Study of an Integral Equation Arising in Detection Theory

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Abstract—This paper considers the solution of a Fredholm equation occurring in detection theory problems. A solution procedure, based on solving differential equations with nonmixed boundary conditions, is described for the case when the kernel of the integral equation is known to be the output covariance of a linear finite-dimensional system excited by white noise. Solutions with discontinuities are considered.

I. Introduction

In this paper, we study the solution of an integral equation arising in detection theory, viz.,

$$s(t) = \int_0^T R(t,\tau)q(\tau)\,d\tau, \qquad (1)$$

where $s(\cdot)$ is a known function of time, $R(\cdot,\cdot)$ is a known covariance function, and $q(\cdot)$ has to be determined.

We shall be concerned with the case when $R(t,\tau)$ is not necessarily stationary, but possesses a finite-dimensionality property of a type that would guarantee, if $R(t,\tau)$ were stationary, that its associated power spectrum was rational. More precisely, we shall assume that $R(t,\tau)$ is the covariance of the scalar output $y(\cdot)$ of a linear system

$$\begin{aligned} \dot{x} &= F(t)x + g(t)u \\ y &= h'(t)x + j_1(t)u + j_2(t)v, \end{aligned} \qquad (2)$$

where $u(\cdot)$ and $v(\cdot)$ are independent zero-mean white-noise processes of covariance $\delta(t-\tau)$, $x(0)$ is a zero-mean random variable, independent of $u(\cdot)$ and $v(\cdot)$, and the superscript prime denotes matrix transposition. This forces $R(t,\tau)$ to have the form [1], [2]

$$R(t,\tau) = c(t)\delta(t-\tau) + h'(t)\Phi(t,\tau)k(\tau)1(t-\tau) + k'(t)\Phi'(\tau,t)h(\tau)1(\tau-t), \qquad (3)$$

where $\Phi(\cdot,\cdot)$ is the transition matrix associated with $F(\cdot)$, and $c(\cdot)$ and $k(\cdot)$ are determinable from (2) and the value of $E[x(0)x'(0)]$ by procedures set out in [1], [2]. Of course, $\delta(\cdot)$ is the Dirac delta function, and $1(\cdot)$ the unit step function.

In most of the work to follow, we shall not assume that the equations of a linear system generating $R(\cdot,\cdot)$ are known, but rather simply that the existence of some such system is known. However, we shall assume that $R(\cdot,\cdot)$ is given in such a form that quantities $F(\cdot)$, $h(\cdot)$, $k(\cdot)$, and $c(\cdot)$ are known satisfying (3). This assumption and the determination of $F(\cdot)$, etc., from $R(\cdot,\cdot)$ is discussed at length in [2].

Historical Remarks: If $T = \infty$ and $R(t,\tau)$ is stationary, (1) becomes a well-studied version of the Wiener–Hopf equation. For $T < \infty$ and $R(t,\tau)$ stationary, there have been many solutions [3], [8], all of which make intrinsic use of the stationarity property. For a finite integration interval and nonstationary covariance, Zadeh and Miller [9] have given a procedure that relies on there being available a differential-equation description of (2), on (2) being in the zero state initially, and on application of special, almost *ad hoc*, techniques to allow for discontinuities of $s(\cdot)$ at the endpoints. Baggeroer [10] solves a vector version of the nonstationary problem under the restrictions that $c(t)$, which is a square matrix, is nonsingular for all t, that $j_1(t)$ in (2) is identically zero, or equivalently, that $R(t,\tau) - c(t)\delta(t-\tau)$ is a covariance, and, finally, that the system matrices in (2) are known. As noted above, in this paper we do not make this last assumption, nor actually do we make the assumption that $j_1(t)$ is zero. We also consider the situation where both $c(t)$ is nonzero and zero. The case of $c(t)$ nonzero, though studied here for scalar, processes only, is trivially extendable to the vector case. (By contrast, the case of a matrix $c(t)$ that is singular or zero introduces much more difficulty than the case of a scalar zero $c(t)$.)

Shinbrot [11] includes a technique for solving an equation similar to (1), again under the assumption that $c(t)$ is never zero. In [12] the solution of (1) is described under the highly restrictive condition that $h(\cdot)$ and $k(\cdot)$ are scalars and that $c(\cdot)$ is identically zero. More recently, one of the authors has presented in outline [13] a procedure for solving (1) when $c(t)$ is never zero, which does not demand stationarity or knowledge of a generating system.

The question of whether or not $c(t)$ is zero plays a vital role in determining the form of the solution of (1) in the stationary and nonstationary cases. If $c(t)$ is never zero, (1) becomes a Fredholm equation of the second kind, and, given certain continuity requirements on $R(\cdot,\cdot)$ and $s(\cdot)$, the solution $q(\cdot)$ is continuous. On the other hand, if $c(t)$ is identically zero, the solution of (1), if it exists, generally possesses discontinuities at the endpoints. These have been observed in case $R(t,\tau)$ is stationary, and the methods for dealing with them are *ad hoc*, see, e.g., [9]. However, the method of this paper provides a systematic technique for the evaluation of the discontinuities.

No general solution is known in case $c(t)$ is sometimes zero, sometimes nonzero. It appears that this case is

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sometimes soluble and sometimes insoluble, depending on the precise functions involved.

Notation: In obvious notation, (1) will sometimes be written as $s = R \circ q$. Given two kernels $R_1(\cdot,\cdot)$ and $R_2(\cdot,\cdot)$, the kernel $R_3 = R_1 \circ R_2$ will be defined by

$$R_3(t,\tau) = \int_0^T R_1(t,\lambda)R_2(\lambda,\tau)\, d\lambda. \quad (4)$$

Unfortunately, the operator $\circ$ is not necessarily associative [14], so care must be exercised in interpreting successive uses of the operator.

Outline of Paper: In Section II we discuss the solution of (1) when in the expression for $R(t,\tau)$, $c(t)$ is never zero. The main result, Theorem 1, explains how an operator R^{-1} can be constructed, through the solution of differential equations with the property that $q = R^{-1} \circ s$. Lying behind the theorem is the result on time-varying spectral factorization in [2]. Section III discusses the case when $c(t)$ is identically zero; in this section, some use is made of results on singular time-varying spectral factorization appearing in [15], together with the procedure of Section II. Section IV contains examples illustrating the ideas of Sections II and III.

II. Solution of Fredholm Equation of the Second Kind

In this section we study the solution of the fundamental equation (1) when it becomes a Fredholm equation of the second kind. This is the case if and only if $R(t,\tau)$ has the form of (3), with $c(t)$ never zero. This means that (1) is equivalent to

$$s(t) = c(t)q(t) + \int_0^T K(t,\tau)q(\tau)\, d\tau, \quad (5)$$

where $K(\cdot,\cdot)$ is continuous, provided the entries of $\boldsymbol{F}(\cdot)$, $\boldsymbol{h}(\cdot)$, and $\boldsymbol{k}(\cdot)$ are continuous.

We seek an operator R^{-1} with the property that the solution of (1) is $q = R^{-1} \circ s$. The procedure for constructing R^{-1} can be thought of as falling into three steps as follows.

1) Construction, by solving a differential equation, of a causal operator $w(\cdot,\cdot)$ satisfying $R = w \circ w^a$; [note: $w(t,\tau)$ is causal if $w(t,\tau) = 0$ for $t < \tau$, and $w^a(t,\tau) = w(\tau,t)$].

2) Construction of a causal operator w^{-1}, satisfying $w \circ w^{-1} = w^{-1} \circ w = \delta$; again, the construction proceeds by solving a differential equation.

3) Construction of R^{-1} by $R^{-1} = (w^a)^{-1} \circ w^{-1}$.

It is immediate that R^{-1} possesses the desired properties if it is true that the operator $\circ$ is associative. For then we have

$$R^{-1} \circ s = R^{-1} \circ (R \circ q) = (w^a)^{-1} \circ w^{-1} \circ w \circ w^a \circ q = q.$$

As we have already remarked, associativity is not always guaranteed. But in some special cases, it is. These cases include, see [14], the case when $q \in \mathscr{L}_2$, and w, w^a, w^{-1}, and $(w^a)^{-1}$ map $\mathscr{L}_2$ into $\mathscr{L}_2$: by continuity and the boundedness of the interval $[0,T]$, q and w, etc., have the properties stated, and the formula $R^{-1} = (w^a)^{-1} \circ w^{-1}$ is validated.

The main result is as follows. See Appendix I for a proof.

Theorem 1: Suppose that $R(\cdot,\cdot)$ is given by (3) with $R(\cdot,\cdot)$ positive definite (or, equivalently [2], the output covariance of some possibly unknown linear finite-dimensional system excited by white noise), with $\boldsymbol{F}(\cdot)$, $\boldsymbol{h}(\cdot)$, $\boldsymbol{k}(\cdot)$, and $c(\cdot)$ possessing continuous entries, and with $c(t)$ nonzero and thus positive for all t. Then 1) a causal operator $w(\cdot,\cdot)$ satisfying $w \circ w^a = R$ is given by

$$w(t,\tau) = c^{1/2}(t)\delta(t-\tau) + \boldsymbol{h}'(t)\boldsymbol{\Phi}(t,\tau)\boldsymbol{g}_1(\tau)1(t-\tau) \quad (6)$$

where

$$\boldsymbol{g}_1(t) = [\boldsymbol{k}(t) - \boldsymbol{P}(t)\boldsymbol{h}(t)]c^{-1/2}(t) \quad (7)$$

and $\boldsymbol{P}(\cdot)$ is the solution of

$$\dot{\boldsymbol{P}} = \boldsymbol{PF}' + \boldsymbol{FP} + (\boldsymbol{Ph} - \boldsymbol{k})c^{-1}(\boldsymbol{Ph} - \boldsymbol{k})', \qquad \boldsymbol{P}(0) = \boldsymbol{0}; \quad (8)$$

2) if the kernel $R^{-1}(t,\tau)$ is defined by

$$R^{-1}(t,\tau) = \int_0^T [w^{-1}(t,\lambda)]^a w^{-1}(\lambda,\tau)\, d\lambda, \quad (9)$$

then

$$R^{-1}(t,\tau) = c^{-1}(t)\delta(t-\tau) + \boldsymbol{h}_1'(t)\boldsymbol{\Phi}_1(t,\tau)\boldsymbol{g}_1(\tau)c^{-1/2}(\tau)1(t-\tau) + c^{-1/2}(t)\boldsymbol{g}_1'(t)\boldsymbol{\Phi}_1'(\tau,t)\boldsymbol{h}_1(\tau)1(\tau-t), \quad (10)$$

where $\boldsymbol{\Phi}_1(\cdot,\cdot)$ is the transition matrix associated with $\boldsymbol{F}_1 = \boldsymbol{F} - \boldsymbol{g}_1c^{-1/2}\boldsymbol{h}'$, and $\boldsymbol{h}_1(\cdot)$ is given by

$$\boldsymbol{h}_1 = \boldsymbol{Qg}_1c^{1/2} - \boldsymbol{h}c^{-1} \quad (11)$$

with $\boldsymbol{Q}(\cdot)$ given by

$$-\dot{\boldsymbol{Q}} = \boldsymbol{F}_1'\boldsymbol{Q} + \boldsymbol{QF}_1 + \boldsymbol{h}c^{-1}\boldsymbol{h}', \quad \boldsymbol{Q}(T) = \boldsymbol{0}. \quad (12)$$

Furthermore, $q(\cdot)$ is uniquely specified by

$$q(t) = \int_0^T R^{-1}(t,\tau)s(\tau)\, d\tau \quad (13)$$

and is continuous.

If (1) is to be solved for a number of functions $s(\cdot)$, then evaluation of $R^{-1}(\cdot,\cdot)$ by the method given in the theorem followed by evaluation of $R^{-1} \circ s$ for each $s(\cdot)$ is probably efficient. But if $q(\cdot)$ is to be evaluated for only one $s(\cdot)$, some slight saving in computation time may perhaps be achievable as follows. It is easy to check that if

$$\boldsymbol{x}_1(t) = \int_0^t \boldsymbol{\Phi}_1(t,\tau)\boldsymbol{g}_1(\tau)c^{-1/2}(\tau)s(\tau)\, d\tau, \quad (14)$$

then

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{F}_1\boldsymbol{x}_1 + \boldsymbol{g}_1c^{-1/2}s, \qquad \boldsymbol{x}_1(0) = \boldsymbol{0} \quad (15)$$

Likewise, if

$$\boldsymbol{x}_2(t) = \int_t^T \boldsymbol{\Phi}_1'(\tau,t)\boldsymbol{h}_1(\tau)s(\tau)\, d\tau, \quad (16)$$

one finds that

$$\dot{\boldsymbol{x}}_2 = -\boldsymbol{F}_1'\boldsymbol{x}_2 - \boldsymbol{h}_1s, \qquad \boldsymbol{x}_2(T) = \boldsymbol{0}. \quad (17)$$

Now examination of (10) and (13) shows that

$$q(t) = c^{-1}(t)s(t) + \boldsymbol{h}_1'(t) \int_0^t \boldsymbol{\Phi}_1(t,\tau)\boldsymbol{g}_1(\tau)c^{-1/2}(\tau)s(\tau)\, d\tau + c^{-1/2}(t)\boldsymbol{g}_1'(t) \int_t^T \boldsymbol{\Phi}_1'(\tau,t)\boldsymbol{h}_1(\tau)s(\tau)\, d\tau,$$

which, by the definition of $\boldsymbol{x}_1(\cdot)$ and $\boldsymbol{x}_2(\cdot)$, is

$$q(t) = c^{-1}(t)s(t) + \boldsymbol{h}_1'(t)\boldsymbol{x}_1(t) + c^{-1/2}(t)\boldsymbol{g}_1'(t)\boldsymbol{x}_2(t). \quad (18)$$

Therefore, in evaluating $q(\cdot)$ it is probably easier to solve (15) and (17) directly, than to evaluate $\boldsymbol{\Phi}_1(\cdot,\cdot)$ first and then perform the integrations implicit in (14) and (16). When $q(\cdot)$ is found this way, $\boldsymbol{P}(\cdot)$ and $\boldsymbol{Q}(\cdot)$ must still be evaluated, in order to obtain $\boldsymbol{F}_1(\cdot)$, $\boldsymbol{g}_1(\cdot)$, and $\boldsymbol{h}_1(\cdot)$.

It is also possible to establish the result of Theorem 1 without recourse to the tricky point involved in validating the associativity of operators. Theorem 2, also established in Appendix I, contains another insight into the derivation of $q(\cdot)$, by claiming that $q(\cdot)$ is obtainable by solving a two-point boundary value problem.

Theorem 2: With the same hypothesis as Theorem 1, the function $q(\cdot)$ exists, is unique and continuous, and can be obtained by solving the two-point boundary value problem

$$\dot{\boldsymbol{x}} = (\boldsymbol{F} - c^{-1}\boldsymbol{kh}')\boldsymbol{x} - c^{-1}\boldsymbol{kk}'\boldsymbol{y} + c^{-1}\boldsymbol{k}s$$
$$\dot{\boldsymbol{y}} = c^{-1}\boldsymbol{hh}'\boldsymbol{x} + (-\boldsymbol{F}' + c^{-1}\boldsymbol{hk}')\boldsymbol{y} - c^{-1}\boldsymbol{h}s \quad (19)$$

with boundary conditions $\boldsymbol{x}(0) = \boldsymbol{y}(T) = \boldsymbol{0}$, and by setting

$$q = c^{-1}s - c^{-1}\boldsymbol{h}'\boldsymbol{x} - c^{-1}\boldsymbol{k}'\boldsymbol{y}. \quad (20)$$

Moreover, this boundary value problem can always be solved and has the unique solution

$$q(t) = \int_0^T R^{-1}(t,\tau)s(\tau)\, d\tau, \quad (13)$$

where $R^{-1}(\cdot,\cdot)$ is defined by (10).

This theorem says that $R^{-1}(t,\tau)$ can be regarded as a type of impulse response for the linear system of (19) and (20), where the boundary condition is not the natural one of $\boldsymbol{x}(0) = \boldsymbol{y}(0) = \boldsymbol{0}$, but rather $\boldsymbol{x}(0) = \boldsymbol{y}(T) = \boldsymbol{0}$. The fact that (19) has mixed boundary conditions means that there is no immediate way of solving these equations; however, the calculation procedure implicit in the statement of Theorem 1 does provide a technique for solving (19).

In Section I we noted that the above results are straightforwardly extendable to the case of matrix $R(\cdot,\cdot)$, and we noted that the extension has been carried out in [10] when a lot more conditions are presupposed. Among these conditions is one requiring knowledge of a spectral factor *not* of $R(\cdot,\cdot)$, but of $R(\cdot,\cdot) - c(t)\delta(t-\tau)$. Now if $R(\cdot,\cdot)$ is a covariance, this quantity need not be. However, if it is a covariance, then by knowing only $R(\cdot,\cdot)$ and $c(\cdot)$ a spectral factor can be found via a singular spectral factorization procedure [15] and then the procedure of [10] could be used to solve the integral equation of interest. This would be a long way round to the same endpoint as reached in this section, since the singular spectral factorization is certainly a good deal more complex than the spectral factorization used in this section. Moreover, the complexity of the calculations of this section is about the same as the complexity of the calculations in [10]. Finally, if $R(\cdot,\cdot) - c(t)\delta(t-\tau)$ is not a covariance, it would be impossible to use [10] through inability to obtain a spectral factor of $R(\cdot,\cdot) - c(t)\delta(t-\tau)$.

Two other papers relevant to this section are [20] and [21]. Both papers were concerned with giving procedures for computing the kernel $R^{-1}(\cdot,\cdot)$ rather than results like those of Theorems 1 and 2, which give explicit formulas for $R^{-1}(\cdot,\cdot)$ and the integral equation solution. Note finally that the material of this section was first presented in [13].

It will be noted that even if $R(t,\tau)$ is stationary, the procedures of this section effectively involve the generation of a nonstationary spectral factor. It would appear that this could be avoided by generalizing the theory, as in [10], to permit nonzero boundary conditions on the differential equations for $\boldsymbol{P}$ and $\boldsymbol{Q}$; these boundary conditions could then be selected to ensure that $\boldsymbol{P}$ and $\boldsymbol{Q}$ are constant for all t. To do this would involve frequency-domain spectral factorization and solution of linear matrix equations, as in [10], which might represent more trouble than it is worth.

Finally, we comment that the kernel $R^{-1}(\cdot,\cdot)$ can be thought of as defining the major element of an optimal detector operating in colored noise; it is the impulse response of a block that processes the incoming measurements.

III. Solution of the Fredholm Equation of the First Kind

In this section we study the solution of (1) when the function $c(t)$ appearing in $R(t,\tau)$ [see (3)] is identically zero. An outline of the section is as follows.

First, we shall dispense with the straightforward case when $R(t,\tau)$ has the form

$$R(t,\tau) = \boldsymbol{\alpha}'(t)\boldsymbol{\alpha}(\tau). \quad (21)$$

Then we shall introduce a covariance R^*, computable from R. In terms of this covariance, we can state a condition for the integral equation to be solvable; it turns out that unsolvability of the integral equation corresponds to singularity of an associated detection problem. From s, we shall compute a function s^*, and we shall argue that if q is such that $s = R \circ q$, then $s^* = R^* \circ q$. Finally, using the theory of the previous section, we shall find all solutions q^* of $s^* = R^* \circ q^*$ and then isolate those solutions that are also solutions of $s = R \circ q$.

It might be thought that a logical way to attempt to solve (1) would again be to find a causal operator w such that $R = w \circ w^a$, and then to set $R^{-1} = (w^a)^{-1} \circ w^{-1}$. Even assuming such w can be found[1], the fact that $c(t)$ is zero will force w^{-1} to contain derivatives of delta functions. Accordingly, it will not be true that w^{-1} maps $\mathscr{L}_2$ into $\mathscr{L}_2$.

[1] In general, it cannot (see [15]).

There then appears no way of justifying associativity of the various elements of the product $[(w^a)^{-1} \circ w^{-1}] \circ \{[w \circ w^a] \circ q\}$, which is necessary to validate the formula for R^{-1}.

Simple Special Case

Suppose that $R(\cdot,\cdot)$ has the form of (21). Without loss of generality, we can assume the entries $\alpha_i(\cdot)$ of $\boldsymbol{\alpha}(\cdot)$ are linearly independent over $[0,T]$. We claim that (1) is solvable if and only if $s(t) = \boldsymbol{\alpha}'(t)\gamma$ for some constant vector γ. To see this, suppose first that $q(\cdot)$ is a solution of (1). Then

$$\boldsymbol{\alpha}'(t) \int_0^T \boldsymbol{\alpha}(\tau)q(\tau)\, d\tau = s(t)$$

or, with $\gamma = \int_0^T \boldsymbol{\alpha}(\tau)q(\tau)\, d\tau$, we have $\boldsymbol{\alpha}'(t)\gamma = s(t)$. Conversely, suppose for some constant vector γ that $s(t) = \boldsymbol{\alpha}'(t)\gamma$. Then it is readily verified that the solution of (1) is

$$q(t) = \boldsymbol{\alpha}'(t) \left[\int_0^T \boldsymbol{\alpha}(\tau)\boldsymbol{\alpha}'(\tau)\, d\tau\right]^{-1} \gamma.$$

Note that existence of the inverse follows from the assumption of linear independence of the $\boldsymbol{\alpha}_i(\cdot)$.

Introduction of Covariance R^*

It is now necessary to make an assumption regarding the covariance $R(\cdot,\cdot)$ as follows. Not only is $R(\cdot,\cdot)$ the covariance of the output $y(\cdot)$ of a linear finite-dimensional system excited by white noise, but there exists some m such that

$$\frac{\partial^{2i}}{\partial t^i \partial \tau^i} R(t,\tau)$$

is continuous for all t, τ for $i < m$, but

$$\frac{\partial^{2m}}{\partial t^m \partial \tau^m} R(t,\tau)$$

is of the form $c_m(t)\delta(t-\tau)$ plus a bounded continuous part, with $c_m(t) > 0$ for all t. (This means that the first, second, $\cdots$, $(m-1)$th derivatives of the process y with covariance $R(\cdot,\cdot)$ contain no white noise, but are continuous processes, while the mth derivative $y^{(m)}$ contains white noise.) The significance of this assumption, a very reasonable one, is discussed in [15], [18] at some length; provided there are no structural changes occurring in the generating system as time evolves, either this assumption is satisfied, or $R(\cdot,\cdot)$ has the form of (21). If the generating system is stationary, of course, the occurrence of structural changes is impossible.

We shall denote by $R_{ij}(t,\tau)$ the derivative $[\partial^{i+j}/(\partial t^i \partial \tau^j)] R(t,\tau)$, so then $R_{ij}(t,\tau)$ is $E[y^{(i)}(t)y^{(j)}(\tau)]$. It is straightforward to establish that

$$R_{mm}(t,\tau) = c_m(t)\delta(t-\tau) + \boldsymbol{h}_m'(t)\boldsymbol{\Phi}(t,\tau)\boldsymbol{k}_m(\tau)1(t-\tau) + \boldsymbol{k}_m'(t)\boldsymbol{\Phi}'(\tau,t)\boldsymbol{h}_m(\tau)1(\tau-t), \quad (22)$$

where quantities $\boldsymbol{h}_i$, $\boldsymbol{k}_i$, and c_i are defined recursively by

$$\boldsymbol{h}_i = \dot{\boldsymbol{h}}_{i-1} + \boldsymbol{F}'\boldsymbol{h}_{i-1}, \qquad \boldsymbol{h}_0 = \boldsymbol{h},$$
$$\boldsymbol{k}_i = \dot{\boldsymbol{k}}_{i-1} - \boldsymbol{F}\boldsymbol{k}_{i-1}, \qquad \boldsymbol{k}_0 = \boldsymbol{k},$$

and

$$c_i = \boldsymbol{k}_i'\boldsymbol{h}_{i-1} - \boldsymbol{h}_i'\boldsymbol{k}_{i-1}.$$

As a consequence of the fact that $R_{ii}(t,\tau)$ contains no delta function for $i < m$, one can deduce that $c_i = 0$ for $i < m$, whereas by assumption $c_m(t) > 0$ for all t. We also make the assumption that c_m and the entries of $\boldsymbol{h}_m$ and $\boldsymbol{k}_m$ are continuous.

Now introduce matrices $\boldsymbol{H}_m$ and $\boldsymbol{K}_m$ defined by

$$\boldsymbol{H}_m = [\boldsymbol{h}\boldsymbol{h}_1 \cdots \boldsymbol{h}_{m-1}] \qquad \boldsymbol{K}_m = [\boldsymbol{k}\boldsymbol{k}_1 \cdots \boldsymbol{k}_{m-1}]. \quad (23)$$

It is straightforward to check that with $\mathscr{Y}(t) = [y(t)\dot{y}(t) \cdots y^{(m-1)}(t)]'$,

$$E[\mathscr{Y}(t)\mathscr{Y}'(t)] = \boldsymbol{K}_m'\boldsymbol{H}_m$$

and thus $\boldsymbol{K}_m'\boldsymbol{H}_m$ is symmetric and nonnegative definite.

We can now state a theorem due to Shepp [18] in a slightly extended form. The extension is essentially given in [15] and involves the removal of a restrictive condition on the matrix $\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)$.

Theorem 3: With $R(\cdot,\cdot)$ satisfying the conditions as stated above, there exists a decomposition of $R(\cdot,\cdot)$ as

$$R(t,\tau) = R^*(t,\tau) + \boldsymbol{\alpha}'(t)\boldsymbol{\alpha}(\tau), \quad (24)$$

where[2]

$$\boldsymbol{\alpha}(t) = \{[\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)]^\#\}^{1/2}\boldsymbol{K}_m'(0)\boldsymbol{\Phi}'(t,0)\boldsymbol{h}(t) \quad (25)$$

and

$$R^*(t,\tau) = \boldsymbol{h}'(t)\boldsymbol{\Phi}(t,\tau)k^*(\tau)1(t-\tau) + k^*(t)\boldsymbol{\Phi}'(\tau,t)\boldsymbol{h}(\tau)1(\tau-t) \quad (26)$$

with

$$\boldsymbol{k}^*(t) = \boldsymbol{k}(t) - \boldsymbol{\Phi}(t,0)\boldsymbol{K}_m(0)[\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)]^\#\boldsymbol{K}_m'(0) \cdot \boldsymbol{\Phi}'(t,0)\boldsymbol{h}(t). \quad (27)$$

The entries of $k^*(\cdot)$ and $\boldsymbol{\alpha}(\cdot)$ are continuous. Furthermore, $R^*(t,\tau)$ possesses the following properties.

1) $R^*(t,\tau)$ is the covariance of

$$z(t) = y(t) - \boldsymbol{\alpha}'(t)\{[\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)]^\#\}^{1/2}\mathscr{Y}(0).$$

2) $R_{ij}^*(t,\tau)$ for $0 \le i, j \le m-1$ is continuous in t and τ and for $i = m$, $j \le m-1$ and $i \le m-1$, $j = m$ is bounded in t and τ; furthermore,

$$R_{ij}^*(0,\tau) = R_{ij}^*(t,0) = 0, \quad (28)$$

for all t, $\tau \in [0,T]$, for $0 \le i, j \le m-1$, for $i = m$, $j \le m-1$ and $i \le m-1$, $j = m$.

For a full discussion of the significance of this theorem, see [18]. In effect, $R^*(\cdot,\cdot)$ is the covariance of a process obtained from the process $y(\cdot)$ by pinning this process (and some of its derivatives) at the origin.

[2] The pseudoinverse # in (25) and later equations is the Moore–Penrose pseudoinverse.

A further important result, obtained in [15], is that, as a result of $R(t,\tau)$ being the output covariance of some finite-dimensional linear system driven by white noise, $R_{mm}^*(t,\tau)$ is positive definite.[3] Other interpretations of this condition may be found in [18].

Perfect Detection, and Condition for Solvability of the Integral Equation

The integral equation (1) arises in the following detection problem. We are given a measurement $r(t)$, $0 \leq t \leq T$, which under hypothesis 1, is $r(t) = s(t) + y(t)$ and under hypothesis 0 is $r(t) = y(t)$, where $y(\cdot)$ is a zero-mean Gaussian process with covariance $R(t,\tau)$. The problem is to decide in some optimal fashion whether hypothesis 1 or hypothesis 0 is in force. A sufficient statistic for the associated likelihood ratio is provided by $\int_0^t r(t)q(t)\,dt$ where $q = R^{-1} \circ s$ [19, ch. 4].

Denote the $m \times m$ matrix whose $i - j$ element is $R_{i-1\,j-1}(t,\tau)$ by $\mathscr{R}(t,\tau)$; thus $\mathscr{R}(t,\tau)$ is the same as $E[\mathscr{Y}(t)\mathscr{Y}'(\tau)]$. Also denote by $\mathscr{S}(t)$ the m-vector with ith entry $s^{(i-1)}(t)$; because $y(\cdot)$ is $(m-1)$ times continuously differentiable, we shall assume henceforth, as is reasonable for problems of this nature, that $s(\cdot)$ is m times continuously differentiable,[4] and so $\mathscr{S}(t)$, and in fact $\dot{\mathscr{S}}(t)$, exists. The following result then holds.

Lemma 1: If there exists no constant vector $\boldsymbol{\beta}$ such that $\mathscr{S}(0) = \mathscr{R}(0,0)\boldsymbol{\beta}$, perfect detection is possible.

Proof: If $\mathscr{R}(0,0)$ is nonsingular, the lemma is inapplicable. Therefore, suppose $\mathscr{R}(0,0)$ is singular. Then there exists a constant vector γ such that $\mathscr{R}(0,0)\gamma = 0$, and thus $\gamma'\mathscr{R}(0,0)\gamma = E[\gamma'\mathscr{Y}(0)]^2 = 0$. Hence, unless $\gamma'\mathscr{S}(0) = 0$, by forming γ' multiplied by a vector of the measurement and its first $m - 1$ derivatives, we could determine with certainty the presence or absence of the signal. Hence, absence of perfect detection means that $\mathscr{S}(0)$ is orthogonal to vectors in the nullspace of $\mathscr{R}(0,0)$; therefore, $\mathscr{S}(0)$ is in the range-space of $\mathscr{R}'(0,0) = \mathscr{R}(0,0)$, as required.

In the sequel, we shall assume that $\mathscr{S}(0)$ is in the range space of $\mathscr{R}(0,0)$, i.e., that there exists a constant vector $\boldsymbol{\beta}$ such that

$$\mathscr{S}(0) = \mathscr{R}(0,0)\boldsymbol{\beta}. \tag{29}$$

Introduction of Function s^* and Its Properties

Our purpose is to find an integral equation like (1), but with $R(\cdot,\cdot)$ replaced by $R^*(\cdot,\cdot)$; this integral equation will be set up so that any solution of (1) also satisfies it. However, it will be easier to solve than (1), and by solving it, we shall get at a solution of (1).

Define $s^*(t)$ by

$$s^*(t) = s(t) - \boldsymbol{h}'(t)\boldsymbol{\Phi}(t,0)\boldsymbol{K}_m(0)[\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)]^{\#}\mathscr{S}(0). \tag{30}$$

A very simple property of $s^*(t)$, paralleling a property of $R^*(t,\tau)$, is that

$$s^{*(i)}(0) = 0, \qquad 0 \leq i \leq m - 1. \tag{31}$$

To see this, we observe that

$$[s^*(0)\cdots s^{*(m-1)}(0)]' = \mathscr{S}(0) - \boldsymbol{H}_m'(0)\boldsymbol{K}_m(0) \cdot [\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)]^{\#}\mathscr{S}(0).$$

Since $\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0) = E[\mathscr{Y}(0)\mathscr{Y}'(0)] = \mathscr{R}(0,0)$, and since $\mathscr{S}(0)$ is in the range space of $\mathscr{R}(0,0)$, the equality of the right-hand side to zero follows by a standard property of the pseudoinverse operator.[5] Notice also that $s^*(\cdot)$ inherits the differentiability properties assumed for $s(\cdot)$; in particular, $s^{*(m)}(\cdot)$ is continuous.

With $s^*(\cdot)$ and $R^*(\cdot,\cdot)$ defined, we can state the following important lemma.

Lemma 2: With $R(\cdot,\cdot)$ and $q(\cdot)$ satisfying all assumptions hitherto stated, any solution $q(\cdot)$ of (1) also satisfies

$$s^*(t) = \int_0^T R^*(t,\tau)q(\tau)\,d\tau, \tag{32}$$

but not necessarily conversely.

Proof: From (1) it follows by differentiating and setting $t = 0$ that for $i = 0,1,\cdots,m-1$

$$s^{(i)}(0) = \int_0^T R_{i0}(0,\tau)q(\tau)\,d\tau.$$

From the fact that $R(t,\tau) = \boldsymbol{h}'(t)\boldsymbol{\Phi}(t,\tau)\boldsymbol{k}(\tau)1(t-\tau) + \boldsymbol{k}'(t)\boldsymbol{\Phi}'(\tau,t)\boldsymbol{h}(\tau)1(\tau - t)$, it is straightforwardly found that

$$\mathscr{S}(0) = \int_0^T \boldsymbol{K}_m'(0)\boldsymbol{\Phi}'(\tau,0)\boldsymbol{h}(\tau)q(\tau)\,d\tau.$$

After premultiplying both sides of this equation by the expression $\boldsymbol{h}'(t)\boldsymbol{\Phi}(t,0)\boldsymbol{K}_m(0)[\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)]^{\#}$ and subtracting the result from (1), (32) follows immediately upon using the definitions $s^*(\cdot)$ and $R^*(\cdot,\cdot)$.

Lemma 2 suggests that we might attempt to solve (1) by finding all solutions of

$$s^*(t) = \int_0^T R^*(t,\tau)q^*(\tau)\,d\tau \tag{33}$$

and isolating those solutions of (33) that are solutions also of (1). The next lemma, established in Appendix II, states how to find all solutions of (33).

Lemma 3: With all quantities as defined previously, all solutions of (33) are given by

$$q^*(t) = (-1)^m \left[l_0(t) - l_1(T)\delta(t-T) - \cdots - l_m(T)\delta^{(m-1)}(t-T) - \sum_{i=1}^{m} a_i\delta^{(i-1)}(t) \right], \tag{34}$$

where the a_i are arbitrary constants, $l_m(\cdot)$ is the unique

[3] Note also that if a covariance containing a delta function is not positive definite, it cannot possess a spectral factor; furthermore, the associated detection problem is very probably singular.

[4] Slightly less strong assumptions are possible; e.g., $S(\cdot)$ must be $(m-1)$ times continuously differentiable, and $s^{(m)} \in \mathscr{L}_2$.

[5] The property being used here is $AA^{\#}A = A$.

solution of

$$s^{*(m)}(t) = \int_0^T R_{mm}^*(t,\tau)l_m(\tau)\, d\tau, \qquad (35)$$

the $l_i(\cdot)$ are defined recursively for $0 \le i \le m-1$ by

$$l_i(t) = \frac{d}{dt}[l_{i+1}(t)], \qquad (36)$$

and the singularities of the delta functions are assumed, for the purposes of satisfying (33) to lie entirely within the range of integration $[0,T]$.

The key calculation in obtaining $q^*(\cdot)$ is the solution of (35). The various assumptions on $R(\cdot,\cdot)$ guarantee that $R_{mm}^*(\cdot,\cdot)$ is of the form $c_m(t)\delta(t-\tau)$ plus a term that is continuous in t and τ, and that $R_{mm}^*(\cdot,\cdot)$ is positive definite, Since $s^{*(m)}(\cdot)$ is continuous, this means that $l_m(\cdot)$ can be obtained as a continuous function by the methods of Section II. Equation (36) apparently requires that $l_m(\cdot)$ be m times differentiable. This is not really the case, since $l_i(\cdot)$ for $i < m$ can be regarded as a distribution, and the integral (33) will be well defined with $q^*(\cdot)$ as given in (34) because of the differentiability of $R^*(\cdot,\cdot)$. Put another way, the fact that the $l_i(\cdot)$ may be distributions has no more significance in (34) than does the fact that the $\delta^{(i)}(t-T)$ and $\delta^{(i)}(t)$ are distributions.

The final result of this section, Theorem 4 below, serves to identify those particular solutions of $s^* = R^* \circ q^*$ that are also solutions of $s = R \circ q$. It is clear from (34) that the essential task is to compute the constants a_i.

Theorem 4: Suppose all quantities are as defined previously, and let $\Delta_m(t)$ be defined by

$$\Delta_m'(t) = [\delta(t)\delta^{(1)}(t)\cdots(-1)^{m-1}\delta^{(m-1)}(t)]. \qquad (37)$$

With $q_0(\cdot)$ as that solution of $s^* = R^* \circ q^*$ with the arbitrary constants a_i of (34) all zero, and with the entries of $\boldsymbol{h}'(t)\boldsymbol{\Phi}(t,0)$ linearly independent over $[0,T]$, all solutions of (1) are given by

$$q(t) = q_0(t) + \boldsymbol{\alpha}'\Delta_m(t),$$

where $\boldsymbol{\alpha}$ is any constant m-vector satisfying

$$\boldsymbol{K}_m(0)\boldsymbol{\alpha} = \boldsymbol{K}_m(0)[\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)]^{\#}\mathscr{S}(0) - \boldsymbol{K}_m(0) \cdot [\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)]^{\#}\boldsymbol{K}_m'(0)\int_0^T \boldsymbol{\Phi}'(\tau,0)\boldsymbol{h}(\tau)q_0(\tau)\, d\tau. \qquad (38)$$

Moreover, one solution of this equation is

$$\boldsymbol{\alpha} = [\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)]^{\#}\left[\mathscr{S}(0) - \boldsymbol{K}_m'(0) \cdot \int_0^T \boldsymbol{\Phi}'(\tau,0)\boldsymbol{h}(\tau)q_0(\tau)\, d\tau\right] \qquad (39)$$

and this is the only solution if $\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)$ is nonsingular.

For the proof of this theorem, see Appendix III. Note that the requirement of the theorem that $\boldsymbol{h}'(t)\boldsymbol{\Phi}(t,0)$ have linearly independent entries over $[0,T]$ is not in practice restrictive. If this is not the case, then the procedures of linear system theory (see, e.g., [20]) guarantee the existence and computability of vector functions $\hat{\boldsymbol{h}}(\cdot)$ and $\hat{\boldsymbol{k}}(\cdot)$ and a transition matrix $\hat{\boldsymbol{\Phi}}(\cdot,\cdot)$ such that $\boldsymbol{h}'(t)\boldsymbol{\Phi}(t,\tau)\boldsymbol{k}(\tau) = \hat{\boldsymbol{h}}'(t)\hat{\boldsymbol{\Phi}}(t,\tau)\hat{\boldsymbol{k}}(\cdot)$, with the entries of $\hat{\boldsymbol{h}}'(t)\hat{\boldsymbol{\Phi}}(t,0)$ linearly independent over $[0,T]$; from the start one then works with $\hat{\boldsymbol{F}}$, $\hat{\boldsymbol{h}}$, and $\hat{\boldsymbol{k}}$ rather than $\boldsymbol{F}$, $\boldsymbol{h}$, and $\boldsymbol{k}$.

In summary, the solution algorithm is as follows: 1) differentiate $R(t,\tau)$ until m is such that $R_{mm}(t,\tau)$ contains a delta function; 2) form the covariance R^* and the function s^*; 3) Solve $s^{*(m)}(t) = \int_0^T R_{mm}^*(t,\tau)l_m(\tau)\, d\tau$ and form $q^*(\cdot)$ using $l_m(\cdot)$; 4) identify the arbitrary constants in $q^*(\cdot)$ to obtain $q(\cdot)$.

IV. Examples

As a first example, we shall compute for

$$R(t,\tau) = \delta(t-\tau) + \tau 1(t-\tau) + t1(\tau - t) \qquad (40)$$

the operator $R^{-1}(\cdot,\cdot)$ for the interval $[0,2]$. Notice that $R(\cdot,\cdot)$ is the covariance of the sum of white noise and an independent Wiener process. This example will illustrate the procedures of Section II.

Evidently, we can set

$$c = 1, \qquad \boldsymbol{h} = [1], \qquad \boldsymbol{F} = [0], \qquad \boldsymbol{k} = [t].$$

The Riccati equation for P is

$$\dot{P} = (P - t)^2, \qquad P(0) = 0,$$

whence

$$P = \frac{1 - e^{2t}}{1 + e^{2t}} + t.$$

Then

$$\boldsymbol{g}_1 = (\boldsymbol{k} - P\boldsymbol{h})c^{-1/2} = \left[\frac{e^{2t} - 1}{e^{2t} + 1}\right]$$

and

$$\boldsymbol{F}_1 = \boldsymbol{F} - \boldsymbol{g}_1 c^{-1/2}\boldsymbol{h}' = \left[\frac{1 - e^{2t}}{1 + e^{2t}}\right].$$

Next,

$$\boldsymbol{\Phi}_1(t,\tau) = \left[\exp \int_\tau^t \frac{1 - e^{2\sigma}}{1 + e^{2\sigma}}\, d\sigma\right] = \left[\frac{e + e^{-\tau}}{e^t + e^{-t}}\right].$$

Furthermore, the differential equation for Q is

$$-\dot{Q} = 2Q\frac{1 - e^{2t}}{1 + e^{2t}} + 1, \qquad Q(2) = 0,$$

from which

$$Q(t) = \cosh^2 t(\tanh 2 - \tanh t).$$

Since $\boldsymbol{h}_1 = Q\boldsymbol{g}_1 c^{-1/2} - \boldsymbol{h}c^{-1}$, we have in this case

$$h_1 = [\sinh t \cosh t (\tanh 2 - \tanh t] - 1]$$
$$= [\sinh t \cosh t \tanh 2 - \sinh^2 t - 1].$$

Together, $g_1(\cdot)$, $h_1(\cdot)$, and $\Phi_1(\cdot,\cdot)$ define $R^{-1}(\cdot,\cdot)$ [see (9)].

As a second example, we shall solve (1) for $T = 2$, for

$$R(t,\tau) = \tau 1(t - \tau) + t1(\tau - t) \tag{41}$$

and for

$$s(t) = \sin t. \tag{42}$$

With $R(\cdot,\cdot)$ as in (41) we identify $h = [1]$, $F = [0]$, $k = [t]$. The first step is to determine $R^*(\cdot,\cdot)$. In accordance with the procedures of Section III, we have

$$h_1 = [0], \qquad k_1 = [1], \qquad c_1 = 1.$$

Then $m = 1$, $H_m = h$, $K_m = k$, $H_m'(0)K_m(0) = 0$, and so $\alpha(t) = 0$, $R^*(t,\tau) = R(t,\tau)$, and $s^*(t) = s(t)$. Equation (35) becomes

$$s^{*(1)}(t) = \cos t = \int_0^2 R_{11}^*(t,\tau)l_1(\tau)\, d\tau$$
$$= \int_0^2 \delta(t - \tau)l_1(\tau)\, d\tau$$

whence $l_1(t) = \cos t$. It follows that $l_0(t) = -\sin t$ and so, from Lemma 3, all solutions of $s^* = R^* \circ q^*$ are given by

$$q^*(t) = -[-\sin t - \cos 2\delta(t - T) - a_1\delta(t)],$$

where a_1 is an arbitrary constant. Since $s = s^*$ and $R = R^*$, all solutions of (1) agree with this set of $q^*(\cdot)$.

V. Conclusions

We can best sum up the preceding material by commenting how our procedure for solving (1) differs from existing procedures. First, our procedure imposes a finite-dimensionality constraint on $R(\cdot,\cdot)$. Second, it permits nonstationary $R(\cdot,\cdot)$. Third, the method depends on solving differential equations, even linear differential equations. Fourth, the method offers an explicit and not *ad hoc* procedure for obtaining the endpoint singularities in the solution of (1), and even isolating the set of solutions of (1) when more than one exists. Fifth, our proofs do not rely on using the associativity of operators, except when it is known that the operators are associative.

Appendix I

Proof of Theorems 1 and 2

Part 1) of Theorem 1 is the main result of [2] and will not be rederived here. Next we note the form of $w^{-1}(t,\tau)$.

Lemma 4: With $w(t,\tau) = c^{1/2}(t)\delta(t - \tau) + h'(t)\Phi(t,\tau) g_1(\tau)1(t - \tau)$ the inverse kernel is

$$w^{-1}(t,\tau) = c^{-1/2}(t)\delta(t - \tau)$$
$$-c^{-1/2}(t)h'(t)\Phi_1(t,\tau)g_1(\tau)c^{-1/2}(\tau)1(t - \tau), \tag{43}$$

where $\Phi_1(\cdot,\cdot)$ is the transition matrix of $F - g_1c^{-1/2}h'$.

The calculation is straightforward and will be omitted. Using the definition (9) of $R^{-1}(\cdot,\cdot)$ and the above formula (43) for $w^{-1}(t,\tau)$, we easily evaluate

$$R^{-1}(t,\tau) = c^{-1}(t)\delta(t - \tau) - c^{-1}(t)h'(t)\Phi_1(t,\tau)g_1(\tau)c^{-1/2}(\tau)$$
$$\cdot 1(t - \tau) - c^{-1/2}(t)g_1'(t)\Phi_1'(\tau,t)h(\tau)c^{-1}$$
$$\cdot (\tau)1(\tau - t) + c^{-1/2}(t)g_1'(t) \int_{\max(t,\tau)}^T \Phi_1'(\lambda,t)$$
$$\cdot h(\lambda)c^{-1}(\lambda)h'(\lambda)\Phi_1(\lambda,\tau)\, d\lambda g_1(\tau)c^{-1/2}(\tau).$$

With the definition

$$Q(t) = \int_t^T \Phi_1'(\lambda,t)h(\lambda)c^{-1}(\lambda)h'(\lambda)\Phi_1(\lambda,t)\, d\lambda, \tag{44}$$

it is straightforward to check that Q satisfies (12), and that with $h_1 = Qg_1c^{-1/2} - hc^{-1}$,

$$R^{-1}(t,\tau) = c^{-1}(t)\delta(t - \tau) + h_1'(t)\Phi_1(t,\tau)g_1(\tau)c^{-1/2}$$
$$\cdot (\tau)1(t - \tau) + c^{-1/2}(t)g_1'(t)\Phi_1'(\tau,t)h_1(\tau)1(\tau - t).$$

The fact that $q(\cdot)$ is given by (13) follows from the associativity arguments given prior to the theorem statement. Standard results of integral equation theory, see [17, chs. 2, 3], combined with the positive definite nature of $R(\cdot,\cdot)$ guarantee uniqueness (and, in fact, existence) of $q(\cdot)$. The continuity of $q(\cdot)$ follows from the continuity of $c(\cdot)$ and of $R(t,\tau) - c(t)\delta(t - \tau)$, [17, chs. 2, 3], which in turn follows from the continuity of $h(\cdot)$, etc.

The last argument establishes the last statement of Theorem 1 and the first claims of Theorem 2. We turn now to the remaining claims of Theorem 2.

First, we shall establish that if (19) is solved with boundary condition $x(0) = y(T) = \mathbf{0}$, and if $q(\cdot)$ is defined by (20), then $q(\cdot)$ satisfies $s = R \circ q$. Accordingly, suppose we have, by some means, derived function $x(\cdot)$ and $y(\cdot)$ satisfying (19) and the boundary conditions. Using (20), we can eliminate s to write

$$\dot{x} = (F - c^{-1}kh')x - c^{-1}kk'y + kq + c^{-1}kh'x + c^{-1}kk'y$$
$$= Fx + kq.$$

Similarly,

$$\dot{y} = -F'y - hq.$$

From the boundary conditions $x(0) = \mathbf{0}$ and $y(T) = \mathbf{0}$, we then have

$$x(t) = \int_0^t \Phi(t,\tau)k(\tau)q(\tau)\, d\tau$$

$$y(t) = \int_t^T \Phi'(\tau,t)h(\tau)q(\tau)\, d\tau$$

and so, from (20)

$$\begin{aligned} s &= cq + h'x + k'y \\ &= cq + h'(t)\int_0^t \Phi(t,\tau)k(\tau)q(\tau)\,d\tau \\ &\quad + k'(t)\int_t^T \Phi'(\tau,t)h(\tau)q(\tau)\,d\tau \\ &= R \circ q \end{aligned}$$

as required.

We have just proved that if there is a solution to (19) with the appropriate boundary conditions, then it determines a function $q(\cdot)$ for which $s = R \circ q$. That there is always a solution can be established straightforwardly from the fact that $q(\cdot)$ is known to exist. The argument is as follows.

Define vector variables $\bar{x}(t)$ and $\bar{y}(t)$ by

$$\begin{aligned} \bar{x}(t) &= \int_0^t \Phi(t,\tau)k(\tau)q(\tau)\,d\tau \\ \bar{y}(t) &= \int_t^T \Phi'(\tau,t)h(\tau)q(\tau)\,d\tau. \end{aligned} \tag{45}$$

Clearly $\bar{x}(0) = \bar{y}(T) = 0$. We will show that $\bar{x}(\cdot)$ and $\bar{y}(\cdot)$ also satisfy (19). The equation $s = R \circ q$ and the formula for $R(\cdot,\cdot)$, viz., (3), establish that

$$s = cq + h'\bar{x} + k'\bar{y}. \tag{46}$$

From the definition (45) of $\bar{x}$, it is easy to check that

$$\dot{\bar{x}} = F\bar{x} + kq = (F - c^{-1}kh')\bar{x} - c^{-1}kk'\bar{y} + c^{-1}ks \tag{47}$$

with the second equality following by using (46). Similarly

$$\dot{\bar{y}} = c^{-1}hh'\bar{x} + (-F' + c^{-1}hk')\bar{y} - c^{-1}hs,$$

which with (47) establishes the claim.

It remains finally to verify that with $R^{-1}(\cdot,\cdot)$ defined by (10), $q = R^{-1} \circ s$ holds.

What we shall actually do is establish the existence of vector functions $x_1(\cdot)$ and $x_2(\cdot)$ satisfying (15), (17), and (18), repeated for convenience as

$$\dot{x}_1 = F_1x_1 + g_1c^{-1/2}s, \qquad x_1(0) = 0 \tag{15}$$

$$\dot{x}_2 = -F_1'x_2 - h_1s, \qquad x_2(T) = 0 \tag{17}$$

$$q = c^{-1}s + h_1'x_1 + c^{-1/2}g_1'x_2. \tag{18}$$

This is equivalent, by the remarks made in Section II, to the establishing of the result $q = R^{-1} \circ s$. The result we prove is summarized in the following lemma.

Lemma 5: Consider the two-point boundary problem (19) with $x(0) = y(T) = 0$; then the quantities $x_1(\cdot)$ and $x_2(\cdot)$ defined by

$$x_1 = x + Py \qquad x_2 = -Qx - (QP + I)y \tag{48}$$

satisfy (15) and (17), where P and Q are as defined in the statement of Theorem 1 in (8) and (12). Moreover, (18) holds.

Proof: From (48) we have

$$\begin{aligned} \dot{x}_1 &= \dot{x} + P\dot{y} + \dot{P}y \\ &= (F - c^{-1}kh')x - c^{-1}kk'y + c^{-1}ks + Pc^{-1}hh'x \\ &\quad + P(-F' + c^{-1}hk')y - Pc^{-1}hs + PF'y \\ &\quad + FPy + (Ph - k)c^{-1}(Ph - k)'y \\ &= F_1x + F_1Py + g_1c^{-1/2}s \\ &= F_1x_1 + g_1c^{-1/2}s. \end{aligned}$$

The second equality follows by using the expressions for $\dot{x}$, $\dot{y}$, and $\dot{P}$, the third by cancellation and use of the definitions of F_1 and g_1, and the final equality by use of the definition (48) of x_1.

Again from (48), we have

$$\begin{aligned} \dot{x}_2 &= -Q\dot{x} - (QP + I)\dot{y} - \dot{Q}x - \dot{Q}Py - Q\dot{P}y \\ &= -Q(F - c^{-1}kh')x + Qc^{-1}kk'y - Qc^{-1}ks \\ &\quad - (QP + I)c^{-1}hh'x - (QP + I)(-F' + c^{-1}hk')y \\ &\quad + (QP + I)c^{-1}hs + F_1'Qx + QF_1x + hc^{-1}h'x \\ &\quad + F_1'QPy + QF_1Py + hc^{-1}h'Py - QPF'y \\ &\quad - QFPy - Q(Ph - k)c^{-1}(Ph - k)'y \\ &= F_1'Qx + F_1'(I + QP)y - h_1s \\ &= -F_1'x_2 - h_1s. \end{aligned}$$

The second equality follows by using the expressions for $\dot{x}$, $\dot{y}$, $\dot{P}$, and $\dot{Q}$, the third by cancellation and use of the definitions of F_1, g_1, and h_1, and the final equality by use of the definition (48) of x_2.

Next, notice from (48) that $x_1(0) = 0$ because $x(0) = 0$ and $P(0) = 0$; also, $x_2(T) = 0$ because $Q(T) = 0$ and $y(T) = 0$. Thus (15) and (17) are validated. Finally, we must establish (18). From (48), it follows that

$$x = (I + PQ)x_1 + Px_2 \qquad y = -Qx_1 - x_2.$$

Therefore, the known result (20), viz.,

$$q = c^{-1}s - c^{-1}h'x - c^{-1}k'y$$

leads to

$$\begin{aligned} q &= c^{-1}s - c^{-1}[h'(I + PQ) - k'Q]x_1 - c^{-1}(h'P - k')x_2 \\ &= c^{-1}s + h_1'x_1 + c^{-1/2}g_1'x_2 \end{aligned} \tag{18}$$

on using the definitions for g_1 and h_1. This proves the lemma.

Appendix II

Proof of Lemma 3

By the assumptions made on $R(\cdot,\cdot)$, $R_{mm}^*(\cdot,\cdot)$ is positive definite and is the sum of a continuous function of t and τ and a term $c_m(t)\delta(t - \tau)$ with $c_m(\cdot)$ continuous. These facts and the continuity of $s^{*(m)}(t)$ guarantee that $l_m(\cdot)$ exists and is continuous.

Now integrate the right-hand side of (35) by parts. There results

$$s^{*(m)}(t) = R_{m,m-1}^*(t,\tau)l_m(\tau)\Big|_{\tau=0}^{\tau=T} - \int_0^T R_{m,m-1}^*(t,\tau)l_{m-1}(\tau)\,d\tau$$
$$= \int_0^T R_{m,m-1}^*(t,\tau)[-l_{m-1}(\tau) + l_m(T)\delta(\tau - T) - l_m(0)\delta(\tau)]\,d\tau,$$

where the singularities of the delta functions are interpreted as lying strictly within the integration interval $[0,T]$. Now by (28) $R_{m,m-1}^*(t,0)$ is zero. Consequently, the integral is independent of the value of $l_m(0)$, or

$$s^{*(m)}(t) = \int_0^T R_{m,m-1}^*(t,\tau)[-l_{m-1}(\tau) + l_m(T)\delta(\tau - T) + a_m\delta(\tau)], \quad (49)$$

where a_m is an arbitrary constant. What is most important is that the expression

$$r(\tau) = -l_{m-1}(\tau) + l_m(T)\delta(\tau - T) + a_m\delta(\tau),$$

where a_m is arbitrary is the most general expression such that

$$s^{*(m)}(t) = \int_0^T R_{m,m-1}^*(t,\tau)r(\tau)\,d\tau.$$

This is easily seen through integrating by parts to rewrite $\int_0^T R_{m,m-1}^*(t,\tau)r(\tau)\,d\tau$ as $\int_0^T R_{mm}^*(t,\tau)l_m(\tau)\,d\tau$.

Return now to (49). Further integration by parts yields

$$s^{*(m)}(t) = \int_0^T R_{m,m-2}^*(t,\tau)[l_{m-1}(\tau) - l_{m-1}(T)\delta(\tau - T) - l_m(T)\delta^{(1)}(\tau - T) - a_{m-1}\delta(\tau) - a_m\delta^{(1)}(\tau)]\,d\tau,$$

where a_{m-1} is an arbitrary constant. More generally, we find

$$s^{*(m)}(t) = (-1)^m \int_0^T R_{m,0}^*(t,\tau)\Big[l_0(\tau) - l_1(T)\delta(\tau - T) - l_2(T)\delta^{(1)}(\tau - T) - \cdots - l_m(T)\delta^{(m-1)}\cdot(\tau - T) - \sum_1^m a_i\delta^{(i-1)}(\tau)\Big]\,d\tau. \quad (50)$$

Moreover, the expression

$$q^*(\tau) = (-1)^m\Big[l_0(\tau) - l_1(T)\delta(\tau - T) - \cdots - \sum_1^m a_i\delta^{(i-1)}(\tau)\Big]$$

is the most general expression such that

$$s^{*(m)}(t) = \int_0^T R_{m,0}^*(t,\tau)q^*(\tau)\,d\tau. \quad (51)$$

It merely remains to show that (33) and (51) imply one another. This is straightforward, for (51) follows from (33) by differentiation m times with respect to t, while (33) follows (51) by integration m times with respect to t, using the boundary condition $s^{*(i)}(0) = 0$, $0 \le i \le m - 1$ of (31) and $R_{i,0}^*(0,\tau) = 0$ for $0 \le i \le m - 1$ and all τ of (28).

Notice that in the course of proving this lemma, the fact that $R_{ij}^*(0,\tau) = R_{ij}^*(t,0) = 0$ for various i and j has been repeatedly used; the reason why $s = R \circ q$ is not solved directly but rather $s^* = R^* \circ q^*$ is solved is that $R_{ij}(0,\tau)$ and $R_{ij}(t,0)$ do not generally have the helpful property of being zero.

Appendix III

Proof of Theorem 4

Define $s_I(\cdot) = s(t) - s^*(t)$ and $\tilde{R}(t,\tau) = R(t,\tau) - R^*(t,\tau)$. With α yet to be determined, we know from the fact that $s = R \circ q$, and the general structure of $q(\cdot)$ as implied by Lemma 3, that

$$s^*(t) + s_I(t) = \int_0^T [R^*(t,\tau) + \tilde{R}(t,\tau)][q_0(\tau) + \alpha'\Delta_m(\tau)]\,d\tau.$$

By Lemma 2, this means that

$$s_I(t) = \int_0^T \tilde{R}(t,\tau)[q_0(\tau) + \alpha'\Delta_m(\tau)]\,d\tau.$$

Both $s_I(\cdot)$ and $R(\cdot,\cdot)$ are available, from (30) and Theorem 3, respectively. We obtain

$$\begin{aligned}
&\boldsymbol{h}'(t)\boldsymbol{\Phi}(t,0)\boldsymbol{K}_m(0)[\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)]^{\#}\mathscr{S}(0)\\
&= \int_0^T \boldsymbol{h}'(t)\boldsymbol{\Phi}(t,0)\boldsymbol{K}_m(0)[\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)]^{\#}\cdot \boldsymbol{K}_m'(0)\boldsymbol{\Phi}'(\tau,0)\boldsymbol{h}(\tau)q_0(\tau)\,d\tau\\
&\quad + \int_0^T \boldsymbol{h}'(t)\boldsymbol{\Phi}(t,0)\boldsymbol{K}_m(0)[\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)]^{\#}\cdot \boldsymbol{K}_m'(0)\boldsymbol{\Phi}'(\tau,0)\boldsymbol{h}(\tau)\boldsymbol{\Delta}_m'(\tau)\,d\tau\boldsymbol{\alpha}\\
&= \boldsymbol{h}'(t)\boldsymbol{\Phi}(t,0)\boldsymbol{K}_m(0)[\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)]^{\#}\boldsymbol{K}_m'(0)\cdot\int_0^T \boldsymbol{\Phi}'(\tau,0)\boldsymbol{h}(\tau)q_0(\tau)\,d\tau\\
&\quad + \boldsymbol{h}'(t)\boldsymbol{\Phi}(t,0)\boldsymbol{K}_m(0)[\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)]^{\#}\boldsymbol{K}_m'(0)\boldsymbol{H}_m(0)\boldsymbol{\alpha}.
\end{aligned}$$

Using the fact that the entries of $\boldsymbol{h}'(t)\boldsymbol{\Phi}(t,0)$ are linearly independent,

$$\boldsymbol{K}_m(0)[\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)]^{\#}\boldsymbol{K}_m'(0)\boldsymbol{H}_m(0)\boldsymbol{\alpha} = \boldsymbol{K}_m(0)[\boldsymbol{H}_m'(0)\boldsymbol{K}_m(0)]^{\#}\Big[\mathscr{S}(0) - \boldsymbol{K}_m'(0)\int_0^T \boldsymbol{\Phi}'(\tau,0)\boldsymbol{h}(\tau)q_0(\tau)\,d\tau\Big]. \quad (52)$$

The fact that (39) constitutes one solution of this equation is obtained on using a standard property of the pseudoinverse; if $H_m'(0)K_m(0)$ is nonsingular, premultiplication of (52) by $H_m'(0)$ establishes that (39) is the only solution.

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