FORMULAS FOR MULTIDIMENSIONAL OPTIMAL LINEAR FILTERING IN WHITE NOISE

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The multidimensional linear-filtering problem for a signal in white noise is considered, and formulas are given for optimum causal and noncausal filters and the associated errors.

The purpose of this letter is to present a number of error-performance formulas associated with optimal multivariable linear filtering and smoothing in white noise. There is no restriction made as to rationality of the signal spectrum. Filtering formulas for the scalar case were established in References 1 and 2 for rational spectra, and in References 3 and 4 for arbitrary spectra. The argument of Reference 3 is based on solving a discrete-time problem and obtaining continuous-time results by letting the sampling interval tend to zero. The argument here is based on extending that of Reference 4; in so doing, we avoid an apparent technical error in Reference 4, where the series expansion

$$\ln(1-x) = -(x+\tfrac{1}{2}x^2+\ldots)$$

was used without the condition $|x| < 1$ necessarily being satisfied.

Preliminaries: Consider an m-dimensional signal process $s(t)$ with power-spectral-density matrix $\Phi_s(s)$, assumed to approach zero as $s \to \infty$ in such a way that $\lim_{s\to\infty} s^2 \Phi_s(s)$ is finite. Noisy measurements $y(t) = s(t)+n(t)$ are available of the signal process, where $n(t)$ is white noise of covariance $N_0 \delta(t-\tau)$. Define $\Phi_y(s) = \Phi_s(s)+N_0$, so that $\Phi_y(s)$ is the power-spectral-density matrix of the measurement process. Let $H(s)$ be the transfer-function matrix of an arbitrary filter with the z process as the input, and $H_f(s)$ and $H_n(s)$ be the transfer-function matrices of the optimal linear causal filter and the optimal linear noncausal filter or optimal smoother, respectively. The optimal causal filter, besides being causal, has as its output an m vector $\hat{s}_f(t)$ so that $E[\{s(t)-\hat{s}_f(t)\}' A\{s(t)-\hat{s}_f(t)\}]$ is minimised for any positive-definite A. The optimal noncausal filter is, of course, noncausal, and has as its output an m vector $\hat{s}_n(t)$ so that $E[\{s(t)-\hat{s}_n(t)\}' A\{s(t)-\hat{s}_n(t)\}]$ is minimised for any positive-definite A.

References 5 and 6 give the formula for $H_f(s)$:

$$H_f'(s) = \Psi^{-1}(s)[\{\Psi'(-s)\}^{-1}\Phi_s(s)]_+ \qquad (1)$$

In this formula, the prime denotes matrix transposition, and $\Psi(s)$ is a matrix uniquely determined as satisfying

$$\Psi'(-s)\Psi(s) = \Phi_y(s) \qquad (2)$$

with $\Psi(s)$ and $\Psi^{-1}(s)$ analytic in $\mathrm{Re}(s) \geq 0$, and $\Psi(\infty) = N_0^{\frac{1}{2}}$, the positive-definite square root of N_0. (This last constraint can be relaxed if desired.) Finally, $[]_+$ denotes the operation of taking the inverse Laplace transform, multiplying by the unit step function $1(t)$, and taking the Laplace transform. Equivalently, it amounts to expressing the contents of [] as the sum of two parts, one analytic in $\mathrm{Re}(s) \geq 0$, the other in $\mathrm{Re}(s) \leq 0$, and then discarding this latter part. Observe that $H_f(s)$ is analytic in $\mathrm{Re}(s) \geq 0$, and $\lim_{s\to\infty} sH_f(s)$ is finite.

The optimum noncausal filter is given by

$$H_n'(s) = \Phi_s(s)\{\Phi_s(s)+N_0\}^{-1} \qquad (3)$$

This formula may be simply derived by extending arguments for the scalar case, e.g. as in Reference 2.

Alternative specifications of the optimal filter: Starting from eqn. 1, we have

$$\begin{aligned} H_f'(s) &= \Psi^{-1}(s)[\{\Psi'(-s)\}^{-1}\{\Phi_s(s)+N_0\}-\{\Psi'(-s)\}^{-1}N_0]_+ \\ &= \Psi^{-1}(s)\{\Psi(s)\}_+ - \Psi^{-1}(s)[\{\Psi'(-s)\}^{-1}N_0]_+ \\ &= I-\Psi^{-1}(s)N_0^{\frac{1}{2}} \qquad (4) \end{aligned}$$

The second equality follows by using eqn. 2, and the third by using the analyticity of $\{\Psi(s)\}^{-1}$ in $\mathrm{Re}(s) > 0$ and the fact that $\Psi(\infty) = N_0^{\frac{1}{2}}$.

Also of interest is the realisation of $H_f(s)$ and $H_n(s)$ by a closed-loop system with gains $G_f(s)$ and $G_n(s)$ in the forward part of the loop, and with unity negative feedback. The relationship $H(s) = G(s)\{I+G(s)\}^{-1}$ yields

$$\left.\begin{aligned} G_f'(s) &= N_0^{-\frac{1}{2}}\Psi(s)-I \\ G_n'(s) &= \Phi_s(s)N_0^{-1} \end{aligned}\right\} \qquad (5)$$

Observe that $G_f(s)$ is analytic in $\mathrm{Re}(s) \geq 0$, and $\lim_{s\to\infty} sG_f(s)$ is finite.

Error performance—causal filtering: With an arbitrary filter $H(s)$ yielding output $\hat{s}(t)$, it is straightforward to show that

$$\begin{aligned} &E[\{s(t)-\hat{s}(t)\}\{s(t)-\hat{s}(t)\}'] \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} [\{I-H(-j\omega)\}\Phi_s(j\omega)\{I-H'(j\omega)\} \\ &\qquad +H(-j\omega)N_0H'(j\omega)]\,d\omega \qquad (6) \end{aligned}$$

Setting $H(j\omega) = H_f(j\omega)$, calling the left-hand side R_f and after minor manipulation using eqn. 2, we obtain

$$R_f = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\{N_0H_f'(j\omega)+H_f(-j\omega)N_0\}\,d\omega \qquad (7)$$

$$= \tfrac{1}{2}\{N_0h_f'(0+)+h_f(0+)N_0\} \qquad (8)$$

where $h_f(t)$ is the inverse Laplace transform of $H(s)$.

Now make the assumption that $N_0 = n_0 I$, where n_0 is a positive scalar. This does not cause significant loss of generality. Let C be the closed contour comprising the imaginary axis and an infinitely large semicircle extending into $\mathrm{Re}(s) > 0$. Let Γ denote the semicircular part of C. Since $H_f(s)$ is analytic in $\mathrm{Re}(s) > 0$, we have

$$\int_{-\infty}^{+\infty} H_f(j\omega)\,d\omega = -\int_\Gamma H_f(s)\,d\gamma$$

Also, since $I-H(s) = N_0^{\frac{1}{2}}\{\Psi'(s)\}^{-1}$ is analytic together with its inverse in $\mathrm{Re}(s) \geq 0$, we have

$$\begin{aligned} \int_{-\infty}^{+\infty} \ln\{I-H_f(j\omega)\}\,d\omega &= -\int_\Gamma \ln\{I-H_f(s)\}\,d\gamma \\ &= +\int_\Gamma H_f(s)\,d\gamma \end{aligned}$$

The last equality follows from the fact that $H_f(s) \to 0$ as $s \to \infty$, and that $\ln(I-A) = -(A+\frac{1}{2}A^2+\ldots)$ whenever $|\lambda_i(A)| < 1$ (see Reference 7, p. 113). Hence

$$\int_{-\infty}^{\infty} H_f(j\omega)\,d\omega = -\int_{-\infty}^{+\infty}\ln\{1-H_f(j\omega)\}\,d\omega \qquad (9)$$

Observe also, using eqns. 2 and 4 and $N_0 = n_0 I$, that

$$\{I-H_f(-j\omega)\}\left\{\frac{\Phi_s(j\omega)}{n_0}+I\right\}\{I-H_f'(j\omega)\} = I$$

As a consequence,

$$\begin{aligned} &\ln\left[\det\left\{\frac{\Phi_s(j\omega)}{n_0}+I\right\}\right] \\ &= -\ln[\det\{I-H_f(-j\omega)\}]-\ln[\det\{I-H_f'(j\omega)\}] \\ &= -\mathrm{trace}\,[\ln\{I-H_f(-j\omega)\}+\ln\{I-H_f'(j\omega)\}] \end{aligned}$$

using the relationship trace ($\ln A$) = $\ln(\det A)$ (see Reference 8, p. 99). It follows that

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln\left[\det\left\{\frac{\Phi_s(j\omega)}{n_0}+I\right\}\right]d\omega \\ &= -\frac{1}{2\pi}\,\mathrm{trace}\left[\int_{-\infty}^{+\infty}\ln\{I-H_f(-j\omega)\}\,d\omega\right. \end{aligned}$$

$$+ \int_{-\infty}^{+\infty} \ln\{I - H_f'(j\omega)\}\, d\omega \Bigg]$$

$$= \frac{1}{2\pi} \,\mathrm{trace} \left[\int_{-\infty}^{+\infty} \{H_f(-j\omega) + H_f'(j\omega)\}\, d\omega \right]$$

from eqn. 9, and

$$= \frac{1}{n_0} \,\mathrm{trace}\, R_f$$

from eqn. 7. Hence,

$E[\{s(t) - \hat{s}_f(t)\}' \{s(t) - \hat{s}_f(t)\}]$

$$= \frac{n_0}{2\pi} \int_{-\infty}^{\infty} \ln \left[\det \left\{ \frac{\Phi_s(j\omega)}{n_0} + I \right\} \right] d\omega \quad (10)$$

When N_0 is a general positive-definite matrix, one can instead derive the formula

$E[\{s(t) - \hat{s}_f(t)\}' N_0^{-1} \{s(t) - \hat{s}_f(t)\}]$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \ln[\det\{N_0^{-1} \Phi_s(j\omega) + I\}]\, d\omega \quad (11)$$

The manipulations are very similar to those leading to eqn. 10.

The error in noncausal filtering follows from the basic equation (eqn. 6) by taking $H(s) = H_n(s)$. The result is

$$R_s = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \{\Phi_s(j\omega) + N_0\}^{-1}$$

$$\times \{N_0 \Phi_s(j\omega) N_0 + \Phi_s(j\omega) N_0 \Phi_s(j\omega)\} \{\Phi_s(j\omega) + N_0\}^{-1}\, d\omega \quad (12)$$

When $N_0 = n_0 I$ for a scalar constant n_0, we have

$$R_s = \frac{n_0}{2\pi} \int_{-\infty}^{+\infty} \left\{ \frac{\Phi_s(j\omega)}{n_0} + I \right\}^{-1} \frac{\Phi_s(j\omega)}{n_0}\, d\omega \quad . \quad . \quad (13)$$

and

$$\mathrm{trace}\, R_s = E[\{s(t) - \hat{s}_n(t)\}' \{s(t) - \hat{s}_n(t)\}]$$

$$= \frac{n_0}{2\pi} \int_{-\infty}^{+\infty} \mathrm{trace} \left[\left\{ \frac{\Phi_s(j\omega)}{n_0} + I \right\}^{-1} \frac{\Phi_s(j\omega)}{n_0} \right] d\omega \quad (14)$$

When N_0 is a general positive-definite matrix, one has

$E[\{s(t) - \hat{s}_n(t)\}' N_0^{-1} \{s(t) - \hat{s}_n(t)\}]$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{trace}\, [\{N_0^{-1} \Phi_s(j\omega) + I\}^{-1} N_0^{-1} \Phi_s(j\omega)]\, d\omega \quad (15)$$

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B. D. O. ANDERSON — *29th July 1971*
J. B. MOORE

Department of Electrical Engineering
University of Newcastle
New South Wales 2308, Australia

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