Nonlinear Regulator Theory and an Inverse Optimal Control Problem

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***Abstract*—Nonlinear optimal regulators are discussed, and some useful properties are isolated. An inverse problem of nonlinear regulator design is posed and solved.**

I. Introduction

A STRONG motivation for designing optimal systems is that such systems tend automatically to have properties that are desirable from the viewpoint of classical criteria: stability, reduced sensitivity to parameter variations, and many other such properties. In particular, it is well-known [1] that if the linear time-invariant single-input system

$$\dot{x} = Fx + gu \tag{1}$$

is optimized with respect to the performance index

$$\int_{t_0}^{\infty} (u^2 + x'Qx)\, dt \tag{2}$$

then, under mild restrictions on Q, the resulting feedback system is a "good" system by most classical criteria. An equally significant result is as follows [2]. Given a feedback control law $u = -k'x$ for the system (1), which is known to yield an asymptotically stable closed-loop system, then a necessary and sufficient condition for this control law to be optimal with respect to a performance index of the type (2) is that

$$|1 + k'(j\omega I - F)^{-1}g| \geq 1 \tag{3}$$

hold for all real ω. This is the so-called "return difference condition," well known in classical control theory as a desirable condition on a linear feedback control system. The purpose of the present paper is to present corresponding results for a class of nonlinear systems. We restrict ourselves to time-invariant systems, most frequently with time-invariant control laws. With greater notational complexity and with many more ad hoc assumptions, principally in respect to existence of double limits, time-varying systems may be considered, and obvious generalizations of the time-invariant results will be obtained.

We do not, however, restrict consideration to single-input systems. Other results on nonlinear inverse optimal control problems can be found in [3] and [4]. In [3], more general performance indices are considered than those of this paper (which are quadratic in the control, but not necessarily the state), but on the other hand, the main results are restricted to systems possessing an optimal performance that is quadratic in the state. In [4], some structural properties are determined for the optimum control law of a problem of the class considered in this paper.

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II. Solution of the Direct Problem

Consider the system

$$\dot{x}(t) = f[x(t)] + Gu(t) \tag{4}$$

where x and u are, respectively, the system state and control and take as a performance index

$$V[x(t_0), u(\cdot), t_0, T] = \int_{t_0}^{T} \{u'(t)u(t) + m[x(t)]\}\, dt. \tag{5}$$

(The superscript prime denotes matrix transposition.[1]) It is assumed that $m(x)$ is nonnegative for all x, that $m(0) = 0$, and that $f(0) = 0$. In the sequel we shall also require at times the following assumptions.[2]

Assumption 1: The system (4) is *completely controllable;* that is, for any initial state $x(t_0)$ and any other state x_1, there exists a square-integrable control $u(\cdot)$ and a time t_1 such that $x(t_1) = x_1$.

Assumption 2: The functions $f(\cdot)$ and $m(\cdot)$ have sufficient properties including smoothness such that an optimal control exists, and that the optimal performance index satisfies the Hamilton–Jacobi equation.

Assumption 3: The free system

$$\dot{x}(t) = f[x(t)] \qquad y(t) = m[x(t)]$$

is completely observable, in the sense that $m[x(t)] = 0$, for all $t \in (t_1, t_2)$ with t_1 and t_2 arbitrary, implies $x(t) \equiv 0$. Also $m(\cdot)$ is such that the optimal performance index approaches infinity as $\|x(t_0)\|$ approaches infinity.

Under Assumption 2, the optimal feedback control law is seen from the standard Hamilton–Jacobi theory to be

[1] The matrix G, though assumed constant here, may, with very minor smoothness assumptions, be assumed dependent on x. Further, the term $u'(t)u(t)$ in (5) can be weighted by a positive definite smooth function of x with minor adjustment to later arguments.

[2] Although these assumptions are written in global terms, it would be reasonable to restrict them to local regions in which the state was known to lie on *a priori* grounds.

$$u^* = -\tfrac{1}{2}G'\nabla\phi(x,t,T) \tag{6}$$

where $\phi(x,t,T)$ is the solution of

$$\frac{\partial\phi}{\partial t} + \nabla'\phi f(x) - \tfrac{1}{4}\nabla'\phi GG'\nabla\phi + m(x) = 0 \tag{7}$$

with boundary conditions $\phi(0,t,T) = 0$ for all $t \in [t_0, T]$ and $\phi(x,T,T) = 0$ for all x. Moreover,

$$\phi[x(t_0),t_0,T] = \inf_{u(\cdot)} V[x(t_0), u(\cdot),t_0,T].$$

Of special interest is the case $T = \infty$ in (5).

Theorem 1: With Assumptions 1 and 2 holding, the optimal performance index when $T = \infty$ is given by

$$\bar{\phi}(x) = \lim_{T\to\infty} \phi(x,t,T) \tag{8}$$

and the optimal control is given by (6), with $\phi(x,t,T)$ replaced by $\bar{\phi}(x)$. Moreover, $\bar{\phi}(x)$ satisfies (7).

Proof: Following arguments as for the linear-quadratic problem, due to Kalman, see [1, p. 33] and [5], monotonicity of $\phi(x,t,T)$ as a function of T with fixed x and t can be shown, and the boundedness for all T follows from Assumption 1. Consequently the limit on the right side of (8) exists, and is easily checked to be independent of t. That $\bar{\phi}(x)$ is in fact the optimal performance index is easily established, as in [1, pp. 34, 35] and [5]. □

The corresponding results for nonzero terminal weighting are considerably more difficult to obtain, although it is clear that one other situation for which Theorem 1 remains true is if a terminal weighting $n[x(T)]$ added on to the right side of (5) is a steady-state solution of (7). [The boundary condition $\phi(x,T,T) = 0$ is replaced by $\phi(x,T,T) = n[x(T)] = \bar{\phi}(x)$.] However, the presence of $n(x)$ is irrelevant if the optimal solution in Theorem 1 leads to an asymptotically stable system. Sufficient conditions for this to be true are given in the following theorem.

Theorem 2: Under the conditions of Theorem 1, together with Assumption 3, the optimal closed-loop system is globally asymptotically stable.

Proof: The proof precisely parallels that used for the linear-quadratic problem [1, p. 41] and [5]. For from (6) and (7), the rate of change of $\bar{\phi}(x(t))$ along any optimal trajectory is given by

$$\frac{d\bar{\phi}(x(t))}{dt} = \nabla'\bar{\phi}(x)[f(x) - \tfrac{1}{2}GG'\nabla\bar{\phi}(x)]$$
$$= -m(x) - \tfrac{1}{4}\nabla'\bar{\phi}(x)GG'\nabla\bar{\phi}(x).$$

Assumption 3 then implies that this derivative can never be identically zero, so that $\bar{\phi}(x)$ is a Lyapunov function for the optimal system [6, p. 66]. □

Note, however, that verification of Assumptions 1–3 may be far from trivial; a considerable amount of research is still being carried out on the observability and controllability of nonlinear systems [7]–[9]. In practice, it is probably easier in most cases to check for controllability by simple inspection (perhaps, in a given example, from physical reasoning), while the observability requirement may be circumvented, if necessary, by making $m(x)$ positive definite.

III. The Return Difference Condition

In the remainder of this paper, we shall be concerned with feedback control laws for the system (4) with a property to be described as the return difference condition (RDC). The reason for this name is that, for linear systems, the condition reduces to the inequality (3). The precise statement of this condition is as follows.

Definition: A function $k(x)$ of the state of the system (4) is said to satisfy the RDC if, *whenever* $x(t_0) = 0$,

$$\int_{t_0}^{\infty} [u + k(x)]'[u + k(x)]\,dt \geq \int_{t_0}^{\infty} u'u\,dt \tag{9}$$

for all square integrable $u(\cdot)$ *such that* $x(\infty) = 0$, *where* $x(\cdot)$ *is the solution trajectory of* (4).

Note that the inequality (9) is only required to hold for zero initial states; no direct condition is placed on the response of (4) for nonzero $x(t_0)$, although of course the response in this case is implicitly constrained by specifying the response for $x(t_0) = 0$ and arbitrary $u(t)$.

As remarked above, in the linear system case, (9) reduces to (3). This is most easily seen by converting (9) to a frequency domain inequality, using Parseval's theorem, and then making use of the essentially arbitrary nature of $u(\cdot)$. Note that just as (3) is a condition on the input–output behavior of a plant, and independent therefore of initial conditions, so is this true of (9).

One can construct physical interpretations for (9). For example, one could argue that (9) implies that a feedback law of $-k(x)$ constitutes a negative feedback law in a sense somewhat akin to that of classical control: with u_{ext} *denoting an external control applied to the system* (4), and with $-k(x)$ simultaneously a feedback control, one has $u_{\text{ext}} = u + k(x)$, and (9) implies that $u(\cdot)$ is smaller than $u_{\text{ext}}(\cdot)$ as measured by an L_2 norm. In other words, the feedback cancels out some of the applied control.

The importance of the RDC is shown in the following theorem.

Theorem 3: For the system

$$\dot{x}(t) = f[x(t)] + Gu(t) \tag{10}$$

the asymptotically stable control law

$$u(t) = -k[x(t)] \tag{11}$$

is optimal for the problem of minimizing, subject to $\lim_{T\to\infty} x(T) = 0$, a performance index of the form

$$V\{x_0, u(\cdot)\} = \lim_{T\to\infty} \int_0^T [m(x) + u'u]\,dt \tag{12}$$

with $m(x)$ nonnegative for all x if, and only if, $k(x)$ satisfies the RDC.

Proof of Necessity: Suppose that, for some nonnegative function $m(x)$, the control law (11) has been found to be an optimal control for the system (10) that minimizes (12), and that $\bar{\phi}(x)$ is the associated minimum. Then $\bar{\phi}(x)$ satisfies (7), which may be rewritten as

$$m(x) = \tfrac{1}{4}\nabla'\bar{\phi}(x)GG'\nabla\bar{\phi}(x) - \nabla'\bar{\phi}(x)f(x).$$

Now for any $u(\cdot)$, *not* necessarily optimal, but for which $\lim_{t\to\infty} x(t) = 0$, we have

$$\begin{aligned}\int_{t_0}^{\infty} [m(x) + u'u]\,dt &= \int_{t_0}^{\infty} [\tfrac{1}{4}\nabla'\bar{\phi}GG'\nabla\bar{\phi} - \nabla'\bar{\phi}f(x) \\ &\quad + u'u]\,dt \\ &= \int_{t_0}^{\infty} \Big\{(u + \tfrac{1}{2}G'\nabla\bar{\phi})'(u + \tfrac{1}{2}G'\nabla\bar{\phi}) \\ &\quad - \frac{d}{dt}[\bar{\phi}(x(t))]\Big\}\,dt \\ &= \bar{\phi}(x(t_0)) + \int_{t_0}^{\infty} (u + \tfrac{1}{2}G'\nabla\bar{\phi})' \\ &\quad \cdot (u + \tfrac{1}{2}G'\nabla\bar{\phi})\,dt\end{aligned}$$

where the second line is obtained by using the identity

$$\frac{d}{dt}[\bar{\phi}(x(t))] = \nabla'\bar{\phi}f(x) + \nabla'\bar{\phi}Gu.$$

Recalling from Theorem 1 that $k(x) = \tfrac{1}{2}G'\nabla\bar{\phi}$, we obtain *whenever* $x(t_0) = 0$ *and* $x(\infty) = 0$,

$$\int_{t_0}^{\infty} (u + k(x))'(u + k(x))\,dt \geq \int_{t_0}^{\infty} u'u\,dt$$

for all $u(\cdot)$, *where* $x(\cdot)$ *is the solution trajectory of* (4). □

This completes the proof that the RDC is necessarily satisfied by an optimal system. The converse, namely, that satisfaction of the condition implies the existence of a nonnegative function $m(\cdot)$ in (12) such that (11) is optimal, is rather more difficult to prove, and it will first be necessary to establish some preliminary results concerning properties of systems for which the RDC is satisfied. To begin with, it will be necessary to have a finite-time version of the RDC.

Lemma 1: If $k(x)$ is such that the RDC holds and such that $u(t) = -k[x(t)]$ makes (10) asymptotically stable, then, whenever $x(t_0) = 0$, the inequality

$$\int_{t_0}^{T} [u + k(x)]'[u + k(x)]\,dt \geq \int_{t_0}^{T} u'u\,dt \qquad (13)$$

holds for all $T \geq t_0$, and all square-integrable $u(t)$, where $x(\cdot)$ is the solution trajectory of (4).

Proof: For any $T \geq t_0$, consider the class of controls

$$U_T = \Big\{u \Big| \int_{t_0}^{\infty} u'(t)u(t)\,dt < \infty, \quad u(t) = -k[x(t)], \quad \forall t > T\Big\}.$$

There is no restriction on $x(T)$. By the assumed asymptotic stability and stationarity of the control law $u(t) = -k[x(t)]$, use of any $u \in U_T$ (for any T) will lead to $\lim_{t\to\infty} x(t) = 0$, so that U_T is a subset of the class of controls for which (9) holds. Consequently,

$$\int_{t_0}^{T} [u + k(x)]'[u + k(x)]\,dt \geq \int_{t_0}^{\infty} u'u\,dt \geq \int_{t_0}^{T} u'u\,dt. \qquad \square$$

Note, incidentally, that the above lemma requires no direct restriction on $x(T)$. It obviously is of independent interest, since it makes more solid the physical interpretation of the RDC given earlier.

Now let us introduce, in preparation for completing the proof of Theorem 3, the feedback system

$$\dot{x} = f(x) - \tfrac{1}{2}Gk(x) + Gu_1 = f(x) + G[u_1 - \tfrac{1}{2}k(x)]. \quad (14)$$

The relation between (4) and (14) resulting in both systems having the same trajectory $x(\cdot)$ is $u = u_1 - \tfrac{1}{2}k(x)$. Condition (13) then maps into the following *passivity condition* for (14). Whenever $x(t_0) = 0$ and $u_1(\cdot)$ is square integrable

$$\int_{t_0}^{T} u_1'k(x)\,dt \geq 0 \qquad (15)$$

for all $T \geq t_0$.

The reason for describing (15) as a passivity condition is as follows. Consider the system (14) with input u_1 and with output $y_1 = k(x)$. Then, thinking of u_1 as, say, a voltage and y_1 as a current, (15) is the condition that the system be passive, i.e., that when initially unexcited, it acts as an energy sink. Passivity has proved an important concept in, for example, stability theory of control systems, see, e.g., [10], and we shall return to this in Section V.

Inequality (15) has a natural and important extension to cope with the case of a nonzero initial state, given in Lemma 2. A version of this lemma for the linear-quadratic case appears in [11], where connections with optimal control and stability are noted. For our purposes though, Lemma 2 is simply a tool for proving the sufficiency statement of Theorem 3.

This inequality can be extended to cope with the case of a nonzero initial state in the following way.

Lemma 2: Suppose that the function $k(x)$ and the system (4) are such that the RDC is satisfied. Then for the system (14) with arbitrary initial state $x(0) = x_0$, there exists a number $C(x_0)$ such that for any $\epsilon \geq 0$ and any control $u_1(\cdot) \in L_2[0,T]$,

$$\int_{0}^{T} (2u_1'k(x) + \epsilon\, u_1'u_1)\,dt \geq -C(x_0). \qquad (16)$$

Proof: Take t_0 to be sufficiently negative that for arbitrary x_0, there exists by the controllability of (4) and therefore (14) a control $u_1(\cdot)$ defined over $[t_0,0]$ such that with $x(t_0) = 0$, there results $x(0) = x_0$. Let $u_1(\cdot)$ be arbitrary over $[0,T]$; (15) holds for all $T \geq 0$, whence

$$\int_0^T 2u_1'k(x)\,dt \geq -\int_{t_0}^0 2u_1'k(x)\,dt.$$

The right side of this inequality is independent of $u_1(\cdot)$ over $[0,T]$, so that it is possible to find a function $C(x_0)$ of x_0 such that

$$\int_0^T 2u_1'k(x)\,dt \geq -C(x_0)$$

where $x(0) = x_0$, and T is arbitrary. Since $\epsilon u_1'u_1$ is always positive, the required result follows. □

IV. Solution of the Inverse Problem

In this section, we shall pose an optimal control problem for the system (14). By examining a limiting form of the solution of this problem, we shall recover functions $m(\cdot)$ and $n(\cdot)$, which yield a constructive proof of the remainder of Theorem 3.

Consider the system (14), with performance index

$$V[x_0,u_1(\cdot),T,\epsilon] = \int_0^T [2u_1'k(x) + \epsilon\, u_1'u_1]\,dt, \qquad \epsilon > 0. \tag{17}$$

Assume there is sufficient smoothness to apply the Hamilton–Jacobi theory. Then, using the bound (16), one can conclude the existence of an optimal control law $u_1^*(\cdot)$ and associated performance index

$$V[x_0,u_1^*,T,\epsilon] = -\phi_\epsilon(x_0,0,T) \geq 0$$

where the minus sign is a notational covenience, and $\phi_\epsilon(\cdot,\cdot,\cdot)$ satisfies the Hamilton–Jacobi equation with appropriate boundary conditions. Next, one can argue that ϕ_ϵ is monotone increasing with T and bounded, and accordingly, existence is guaranteed of

$$\bar{\phi}_\epsilon(x) = \lim_{T\to\infty} \phi_\epsilon(x,t,T) \leq 0.$$

Further, following arguments like those of Theorem 1, $\bar{\phi}_\epsilon(\cdot)$ satisfies the limiting Hamilton–Jacobi equation

$$\nabla'\bar{\phi}_\epsilon[f(x) - \tfrac{1}{2}Gk(x)] + \epsilon^{-1}[k(x) - \tfrac{1}{2}G'\nabla\bar{\phi}_\epsilon]' \cdot [k(x) - \tfrac{1}{2}G'\nabla\bar{\phi}_\epsilon] = 0. \tag{18}$$

Define

$$m_\epsilon(x) = -\nabla'\bar{\phi}_\epsilon[f(x) - \tfrac{1}{2}Gk(x)]. \tag{19}$$

From (18), we see that $m_\epsilon(x) \geq 0$ for all x.

Next, one can show that $\bar{\phi}_\epsilon(x)$ is monotonic increasing with ϵ, and bounded. Accordingly, there exists

$$\hat{\phi}(x) = \lim_{\epsilon\to 0} \bar{\phi}_\epsilon(x) \geq 0 \tag{20}$$

and, assuming[3] $\lim_{\epsilon\to 0} \nabla\bar{\phi}_\epsilon(x) = \nabla[\lim_{\epsilon\to 0} \bar{\phi}_\epsilon(x)] = \nabla\hat{\phi}(x)$,

$$-\nabla'\hat{\phi}(x)[f(x) - \tfrac{1}{2}Gk(x)] = \hat{m}(x) = \lim_{\epsilon\to 0} m_\epsilon(x) \geq 0. \tag{21}$$

Letting $\epsilon \to 0$ in (18), we also obtain

$$k(x) = \tfrac{1}{2}G'\nabla\hat{\phi}(x). \tag{22}$$

The foregoing results will now be used to complete the proof of Theorem 3. Specifically, it will be shown that, if $k(x)$ is chosen such that RDC is satisfied, then there exists a nonnegative function $m(x)$ in (12) such that the control law (11) is optimal.

Proof of Theorem 3 (continued): The necessity part of the proof has already been outlined. To show that the RDC is sufficient for optimality, we proceed as follows.

Using the technique outlined above, we construct the nonnegative functions $\hat{m}(x)$ and $\hat{\phi}(x)$. Recall that

$$\hat{m}(x) = -\nabla'\hat{\phi}(x)[f(x) - \tfrac{1}{2}Gk(x)]$$

and

$$\tfrac{1}{2}G'\nabla\hat{\phi}(x) = k(x).$$

Now consider the following finite performance index associated with (10):

$$V[x(t_0),u(\cdot),t_0,T] = \hat{\phi}[x(T)] + \int_{t_0}^T [u'u + m(x)]\,dt. \tag{23}$$

The optimal performance index is obtained via the Hamilton–Jacobi theory as the solution of

$$\frac{\partial\phi}{\partial t} + \nabla'\phi f(x) - \tfrac{1}{4}\nabla'\phi GG'\nabla\phi - \nabla'\hat{\phi}f(x) + \tfrac{1}{4}\nabla'\hat{\phi}GG'\nabla\hat{\phi} = 0$$

with the boundary conditions

$$\phi(x,T,T) = \hat{\phi}(x)$$

$$\phi(0,t,T) = 0.$$

It is clear that $\phi(x,t,T) = \hat{\phi}(x)$ is one solution of this equation, and the smoothness assumptions imposed so far imply that it is the unique solution. Letting T approach infinity in (23), and using the asymptotic stability of the control law $u(t) = -k[x(t)]$, it follows that with $m(x) = \hat{m}(x)$ in (12), the minimum value of (12) is

$$\bar{\phi}(x) = \lim_{T\to\infty} \hat{\phi}(x) = \hat{\phi}(x)$$

and obviously $\hat{\phi}(x)$ is the optimal performance index. That the optimal control is $u = -k(x)$ now follows immediately from (22). □

V. Properties of Optimal Systems

A key development in the preceding proof was the observation that (14) represents a passive system. It is

[3] The full significance of this condition escapes us. It would seem that $\bar{\phi}_\epsilon(x)$ is analytic in ϵ, and that the assumption of sufficient smoothness on $f(\cdot)$ and $m(\cdot)$ would imply smoothness of $\bar{\phi}_\epsilon(x)$ as a function of x. But the *joint* smoothness in ϵ and x necessary here seems harder to provide for, via conditions on $f(\cdot)$, etc.

possible to extend this result further, as shown in the following lemma.

Lemma 3: Let $\psi(\cdot,t)$ be any time-varying gain or nonlinearity such that

$$\psi(0,t) = 0, \qquad \text{for all } t$$

$$\sigma'\psi(\sigma,t) \geq \tfrac{1}{2}\sigma'\sigma, \qquad \text{for all } \sigma,t. \tag{24}$$

Then if $x(t)$ is a solution of

$$\dot{x}(t) = f[x(t)] + Gu(t), \qquad x(t_0) = 0$$

where

$$u(t) = u_{\text{ext}}(t) - \psi\{k[x(t)], t\}$$

then the RDC implies that

$$\int_{t_0}^{T} 2u_{\text{ext}}'k(x)\, dt \geq 0$$

for all $T \geq t_0$, and any square-integrable $u_{\text{ext}}(\cdot)$.

Proof: The proof follows directly from the inequality (13). □

Since the conditions (24) include the special case $\psi(\sigma,t) = \sigma$, Lemma 3 shows that an optimal system is always passive in the sense defined above. However, it also shows that the system remains passive when a wide range of nonlinearities is introduced into the feedback loop.

If condition (24) is strengthened slightly to

$$(\tfrac{1}{2} + \epsilon_1)\sigma'\sigma \leq \sigma'\psi(\sigma,t) \leq \frac{1}{\epsilon_2}\sigma'\sigma$$

for some positive constants ϵ_1 and ϵ_2, then the closed-loop system obtained by setting $u = -\psi\{k[x(t)],t\}$ is also (Lyapunov) asymptotically stable. The proof, using $\phi[x(t)]$ as a Lyapunov function, may be found in [12]. (A partial result also appears in [3].) Since the closed-loop system corresponds to setting $u_{\text{ext}}(t) \equiv 0$ in Lemma 3, the stability is simply Lyapunov stability of a passive system, which is to be expected as a consequence of the passivity property.

In the linear quadratic problem, it is well known that the RDC implies that the optimal closed-loop system exhibits a lower sensitivity to parameter variation than the equivalent open-loop system, see, e.g., [1]. The analogous result here is more complicated. This sensitivity improvement follows in case $H_{xx} \geq 0$, where H is the Hamiltonian of the system, see [14]; for nonlinear $f(x)$, this condition is awkward, involving the costate vector. However, for linear $f(x) = Fx$, it becomes $m_{xx} \geq 0$, and the sensitivity improvement can be thought of as resulting via a small-signal, or linearized, version of the RDC.

VI. Conclusions

In this paper an attempt has been made to isolate those properties of a control system that are linked to a certain type of optimality, and it has been shown that a return difference condition, closely related to loop gain concepts in linear systems, provides a criterion for deciding the optimality or otherwise of a feedback law.

It should be noted that although the proof of optimality of a control law as given here is a constructive one, it does not normally lead to computationally attractive construction procedures. Also, only one of the many performance indices for which the control law is optimal has been presented. An example where the procedure can be reasonably used to construct the performance index is in the case of a linear system with linear feedback. This case is, of course, included in the present results.

It is vital to the results of this paper that the control appear quadratically in the performance index; if a more general class of loss functions is considered, properties such as stability and tolerance of input nonlinearities could no longer, in general, be guaranteed. (In particular, the inclusion of cross products between the control and the states would allow the trivial result, apparently first noted in [2], that *any* control law is optimal.) Obviously our results extend to the case where in the loss function, $u'(t)u(t)$ is replaced by $u'(t)Ru(t)$, with R a positive definite matrix; however, it is also interesting to speculate, on the basis of the results in [12], that many of the results of this paper could be extended to the case of a more general loss function of the form $r(u) + m(x)$, with some positivity constraint on $r(u)$.

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