=k-N+1}^{k} D_i^{-1}\right)K_{k+1}H'_{k+1}\phi_k\hat{x}_f(k/k) -(D_{k-N}-\phi_{k-N})\hat{x}_f(k-N/k-N) +\left(\prod_{i=k-N+1}^{k} D_i^{-1}\right)K_{k+1}y(k+1). \qquad (13)$$

The initial condition for (13) is normally noted to be $\hat{x}_s(k_0/k_0+N)$, a quantity which is obtained using the measurements $y(k_0+1), \ldots y(k_0+N)$ in combination with a *fixed-point* smoothing algorithm. We shall not bother here with the precise expression for $\hat{x}_s(k_0/k_0+N)$.

* The physically reasonable assumptions which guarantee stability of the filter normally guarantee nonsingularity of P_k (and $\bar{P}_k$) for all $k \geq$ some k_1, [6, 7]. Thus D_k is well defined, at least for $k \geq k_1$, and is also invertible.

The algorithm of (13) suggests that $\hat{x}_s(\cdot-N/\cdot)$ be regarded as the state and output of a linear finite-dimensional system whose inputs comprise $\hat{x}_f(\cdot/\cdot)$, $\hat{x}_f(\cdot-N/\cdot-N)$ and $y(\cdot+1)$. This appears prima facie reasonable, since the third quantity of the three inputs is directly available, while the first two are certainly computable. The difficulty with this arrangement, as noted in section 1, is that the algorithm is unstable [3]. In the time-invariant case, stability of the algorithm is determined by the stability of the homogeneous equation $z(k+1)=Dz(k)$. Simple manipulations will show that the matrix D^{-1} is similar in the algebraic sense to F, which, roughly speaking, says that if the filter homogeneous equation $z(k+1)=Fz(k)$ is asymptotically stable, as we know to be the case, then $z(k+1)=Dz(k)$ is unstable. In the nonstationary case too, stability of the filter implies instability of the smoother.

3. TRANSFER FUNCTION MATRIX OF THE STATIONARY OPTIMAL SMOOTHER

In this section, we consider stationary optimal smoothers only, and derive a transfer function matrix relating the measurement $y(\cdot)$ to the smoother output $\hat{x}_s(\cdot-N/\cdot)$. By doing this it should be noted that implicitly we are regarding the smoother as a device driven by an input consisting of $y(\cdot)$ alone; the smoother may internally generate $\hat{x}_f(\cdot/\cdot)$. Then at time k, the smoother output is $\hat{x}_s(k-N/k)$, while its input is $y(k)$. Accordingly, we want a transfer function relating $\hat{X}_s(z)=Z\{\hat{x}_s(\cdot-N/\cdot)\}$ and $Y(z)=Z\{y(\cdot)\}$.

In the stationary case as we have noted, P_k, $\bar{P}_k$ and K_k are independent of k. This means that F_k and D_k are also independent of k, and z-transforms of (10) and (13) exist. Thus from (13)

$$z\hat{X}_s(z)=D\hat{X}_s(z)-D^{-N}KH'\phi\hat{X}_f(z) -z^{-N}(D-\phi)\hat{X}_f(z)+D^{-N}KzY(z)$$

while from (10),

$$z\hat{X}_f(z)=F\hat{X}_f(z)+KzY(z).$$

Eliminating $\hat{X}_f(z)$ leads to

$$\hat{X}_s(z)=(zI-D)^{-1}[-D^{-N}KH'\phi-(D-\phi)z^{-N} +D^{-N}(zI-F)](zI-F)^{-1}KzY(z).$$

On using the definition $F=(I-KH')\phi$, we see that the desired transfer function matrix is

$$W_s(z)=(zI-D)^{-1}[-Dz^{-N}+\phi z^{-N}+D^{-N}z -D^{-N}\phi](zI-F)^{-1}Kz.$$

Note that the poles of entries of $W_s(z)$—other than those at the origin—are determined, formally anyway, by the eigenvalues of F and D, and it is the poles associated with the latter matrix that suggest that $W_s(z)$ will be associated with an unstable system. However, we have

$$\begin{aligned}
&-Dz^{-N}+\phi z^{-N}+D^{-N}z-D^{-N}\phi \\
&\quad=z^{-N}[(z^{N+1}I-D^{N+1})D^{-N}-(z^NI-D^N)D^{-N}\phi] \\
&\quad=z^{-N}(zI-D)[(z^NI+z^{N-1}D+\ldots+D^N)D^{-N} \\
&\qquad-(z^{N-1}I+z^{N-2}D+\ldots+D^{N-1})D^{-N}\phi]
\end{aligned}$$

so that

$$\begin{aligned}
W_s(z)=[&D^{-N}+z^{-1}D^{-N}(D-\phi)+z^{-2}D^{-N+1}(D-\phi) \\
&+\ldots+z^{-N}D^{-1}(D-\phi)](zI-F)^{-1}Kz. \quad (14)
\end{aligned}$$

The poles of entries of $W_s(z)$ in this reduced form now all lie inside the unit circle. One could therefore construct an asymptotically stable state-space description of the optimal smoother directly from the expression for $W_s(z)$. We shall however obtain such a description in a slightly more instructive fashion in the next section.

4. STABLE FINITE-DIMENSIONAL REALIZATION OF THE OPTIMAL SMOOTHER

In this section, our aim is to develop the state-space equations of a finite-dimensional linear system, with input the $y(\cdot)$ process and output the $\hat{x}_s(\cdot-N/\cdot)$ process. To begin with, we consider the case where the smoother is stationary. This will be done in detail, while the time-varying case will subsequently be discussed in outline form.

Our starting point is the pair of equations (10) and (13), with F_k, etc. assumed independent of k. These equations readily imply

$$\begin{bmatrix} \hat{x}_s(k-N+1/k+1) \\ \hat{x}_f(k+1/k+1) \\ \hat{x}_f(k/k) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \hat{x}_f(k-N+1/k-N+1) \end{bmatrix} = \begin{bmatrix} D & -D^{-N}KH'\phi & 0 & \ldots & 0 & -(D-\phi) \\ 0 & F & 0 & \ldots & 0 & 0 \\ 0 & I & 0 & \ldots & 0 & 0 \\ 0 & 0 & I & \ldots & 0 & 0 \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & I & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_s(k-N/k) \\ \hat{x}_f(k/k) \\ \hat{x}_f(k-1/k-1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \hat{x}_f(k-N/k-N) \end{bmatrix}$$

$$+\begin{bmatrix} D^{-N}K \\ K \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} y(k+1). \quad (15)$$

and

$$\hat{x}_s(k-N/k)=[I\ 0\ \ldots\ 0]\begin{bmatrix} \hat{x}_s(k-N/k) \\ \hat{x}_f(k/k) \\ \hat{x}_f(k-1/k-1) \\ \cdot \\ \cdot \\ \cdot \\ \hat{x}_f(k-N/k-N) \end{bmatrix} \quad (16)$$

Observe that (15) and (16) may be written in the form

$$w(k+1)=Aw(k)+By(k+1) \quad (17a)$$

$$\hat{x}_s(k-N/k)=Cw(k) \quad (17b)$$

where $w(\cdot)$, A, B and C have obvious definitions. In other words, they provide a description of the smoother as a linear finite-dimensional system. It is easy to see that the nonzero eigenvalues of A are the eigenvalues of D and of F. As a consequence, (15) and (16) as they stand would provide an unstable algorithm. With the ideas of the last section in mind however, we ask whether the instability could be eliminated, by eliminating an uncontrollable or unobservable part of (15) and (16). (As is well known, the elimination of the uncontrollable or unobservable part of a system makes no difference to the input–output performance of the system.)

Let us define the nonsingular matrix T, of dimension equal to that of A, by

$$T=\begin{bmatrix} I & -D^{-N} & -D^{-N}(D-\phi) & \dots & -D^{-1}(D-\phi) \\ 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \end{bmatrix}. \tag{18}$$

The matrix T defines a state-space coordinate transformation in which the new state vector is $\bar{w}(k)=Tw(k)$, and (17) is replaced by

$$\bar{w}(k+1)=TAT^{-1}\bar{w}(k)+TBy(k+1) \tag{19a}$$

$$\hat{x}_s(k-N/k)=CT^{-1}\bar{w}(k). \tag{19b}$$

Explicit evaluation of TAT^{-1} and TB yields

$$TAT^{-1}=\begin{bmatrix} D & 0 & 0 & \dots & 0 & 0 \\ 0 & F & 0 & \dots & 0 & 0 \\ 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 \end{bmatrix} \qquad TB=\begin{bmatrix} 0 \\ K \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{20}$$

while

$$CT^{-1}=[I\; D^{-N}\; D^{-N}(D-\phi)\dots D^{-1}(D-\phi)]. \tag{21}$$

Inspection of the pair of matrices TAT^{-1} and TB reveals immediately that the first block of entries of $\bar{w}(\cdot)$ is uncontrollable. Accordingly (19) may be replaced by equations obtained by eliminating this block from consideration. Examination of T will reveal that $\bar{w}(k)=Tw(k)$ less its first block, is identical to $w(k)$ less its first block, and so we have

Asymptotically stable stationary fixed-lag smoothing equations

$$\begin{bmatrix} \hat{x}_f(k+1/k+1) \\ \hat{x}_f(k/k) \\ \vdots \\ \hat{x}_f(k-N+1/k-N+1) \end{bmatrix}=\begin{bmatrix} F & 0 & \dots & 0 & 0 \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & & I & 0 \end{bmatrix}\begin{bmatrix} \hat{x}_f(k/k) \\ \hat{x}_f(k-1/k-1) \\ \vdots \\ \hat{x}_f(k-N/k-N) \end{bmatrix}+\begin{bmatrix} K \\ 0 \\ \vdots \\ 0 \end{bmatrix} y(k+1) \tag{22a}$$

$$\hat{x}_s(k-N/k)=[D^{-N}\ D^{-N}(D-\phi)\ldots D^{-1}(D-\phi)]\begin{bmatrix}\hat{x}_f(k/k)\\ \hat{x}_f(k-1/k-1)\\ \cdot\\ \cdot\\ \cdot\\ \hat{x}_f(k-N/k-N)\end{bmatrix}. \tag{22b}$$

We now make a number of points:

(1) Equation (22) provides not only a finite-dimensional, but also an asymptotically stable optimum smoother. The last point follows by examining the homogeneous version of (22a), and using the fact that $z(k+1)=Fz(k)$ is asymptotically stable, this being the homogeneous equation associated with the filter.

(2) Simple manipulations will show that the transfer function matrix associated with (22) is the transfer function matrix $W_s(z)$ of (14); conversely, (22) might well have resulted directly from (14), rather than through eliminating the uncontrollable states from (17).

(3) Equation (22b) shows that the smoothed estimate is simply a linear combination of various filtered estimates.

(4) The dimension of the linear system (22) is $(N+1)n$ where n is the dimension of $x(\cdot)$. Therefore, the longer the lag, the greater the complexity of the smoother, as measured by its dimension. However, the sparseness of the system matrices in (22a), coupled with the fact that an increase in N does not increase the *feedback* required in a realization of (22a), means that the problem of complexity with large N is not so great as might at first appear.

In fact an increase in N corresponds to the addition of a block of shift registers and two appropriate gain matrix blocks. A physical realization of the algorithm (22) is illustrated by the block-diagram of Fig. 1.

(5) We have glossed over the question of initial conditions for (15) and equations derived from it, e.g. (22). It might be that the uncontrollable part of (5) has a nonzero initial condition, the effect of which would be observed at the output, thereby rendering (22b) in error. This is not actually the case. It is possible to argue this by studying initial conditions for the original smoothing algorithm, converting these to an initial condition on $w(k)$, and then on $\bar{w}(k)$. If this is done, one will find that the first block of $\bar{w}(k)$ will have a zero initial condition. This fact, coupled with the uncontrollability, will mean that the uncontrollable part of $\bar{w}(k)$ will not show up at the output. But a neater way of arguing is to realize that if the uncontrollable part of $\bar{w}(k)$ gave a nonzero contribution to $\hat{x}_s(\cdot-N/\cdot)$, then $\hat{x}_s(\cdot-N/\cdot)$ would become unbounded as $k\to\infty$ because of the instability of this uncontrollable part of $\bar{w}(\cdot)$. Unboundedness of $\hat{x}_s(\cdot-N/\cdot)$ would be inconsistent with the fact that the covariance matrix of $x(\cdot)-\hat{x}_s(\cdot-N/\cdot)$ is bounded by the covariance matrix of $x(\cdot)-\hat{x}_f(\cdot/\cdot)$, which is itself bounded. Yet a further and independent argument will be provided in the next section. This argument does depend however on detailed manipulations.

Turning now to the writing down of initial conditions for (22a), observe that at time k the state

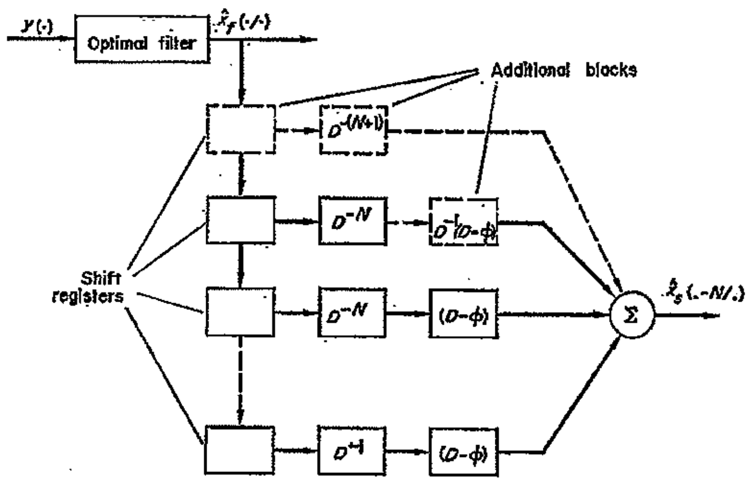

FIG. 1. Implementation of algorithm (22) and the expansion due to an increase in N.

vector of (22) is $[\hat{x}_f(k/k) \ldots \hat{x}_f(k-N/k-N)]'$. The quantities $\hat{x}_f(k/k) \ldots \hat{x}_f(k-N/k-N)$ are all available at time k, and so the initial condition problem is trivially solved. One commences (22a) at time k such that $k-N=k_0$, or $k=k_0+N$, with initial condition

$$[\hat{x}_f(k_0+N/k_0+N)\ \hat{x}_f(k_0+N-1/k_0+N-1) \ldots \hat{x}_f(k_0/k_0)].$$

(6) The arrangement of (22) *may* still be uncontrollable, or unobservable. It may then be possible to obtain state-space equations describing a smoother with a state vector of lower dimension than that of (22). However, this would depend on the particular case on hand. The arrangement of (22) is completely general

Equation (22) does *not* provide a unique description of the smoother; any set of equations obtained from (22) via an invertible coordinate transformation of the state-space would be equally valid. In another coordinate basis, not all of the preceding remarks would be valid however.

(7) So far, we have not indicated how the variance of $x(k)-\hat{x}_s(k/k+N)$ behaves. This matter will be taken up in the next section.

We turn now to the time-varying case, and we shall content ourselves with summarizing the main steps. Equations (10) and (13) are used to define a set of nonstationary equations like (15) and (16). A coordinate transformation of the state-space is then made, defined by

$$ {}_k = \begin{bmatrix} I & -\left(\prod_{k-N}^{k-1} D_i^{-1}\right) & -\left(\prod_{k-N}^{k-1} D_i^{-1}\right)(D_{k-1}-\phi_{k-1}) & \cdots & -D_{k-N}^{-1}(D_{k-N}-\phi_{k-N}) \\ 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ & & & & 0 \\ 0 & 0 & 0 & \cdots & I \end{bmatrix}. \quad (23)$$

In the new coordinate basis, the first block of entries of the state vector is again uncontrollable, and on elimination of the uncontrollable part, one derives an asymptotically stable set of equations for the nonstationary smoother as follows:

Asymptotically stable nonstationary fixed-lag smoother

$$\begin{bmatrix} \hat{x}_f(k+1/k+1) \\ \hat{x}_f(k/k) \\ \vdots \\ \hat{x}_f(k-N+1/k-N+1) \end{bmatrix} = \begin{bmatrix} F_k & 0 & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & & \ddots & 0 & \vdots \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_f(k/k) \\ \hat{x}_f(k-1/k-1) \\ \vdots \\ \hat{x}_f(k-N/k-N) \end{bmatrix} + \begin{bmatrix} K_{k+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} y(k+1) \quad (24a)$$

$$\hat{x}_s(k-N/k)=\left[\prod_{i=k-N}^{k-1} D_i^{-1} \prod_{i=k-N}^{k-1} D_i^{-1}(D_{k-1}-\phi_{k-1}) \prod_{i=k-N}^{k-2} D_i^{-1}(D_{k-2}-\phi_{k-2}) \ldots D_{k-N}^{-1}(D_{k-N}-\phi_{k-N}\right] \begin{bmatrix} \hat{x}_f(k/k) \\ \hat{x}_f(k-1/k-1) \\ \cdot \\ \cdot \\ \cdot \\ \hat{x}_f(k-N/k-N) \end{bmatrix} \quad (24b)$$

Remarks made for the stationary smoother apply *mutatis mutandis* to the nonstationary smoother. In particular, the smoother has the same homogeneous equation determining its stability as does the filter.

5. RELATION WITH THE INNOVATIONS PROCESS; COMPUTATION OF THE ERROR COVARIANCE MATRIX

Retaining the same notation as before, we recall the definition of the innovations process $\nu(\cdot)$, see [8]

$$\nu(k+1)=y(k+1)-H_{k+1}'\phi_k\hat{x}_f(k/k). \quad (25)$$

The innovation process is white, and

$$E[\nu(k+1)\nu'(k+1)]=H_{k+1}'\bar{P}_kH_{k+1}+R_{k+1}. \quad (26)$$

Reference [5] quotes the following formula:

$$\begin{aligned}\hat{x}_s(k-N/k)=&E\{x(k-N)\nu'(k)\}[H_k'\bar{P}_{k-1}H_k\\ &+R_k]^{-1}\nu(k)+\ldots\\ &+E\{x(k-N)\nu'(k-N+1)\}[H_{k-N+1}'\bar{P}_{k-N}H_{k-N+1}\\ &+R_{k-N+1}]^{-1}\nu(k-N+1)+\hat{x}_f(k-N/k-N). \quad (27)\end{aligned}$$

This is connected to the formula (24b) in the following way. From (24b), we have

$$\begin{aligned}\hat{x}_s(k-N/k)=&\left(\prod_{i=k-N}^{k-1} D_i^{-1}\right)\hat{x}_f(k/k)\\ &+\left(\prod_{i=k-N}^{k-1} D_i^{-1}\right)(D_{k-1}-\phi_{k-1})\hat{x}_f(k-1/k-1)\\ &+\ldots+D_{k-N}^{-1}(D_{k-N}-\phi_{k-N})\hat{x}_f(k-N/k-N)\\ =&\left(\prod_{i=k-N}^{k-1} D_i^{-1}\right)[\hat{x}_f(k/k)-\phi_{k-1}\hat{x}_f(k-1/k-1)]\\ &+\left(\prod_{i=k-N}^{k-2} D_i^{-1}\right)[\hat{x}_f(k-1/k-1)\\ &-\phi_{k-2}\hat{x}_f(k-2/k-2)]+\ldots\\ &+D_{k-N}^{-1}[\hat{x}_f(k-N+1/k-N+1)\\ &-\phi_{k-N}\hat{x}_f(k-N/k-N)]+\hat{x}_f(k-N/k-N)\end{aligned}$$

with the second equality following by rearrangement. Now from the filter equation (9), we have $\hat{x}_f(k/k)-\phi_{k-1}\hat{x}_f(k-1/k-1)=K_k\nu(k)$, and therefore the last expression for $\hat{x}_s(k-N/k)$ becomes

$$\begin{aligned}\hat{x}_s(k-N/k)=&\left(\prod_{i=k-N}^{k-1} D_i^{-1}\right)K_k\nu(k)+\ldots\\ &+D_{k-N}^{-1}K_{k-N+1}\nu(k-N+1)+\hat{x}_f(k-N/k-N). \quad (28)\end{aligned}$$

Equation (28) has the same form as (27). That the two equations can be completely reconciled follows after some intricate manipulations outlined in Appendix 1. The fact that they can be reconciled provides another justification of our claim in section 4 that the uncontrollable part of a certain state-space realization of the smoother cannot contribute to $\hat{x}_s(\cdot-N,\ \cdot)$ through a nonzero initial condition, for if this uncontrollable part did contribute to $\hat{x}_s(\cdot-N/\cdot)$ equation (28) would not be true.

We next formulate the algorithm for the error covariance. As is well known, $\hat{x}_s(k-N/k)$ can be defined by requiring it to be that linear function of the measurements up to $y(k)$ such that $E\{[x(k-N)-\hat{x}_s(k-N/k)]y'(l)\}=0$ for all $l\le k$. Because of the essential equivalence between the innovations process and the measurements process [8], it follows that

$$E\{[x(k-N)-\hat{x}_s(k-N/k)]\nu'(l)\}=0 \quad l\le k. \quad (29)$$

Now from (28), we have

$$\begin{aligned}x(k-N)-\hat{x}_f(k-N/k-N)=&x(k-N)-\hat{x}_s(k-N/k)\\ &+\left(\prod_{i=k-N}^{k-1} D_i^{-1}\right)K_k\nu(k)+\ldots\\ &+D_{k-N}^{-1}K_{k-N+1}\nu(k-N+1). \quad (30)\end{aligned}$$

Now the variables

$$x(k-N)-\hat{x}_s(k-N/K),\ \nu(k)\ldots\nu(k-N+1)$$

are all uncorrelated, by (29) and the whiteness of the innovations. The covariance matrix of the left side of (30) is therefore the sum of the covariance matrices of the right side, or

$$\begin{aligned}P_{k-N}=&P_s(k-N/k)+\left(\prod_{i=k-N}^{k-1} D_i^{-1}\right)K_k(H_k'\bar{P}_{k-1}H_k\\ &+R_k)K_k'\left(\prod_{i=k-N}^{k-N} D_i^{-1}\right)'+\ldots\\ &+D_{k-N}^{-1}K_{k-N+1}(H_{k-N+1}'P_{k-N}H_{k-N+1}+R_{k-N+1})\\ &\times K_{k-N+1}'(D_{k-N}^{-1})'\end{aligned}$$

with $P_s(k-N/k)$ the error covariance associated with the smoothed estimate.

We note from (6) and (8) that

$$K_j(H_j'\bar{P}_{j-1}H_j+R_j)K_j'=\bar{P}_{j-1}-P_j.$$

We write:

Nonstationary fixed-lag smoother error covariance

$$\begin{aligned}P_s(k-N/k)&=P_{k-N}-\left(\prod_{i=k-N}^{k-1}D_i^{-1}\right)(\bar{P}_{k-1}\\&-P_k)\left(\prod_{i=k-N}^{k-1}D_i^{-1}\right)'-\dots\\&-D_{k-N}^{-1}(\bar{P}_{k-N}-P_{k-N+1})(D_{k-N}^{-1})'.\end{aligned}\tag{31}$$

In the stationary case we simply have a tidier form of the above.

Stationary fixed-lag smoother error covariance

$$P_s(k-N/k)=P-\sum_{j=1}^{N}D^{-j}(\bar{P}-P)(D^{-j})'.\tag{32}$$

Besides their intrinsic value as formulas yielding the error covariance of a smoother, (31) and (32) offer a basis on which to pick a value of N, if this quantity is adjustable. As (32) particularly shows when one recalls that D^{-1} is similar to F and thus has eigenvalues less than unity in modulus, *there will be a certain value of N beyond which reduction in $P_s(k-N/k)$ will be so small as to be irrelevant.* This value of N is evidently determined as several times the dominant time constant of the optimal filter. The argument extends to the non-stationary case, and has been described in detail for continuous time smoothing in [2]. The arguments of this reference in respect of nonstationary smoothing apply here, *mutatis mutandis.*

6. EXAMPLE

We consider the system

$$x(k+1)=ax(k)+w(k)$$
$$y(k+1)=hx(k+1)+v(k+1).$$

Here $w(\cdot)$ and $v(\cdot)$ are independent scalar white gaussian sequences with zero means and variances $q>0$ and $r>0$ respectively. We assume a is a scalar in the range $0<a<1$, and we assume $k_0=-\infty$; The scalars r, q, a and h are all constant. With these assumptions, the filter and smoother are time-invariant.

We cite the four pertinent relations from (6), (7), (8) and (12):

$$\bar{P}=a^2P+q$$
$$K=\bar{P}h(h^2\bar{P}+r)^{-1}$$
$$P=(1-Kh)\bar{P}$$
$$D=\bar{P}/aP.$$

From these we note that

$$K=Ph/r$$

$$K=-\frac{h^2q+r(1-a^2)}{2a^2rh}+\frac{\sqrt{[h^4q^2+r^2(1-a^2)^2+2h^2qr(1+a^2)]}}{2a^2rh}.$$

Filtering

The filtering equations are obtained using (10): $F=(1-Kh)a$ and

$$\hat{x}_f(k+1/k+1)=(1-Kh)a\hat{x}_f(k/k)+Ky(k+1).\tag{33}$$

Fixed-lag smoothing

By way of comparison we indicate two algorithms, the unstable one noted in section 3, see equation (13), and the stable algorithm of section 5, see equation (22). First, from (13),

$$\begin{aligned}\hat{x}_s(k-N+1/k+1)&=(\bar{P}/aP)\hat{x}_s(k-N/k)\\&+(\bar{P}/aP)^{-N}y(k+1)-(\bar{P}/aP)^{-N}Kha\hat{x}_f(k/k)\\&-(\bar{P}/aP-a)\hat{x}_f(k-N/k-N).\end{aligned}\tag{34}$$

We note that $\bar{P}/aP=1/(1-Kh)a$ from the relations cited above. Stability of the filter (33) implies $(1-Kh)a<1$, which then implies the instability of (34) since $\bar{P}/aP=1/(1-Kh)a>1$.

Next, from (22), we have

$$\begin{aligned}\hat{x}_s(k-N+1/k+1)&=[(1-Kh)a]^N\hat{x}_f(k+1/k+1)\\&+[(1-Kh)a]^N[1/(1-Kh)a-a]\hat{x}_f(k/k)\\&+\dots\\&+(1-Kh)a[1/(1-Kh)a\\&-a]\hat{x}_f(k-N+1/k-N+1).\end{aligned}\tag{35}$$

One can see that $\hat{x}_s(\cdot-N/\cdot)$ remains finite whenever $\hat{x}_f(\cdot/\cdot)$ does.

Performance

The measure of the possible reduction in the error covariance obtained from fixed-lag smoothing as opposed to filtering is obtained from relation (32) by letting $N\to\infty$:

$$P_{s\infty}=P-\sum_{j=1}^{\infty}[(1-Kh)a]^{2j}(\bar{P}-P).\tag{36}$$

We shall consider three cases. We define N^* as being the smallest value of N for which the smoothing error covariance equals $P_{s\infty}$, to the accuracy of five decimal places.

Case 1. $a=0{\cdot}95, h=1, q=1, r=10$

$P_{s\infty}=1{\cdot}5811$	Filtering error covariance$=2{\cdot}4098$	$F=0{\cdot}7211$	$N^*=17$
Values of N	Smoothing error covariance		
1	2·0120		
2	1·8051		
3	1·6976		
4	1·6417		
5	1·6126		
10	1·5823		
16	1·5812		
17	1·5811		

Case 2. $a=0{\cdot}95, h=1, q=10, r=1$

$P_{s\infty}=0{\cdot}8511$	Filtering error covariance$=0{\cdot}9154$	$F=0{\cdot}0803$	$N^*=2$
Values of N	Smoothing error covariance		
1	0·8515		
2	0·8511		

Case 3. $a=0{\cdot}1, h=1, q=1, r=1$

$P_{s\infty}=0{\cdot}5000$	Filtering error covariance$=0{\cdot}5012$	$F=0{\cdot}0499$	$N^*=1$
Values of N	Smoothing error covariance		
1	0·5000		
2	0·5000		

In case 1 fixed-lag smoothing offers substantial improvement over the performance obtainable from filtering. In all cases the value of F determines the rate of convergence as $N\to\infty$ of the smoothing error covariance.

7. SUMMARY OF RESULTS AND CONCLUSIONS

For convenience, we gather together the principal definitions and results, including a statement of the fixed-lag smoothing algorithm of section 4.

System and noise description

The system considered is

$$x(k+1)=\phi_k x(k)+G_k w(k)$$
$$y(k+1)=H'_{k+1}x(k+1)+v(k+1)$$

with

$$E[w(k)w'(l)]=Q_k\delta(k-l)$$
$$E[v(k)v'(l)]=R_k\delta(k-l)$$
$$E[x(k_0)]=\bar{x}_{k_0}$$
$$E\{[x(k_0)-\bar{x}(k_0)][x(k_0)-\bar{x}(k_0)]'\}=P_{k_0}.$$

Here, $\delta(m)$ is zero except for $m=0$, with $\delta(0)=1$; $w(\cdot)$, $v(\cdot)$ and $x(k_0)$ are independent and gaussian, and $w(k)$ and $v(k)$ have zero mean for all k. The matrix ϕ_k is assumed invertible, and Q_k, G_k etc. satisfy certain conditions outlined in Ref. [6] which ensure that the associated filter is exponentially asymptotically stable, with bounded error covariance matrix.

The time-invariant case is defined by ϕ_k, G_k, H_k, Q_k and R_k constant, $k_0=-\infty$, and $x(k+1)=\phi_k x(k)$ asymptotically stable.

Definitions of estimates

We seek filtered and smoothed estimates

$$\hat{x}_f(k/k)=E[x(k)/y(k), y(k-1), \ldots y(k_0+1)]$$
$$\hat{x}_s(k-N/k)=E[x(k-N)/y(k), y(k-1), \ldots y(k_0+1)]$$

and the associated error covariance matrices

$$P_k=E\{[x(k)-\hat{x}_f(k/k)][x(k)-\hat{x}_f(k/k)]'\}$$
$$P_s(k-N/k)=E\{[x(k-N)-\hat{x}_s(k-N/k)][x(k-N)-\hat{x}_s(k-N/k)]'\}.$$

Definition of filter

$$\hat{x}_f(k+1/k+1)=F_k\hat{x}_f(k/k)+K_{k+1}y(k+1)$$

with

$$F_k=[I-K_{k+1}H'_{k+1}]\phi_k$$

and with K_k, P_k and a further quantity $\bar{P}_k$ determined recursively by

$$\bar{P}_k=\phi_k P_k\phi'_k+G_kQ_kG'_k$$
$$K_{k+1}=\bar{P}_kH_{k+1}(H'_{k+1}\bar{P}_kH_{k+1}+R_{k+1})^{-1}$$
$$P_{k+1}=(I-K_{k+1}H'_{k+1})\bar{P}_k.$$

The iterations are commenced by $\hat{x}_f(k_0/k_0)=\bar{x}_{k_0}$ and by P_{k_0}.

In the time-invariant case, F_k, $\bar{P}_k$, K_k and P_k are independent of k.

Definition of smoother

Define

$$D_k=\bar{P}_k(P_k\phi'_k)^{-1}.$$

The estimate equation is

$$\begin{bmatrix} \hat{x}_f(k+1/k+1) \\ \hat{x}_f(k/k) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \hat{x}_f(k-N+1/k-N+1) \end{bmatrix} = \begin{bmatrix} F_k & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ I & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & I & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & I & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_f(k/k) \\ \hat{x}_f(k-1/k-1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \hat{x}_f(k-N/k-N) \end{bmatrix} + \begin{bmatrix} K_{k+1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} y(k+1)$$

$$\hat{x}_s(k-N/k)=\left[\prod_{i=k-N}^{k-1} D_i^{-1} \prod_{i=k-N}^{k-1} D_i^{-1}(D_{k-1}-\phi_{k-1}) \prod_{i=k-N}^{k-2} D_i^{-1}(D_{k-2}-\phi_{k-2}) \ldots D_{k-N}^{-1}(D_{k-N}-\phi_{k-N})\right] \begin{bmatrix} \hat{x}_f(k/k) \\ \hat{x}_f(k-1/k-1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \hat{x}_f(k-N/k-N) \end{bmatrix}.$$

In the time-invariant case, F_k is constant and so is D_k. The last equation becomes

$$\hat{x}_s(k-N/k)=[D^{-N}\, D^{-N}(D-\phi) \ldots D^{-1}(D-\phi)] \begin{bmatrix} \hat{x}_f(k/k) \\ \hat{x}_f(k-1/k-1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \hat{x}_f(k-N/k-N) \end{bmatrix}.$$

The error covariance matrix equation is

$$P_s(k-N/k)=P_{k-N}-\left(\prod_{i=k-N}^{k-1} D_i^{-1}\right)(\bar{P}_{k-1} - P_k)\left(\prod_{i=k-N}^{k-1} D_i^{-1}\right)' - \ldots - (D_{k-N}^{-1})(\bar{P}_{k-N}-P_{k-N+1})(D_{k-N}^{-1})'$$

which in the time-invariant case becomes

$$P_s(k-N/k)=P-\sum_{j=1}^{N}(D^{-j})(\bar{P}-P)(D^{-j})'.$$

CONCLUSIONS

The key features of our algorithm are that it is stable, and that it represents the smoother as a linear finite-dimensional dynamic system, with the natural input and output. The equations are not unique, if for no other reason than they will vary through coördinate transformation in the state-space.

The complexity of the smoother can be measured by the dimension of its statespace, and is $n(N+1)$ where n is the dimension of the state vector of the system generating the measurements, and N is the smoothing lag. The choice of N will normally be governed by the following three factors: (1) complexity of the smoother (2) permissible time lag in producing an estimate (3) improvement through use of smoothing rather than filtering. In any case, there will always be a value of N at which essentially no improvement in error covariance can be obtained by increase of N.

One might also conceive of doing fixed-lag smoothing by carrying out a sequence of fixed-point, or fixed-interval, smoothing calculations. It could reasonably be conjectured that the fixed-point smoothing, when carried out sequentially, would be little different from the fixed-lag smoothing. Fixed-interval smoothing would certainly be as complicated, since storage of the filtered estimate is required for all points in the interval of interest [4].

The question still remains as to how one might design a continuous time smoother. If it is to be finite-dimensional and stable, then it must be sub-optimal.

REFERENCES

[1] R. E. KALMAN: A new approach to linear filtering and prediction problems. *Trans. ASME, Series D: J. bas. Engng* 82, 35–45 (1960).

[2] B. D. O. ANDERSON: Properties of optimal linear smoothing. *IEEE Trans. Aut. Control* AC-14, 114–115 (1969).

[3] C. N. KELLY and B. D. O. ANDERSON: On the stability of fixed-lag smoothing algorithms. *J. Franklin Inst.* to appear.

[4] J. S. MEDITCH: *Stochastic Optimal Linear Estimation and Control*. McGraw-Hill, New York (1969).

[5] T. KAILATH and P. FROST: An innovations approach to least-squares estimation: Part II—Linear smoothing in additive white noise. *IEEE Trans. Aut. Control* AC-13, 646–655 (1968).

[6] R. E. KALMAN: New methods and results in linear filtering and prediction theory. *Proceedings of the Symposium on Engineering Applications of Probability and Random Functions.* John Wiley, New York (1961).

[7] B. D. O. ANDERSON: Stability properties of Kalman-Bucy filters. *J. Franklin Inst.* to appear.

[8] T. KAILATH: An innovations approach to least-squares estimation: Part I—Linear filtering in additive white noise. *IEEE Trans. Aut. Control* AC-13, 655–660 (1968).

APPENDIX 1

Equivalence of equations (27) and (28). As a preliminary we shall establish the relation

$$P_{k+1}(F_k')^{-1}P_k^{-1}=D_k. \tag{A.1}$$

From equation (8), we have

$$\phi_k^{-1}[I-K_{k+1}H_{k+1}']^{-1}=\phi_k^{-1}\bar{P}_kP_{k+1}^{-1}.$$

We recognize the left term as being F_k^{-1} from equation (11). Therefore

$$F_k^{-1}=\phi_k^{-1}\bar{P}_kP_{k+1}^{-1}.$$

Pre-multiplying by P_k^{-1}, post-multiplying by P_{k+1} and taking the transpose leads to

$$P_{k+1}(F_k')^{-1}P_k^{-1}=\bar{P}_k(P_k\phi_k')^{-1}=D_k. \tag{A.1}$$

The following equation—straightforwardly derived from the projection theorem, will also be used.

$$E\{\hat{x}_f(l/l)[x(k)-\hat{x}_f(k/k)]'\}=0$$

$$\text{for all } k,\, l \text{ with } l\le k. \tag{A.2}$$

Now consider the term $E\{x(k-N)v'(k-N+1)\}$. We have

$$\begin{aligned}E\{x(k-N)v'(k-N+1)\}&=E\{x(k-N)[y(k-N+1)\\&\quad -H_{k-N+1}'\phi_{k-N}\hat{x}_f(k-N/k-N)]'\}\\&=E\{x(k-N)[H_{k-N+1}'\phi_{k-N}x(k-N)\\&\quad +H_{k-N+1}'G_{k-N}w(k-N)+v(k-N+1)\\&\quad -H_{k-N+1}'\phi_{k-N}\hat{x}_f(k-N/k-N)]'\}.\end{aligned}$$

Since $w(\cdot)$ and $v(\cdot)$ are independent white sequences, this reduces to

$$\begin{aligned}E\{x(k-N)v'(k-N+1)\}&=E\{x(k-N)[x(k-N)\\&\quad -\hat{x}_f(k-N/k-N)]'\phi_{k-N}'H_{k-N+1}\}\\&=E\{[(x(k-N)-\hat{x}_f(k-N/k-N))\\&\quad +\hat{x}_f(k-N/k-N)][x(k-N)\\&\quad -\hat{x}_f(k-N/k-N)]\phi_{k-N}'H_{k-N+1}\}\\&=P_{k-N}\phi_{k-N}'H_{k-N+1} && \text{(by A.2)}\\&=P_{k-N}\phi_{k-N}'(\bar{P}_{k-N})^{-1}\bar{P}_{k-N}H_{k-N+1}\\&=D_{k-N}^{-1}\bar{P}_{k-N}H_{k-N+1}. && \text{(by A.1)}\end{aligned}$$

Now using equation (6) for K_{k-N+1}, we have immediately the desired relation

$$E\{x(k-N)v'(k-N+1)\}[H_{k-N+1}'\bar{P}_{k-N}H_{k-N+1}+R_{k-N+1}]^{-1}=D_{k-N}^{-1}K_{k-N+1}. \tag{A.3}$$

Next consider the term $E\{x(k-N)v'(k-N+2\}$,

$$\begin{aligned}E\{x(k-N)v'(k-N+2)\}&=E\{x(k-N)[y(k-N+2)\\&\quad -H_{k-N+2}'\phi_{k-N+1}\hat{x}_f(k-N+1/k-N+1)]'\}.\end{aligned}$$

Arguments similar to those already used lead to

$$\begin{aligned}E\{x(k-N)v'(k-N+2)\}&=P_{k-N}\phi_{k-N}'[I\\&\quad -K_{k-N+1}H_{k-N+1}']'\phi_{k-N+1}'H_{k-N+2}.\end{aligned}$$

Using (A.1) we get

$$\begin{aligned}&E\{x(k-N)v'(k-N+2)\}\\&\quad =D_{k-N}^{-1}D_{k-N+1}^{-1}\bar{P}_{k-N+1}H_{k-N+2}.\end{aligned}$$

Again it is easy to see that

$$E\{x(k-N)v'(k-N+2)\}[H'_{k-N+2}\bar{P}_{k-N+1}H_{k-N+2} + R_{k-N+2}]^{-1} = D_{k-N}^{-1}D_{k-N+1}^{-1}K_{k-N+2}.$$

For the general term

$$E\{x(k-N)v'(k-j+1)\}[H'_{k-j+1}\bar{P}_{k-j}H_{k-j+1} + R_{k-j+1}]^{-1}, \quad N \geq j \geq 1$$

a straightforward induction argument will establish

$$E\{x(k-N)v'(k-j+1)\}[H'_{k-j+1}\bar{P}_{k-j}H_{k-j+1} + R_{k-j+1}]^{-1} = \left(\prod_{i=k-N}^{k-j} D_i^{-1}\right)K_{k-j+1}.$$

Hence the coefficients of $v(\cdot)$ in equations (27) and (28) are identical, which proves the assertion.

Résumé—L'article étudie le lisseur optimal à retard fixe et le considère comme un système linéaire aux dimensions finies. Ceci permet de construire un algorithme de lissage stable, alors que jusqu'ici seuls des algorithmes instables ont été publiés. De plus, la complexité du lisseur peut être examinée quantitativement et des critères de calcul peuvent être suggérés pour le choix de l'intervalle de retard. L'article discute la réalisation pratique du lisseur.

Zusammenfassung—Die optimale Glättungseinrichtung mit fester Verzögerung wird betrachtet und als lineares System endlicher Dimension angesehen. Das erlaubt die Konstruktion eines stabilen Glättungsalgorithmus, obgleich bisher nur unstabile Algorithmen veröffentlicht wurden. Ferner können die Komplexität der Glättungseinrichtung quantitativ geprüft und Entwurfskriterien für die Wahl des Verzögerungsintervalles vorgeschlagen werden. Die praktische Durchführung der Glättungseinrichtung wird diskutiert.

Резюме—Статья изучает оптимальный сглаживатель с постоянным запаздыванием и рассматривает его как линейную систему с конечными размерами. Это позволяет построить устойчивый алгоритм сглаживания, в то время как все опубликованные до сих пор алгоритмы были неустойчивы. Сверх того, сложность сглаживателя может быть рассмотрена количественно и могут быть предложены критерии рассчета для выбора интервала запаздывания. Статья обуждает практическое осуществление сглаживания.