SMOOTHING AS AN IMPROVEMENT ON FILTERING: A UNIVERSAL BOUND

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Comparison is made between the mean-square errors P_f and P_s associated with the linear least-squares filtered and smoothed estimates of a stationary process of spectral density $S(\omega)$ in white noise of spectral density N_0. A universal curve is obtained which relates the minimum possible value of P_s/P_f to $\max_\omega \{S(\omega) N_0\}$. The curve sets a bound on the maximum improvement over filtering which smoothing will offer, in terms of the maximum signal/noise ratio.

Let $s(.)$ be a stationary scalar random process with power spectrum $S(\omega)$, and let noisy measurements $z(t) = s(t)+n(t)$ of $s(t)$ be available for $-\infty < t < \infty$. The noise $n(.)$ is stationary and white, with covariance $N_0\delta(t-\tau)$ and power spectrum N_0. The noise process $n(.)$ is assumed to be independent of the process $s(.)$.

It is well known how to obtain linear least-squares filtered and smoothed estimates of $s(.)$ from the measurement process, and this aspect of the estimation problem will not concern us. Instead, we study the errors

$$P_f = E\{s(t)-\hat{s}_f(t)\}^2 \quad P_s = E\{s(t)-\hat{s}_s(t)\}^2 \quad . \quad . \quad (1)$$

where $\hat{s}_f(t)$ and $\hat{s}_s(t)$ are the filtered and smoothed estimates, respectively.

With reasonable conditions on $S(\omega)$, it is found (see Reference 1, pp. 496 and 501) that

$$P_f = \frac{N_0}{2\pi}\int_{-\infty}^{+\infty} \ln\left\{1+\frac{S(\omega)}{N_0}\right\} d\omega \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$P_s = \frac{N_0}{2\pi}\int_{-\infty}^{+\infty} \frac{S(\omega)/N_0}{1+S(\omega)/N_0} d\omega . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Our purpose is to prove the following result.

Smoothing–filtering comparison: Let $\omega = \omega_0$ be the frequency for which $x(\omega) = S(\omega)/N_0$ is maximum. Then

$$\frac{P_s}{P_f} \geqslant \frac{x(\omega_0)}{\{1+x(\omega_0)\}\ln\{1+x(\omega_0)\}} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This relationship is further interpreted below. Its proof depends on the following facts:

(*a*) If

$$f(x) = \frac{x}{(1+x)\ln(1+x)}$$

$f'(x) < 0$ for all $x > 0$. This is straightforward to check.

(*b*)

$$\frac{x(\omega)}{1+x(\omega)} \geqslant \gamma \ln\{1+x(\omega)\}$$

where

$$\gamma = \frac{x(\omega_0)}{\{1+x(\omega_0)\}\ln\{1+x(\omega_0)\}}$$

To see this, recall that ω_0 maximises $(x\omega) = S(\omega)/N_0$, and then use the fact that $f\{x(\omega)\}$ will be minimised when $\omega = \omega_0$.

Eqn. 4 follows from the inequality in (*b*) by integrating with respect to ω.

An advantage of smoothing over filtering is that it leads to a lower error variance. Eqn. 4, by putting a lower limit on P_s/P_f, puts an upper bound on the amount of improvement which it is possible to obtain from smoothing. The bound will not necessarily be attained. Computation of the upper bound is straightforward, and is, in terms of simply $\max_\omega \{S(\omega)/N_0\}$, the maximum signal/noise ratio at any frequency. By plotting the function $f(x)$ defined in (*a*) against x, and relabelling the axes, we obtain the universal curve of Fig. 1; this is a graphical representation of the result of eqn. 4, and shows that, the greater the gains which come from smoothing, the higher is $\max_\omega \{S(\omega)/N_0\}$. At a lower signal/noise ratio, the gains are very small.

If $s(t)$ is a vector process, we can argue as follows. First, suppose that $N_0 = n_0 I$. Then[2]

$$P_f = E[\{s(t)-\hat{s}_f(t)\}' \{s(t)-\hat{s}_f(t)\}]$$

$$= \frac{n_0}{2\pi}\int_{-\infty}^{+\infty} \ln\left[\det\left\{\frac{S(\omega)}{n_0}+I\right\}\right] d\omega \quad (5)$$

with

$$P_s = E[\{s(t)-\hat{s}_s(t)\}'\{s(t)-\hat{s}_s(t)\}]$$

$$= \frac{n_0}{2\pi}\int_{-\infty}^{+\infty} \text{trace}\left[\left\{\frac{S(\omega)}{n_0}+I\right\}^{-1}\frac{S(\omega)}{n_0}\right] d\omega \quad (6)$$

Now let $\{\lambda_i(\omega)\}$ denote the eigenvalues of $S(\omega)$. Then

$$P_f = \frac{n_0}{2\pi}\sum_i \int_{-\infty}^{+\infty} \ln\{1+\lambda_i(\omega)/n_0\} d\omega \quad . \quad . \quad . \quad . \quad (7)$$

and

$$P_s = \frac{n_0}{2\pi}\sum_i \int_{-\infty}^{+\infty} \frac{\lambda_i(\omega)/n_0}{1+\lambda_i(\omega)/n_0} d\omega \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Using arguments as in the scalar case, it follows that

$$\frac{P_s}{P_f} \geqslant \frac{\lambda_I(\omega_0)/n_0}{\{1+\lambda_I(\omega_0)/n_0\}\ln\{1+\lambda_I(\omega_0)/n_0\}} \quad . \quad . \quad . \quad (9)$$

where $\lambda_I(\omega_0) = \max_{i,\omega} \lambda_i(\omega)$. (Note that the $\lambda_i(\omega)$ are all real.) In this case, the curve in Fig. 1 still applies, except that the horizontal axis now is associated with $\lambda_I(\omega_0)/n_0$. If N_0 is a general positive-definite matrix, one can use modified error formulas, the modification involving insertion of a weighting matrix:[2]

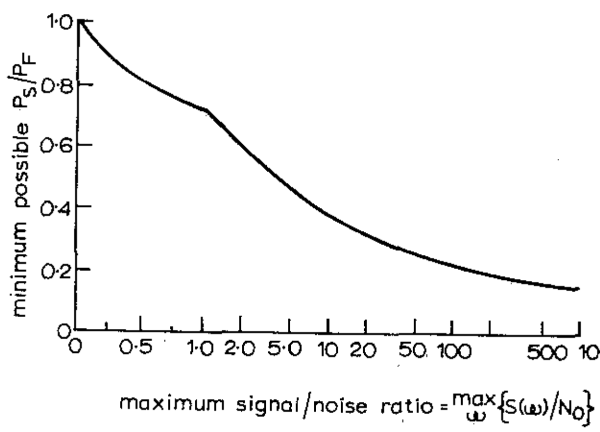

Fig. 1 *Smoothing improvement against maximum signal/noise ratio*

Note the form of the x axis scales: this scale is linear up to 1·0 and then logarithmic

$$P_f = E[\{s(t)-\hat{s}_f(t)\}' N_0^{-1}\{s(t)-\hat{s}_f(t)\}]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} [\ln\det\{N_0^{-1}S(\omega)+I\}] d\omega \quad (10)$$

$$P_s = E[\{s(t)-\hat{s}_s(t)\}' N_0^{-1}\{s(t)-\hat{s}_s(t)\}]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \text{trace}\,[\{N_0^{-1}S(\omega)+I\}^{-1}N_0^{-1}S(\omega)] d\omega \quad (11)$$

and one has

$$\frac{P_s}{P_f} \geqslant \frac{\mu_I(\omega_0)}{\{1+\mu_I(\omega_0)\}\ln\{1+\mu_I(\omega_0)\}} \quad . \quad . \quad . \quad . \quad (12)$$

where

$$\mu_l(\omega_0) = \max_{t,\,\omega} \lambda\{N_0^{-1} S(\omega)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Again, therefore, the curve of Fig. 1 applies. This time, re-interpretation of the variables associated with both axes is required.

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