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SOME INTEGRAL EQUATIONS WITH NONSYMMETRIC SEPARABLE KERNELS*

BRIAN D. O. ANDERSON† AND THOMAS KAILATH‡

Abstract. It is shown that the eigenvalues and eigenfunctions for the class of "separable" or "semidegenerate" kernels can be determined from the solution of a linear differential equation, which is usually more amenable to machine solution. The theory is extended to solve a simultaneous diagonalization problem for two separable kernels. Finally, some new connections are obtained between Riccati differential equations and Fredholm integral equations of the second kind. The results specialize to previously known results for symmetric separable kernels.

1. Introduction. We shall present a method based on the solution of certain differential equations for solving the integral equations

$$\int_0^T K(t,s)\phi(s)\,ds = \lambda\phi(t), \qquad 0 \leqq t \leqq T < \infty, \tag{1}$$

$$\int_0^T K_1(t,s)\phi(s)\,ds = \lambda \int_0^T K_2(t,s)\phi(s)\,ds, \qquad 0 \leqq t \leqq T, \tag{2}$$

$$f(t) + \int_0^T K(t,s)f(s)\,ds = g(t), \qquad 0 \leqq t \leqq T, \tag{3}$$

where the kernels $K(t, s)$ have the separable form[1]

$$K(t,s) = \begin{cases} \alpha'(t)\beta(s), & t \geqq s, \\ \gamma'(t)\delta(s), & s > t. \end{cases} \tag{4}$$

In (4), $\alpha(\cdot)$ and $\beta(\cdot)$ are n-vector functions, and $\gamma(\cdot)$ and $\delta(\cdot)$ m-vector functions. The $\{K_i(t, s)\}$ are of a similar form but with an added delta function, i.e.,

$$K_i(t,s) = \delta(t-s) + \alpha_i'(t)\beta_i(s)l(t-s) + \gamma_i'(t)\delta_i(s)l(s-t), \qquad i = 1, 2. \tag{5}$$

The function $l(t - s)$ is the Heaviside unit-step function, which is unity for $s \leqq t$ and zero for $s > t$. In (4) and (5) we shall assume that the entries of $\alpha(\cdot)$, etc. are continuous functions. In (1), $K(\cdot, \cdot)$ is assumed known, and we seek unknown eigenvalues $\{\lambda\}$ and associated eigenfunctions $\{\phi(\cdot)\}$; in (2), $K_1(\cdot, \cdot)$ and $K_2(\cdot, \cdot)$ are assumed known, and we seek the "simultaneous" eigenvalues $\{\lambda\}$ and eigenfunctions $\{\phi(\cdot)\}$; equation (3) is a Fredholm equation of the second kind, where $K(\cdot, \cdot)$ and $g(\cdot)$ are assumed known and $f(\cdot)$ is sought.

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† Department of Electrical Engineering, University of Newcastle, New South Wales, 2308, Australia.

‡ Department of Electrical Engineering, Stanford University, Stanford, California 94305.

[1] The terms "semiseparable" and "semidegenerate" are also used.

The equations (1)–(3) arise in many situations; in statistical problems the kernels are usually symmetric and often also nonnegative definite. We have discussed the case of symmetric kernels in some detail in previous papers [1], [2], where references to prior work, especially [3]–[7], are also discussed. Section 2 of the present paper constitutes a generalization and a complete discussion of material in [1], where results were given on the calculation of eigenvalues and eigenfunctions of covariance functions. (In contrast to [6], there is no assumption in [1] or this paper that a system generating the covariance is known, in terms of, say, a state variable description.) In § 2 we shall describe procedures for finding the "eigenlengths," or values of T, for which a prescribed λ is an eigenvalue of (1) and (2). We also give procedures for testing if a given λ is an eigenvalue for a prescribed T and we also show how to calculate the eigenfunctions. Similar problems are treated in [4], [5], which claim that the smallest eigenlength associated with a prescribed eigenvalue may be found by determining the escape time of a certain Riccati equation. However, no method is obtained for computing other eigenlengths or computing the eigenfunctions.

In § 3 of this paper we study the Fredholm equation (3), by methods similar to those previously used in [2]–[7]. In [2], [3], [6], only symmetric kernels are treated; [2] includes and interprets the results of [3] and [6] by showing that a least-squares estimation problem may be naturally associated with the integral equation and then noting that for separable kernels, the estimation problem may be solved, following Kalman and Bucy [8], via a Riccati differential equation. In [7], Schumitzky studies the converse equivalence in detail, without restriction to symmetric kernels. More specifically he shows how to associate a Fredholm equation with any Riccati differential equation. However, he does not show how to go from a general Fredholm equation to a Riccati equation. We treat this problem in § 3, where further remarks are made on Schumitzky's work.

In concluding this introductory section we should mention that of course several other methods of solving the equations (1)–(3) are available in the literature. In particular, solutions of varying degrees of explicitness have been found for kernels that are covariance functions of stationary random processes with rational power spectral densities; a recent reference is [9] and a fairly comprehensive list of earlier references is given in [10]. A recent reference on the eigenvalue problem is [11]. Finally, while this paper was being reviewed, some further references [22]–[24] have appeared on the solution of two-point boundary value problems by methods similar to those in [7] and in this paper.

2. Computation of eigenvalues, eigenlengths and eigenfunctions. In Theorem 1 and Corollary 1 we shall show how to solve (1); after some examples, we shall present Theorem 2 which deals with (2).

To study (1), we shall use the following pair of simultaneous equations, written in obvious vector notation:

$$\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} \beta\alpha'/\lambda & \beta\gamma'/\lambda \\ -\delta\alpha'/\lambda & -\delta\gamma'/\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{6}$$

Here $\{\alpha, \beta, x\}$ and $\{\gamma, \delta, y\}$ are each m- and n-vectors respectively and λ is fixed but for the moment arbitrary. Associated with (6) is an $(m+n) \times (m+n)$ transition matrix $\Phi(\cdot, \cdot; \lambda)$ satisfying

$$\frac{d}{dt}\Phi(t, s; \lambda) = \begin{bmatrix} \beta\alpha'/\lambda & \beta\gamma'/\lambda \\ -\delta\alpha'/\lambda & -\delta\gamma'/\lambda \end{bmatrix}\Phi(t, s; \lambda), \qquad \Phi(s, s; \lambda) = I, \tag{7}$$

where the arguments of α, β, γ and δ are all t. The matrix $\Phi(\cdot, \cdot; \lambda)$ may be partitioned as

$$\Phi(t, s; \lambda) = \begin{bmatrix} \Phi_{11}(t, s; \lambda) & \Phi_{12}(t, s; \lambda) \\ \Phi_{21}(t, s; \lambda) & \Phi_{22}(t, s; \lambda) \end{bmatrix}, \tag{8}$$

where Φ_{11} is $m \times m$, Φ_{12} is $m \times n$, Φ_{21} is $n \times m$ and Φ_{22} is $n \times n$.

The main result of this section is the following theorem.

THEOREM 1. *With the preceding definitions, T_0 is an eigenlength associated with eigenvalue λ_0 if and only if $\Phi_{22}(T_0, 0; \lambda_0)$ is singular.*

Proof. First suppose T_0 is an eigenlength associated with an eigenvalue λ_0. Then there exists a scalar function $\phi_0(\cdot)$, not identically zero, such that

$$\alpha'(t)\int_0^t \beta(s)\phi_0(s)\,ds + \gamma'(t)\int_t^{T_0} \delta(s)\phi_0(s)\,ds = \lambda_0\phi_0(t)$$

or

$$\phi_0(t) = \frac{\alpha'(t)}{\lambda_0}x_0(t) + \frac{\gamma'(t)}{\lambda_0}y_0(t), \tag{9}$$

where

$$x_0(t) = \int_0^t \beta(s)\phi_0(s)\,ds \quad \text{and} \quad y_0(t) = \int_t^{T_0} \delta(s)\phi_0(s)\,ds. \tag{10}$$

Differentiation of these expressions for $x_0(t)$ and $y_0(t)$, and then the use of (9) yields the equations

$$\frac{dx_0}{dt} = \frac{\beta\alpha'}{\lambda_0}x_0 + \frac{\beta\gamma'}{\lambda_0}y_0 \quad \text{and} \quad \frac{dy_0}{dt} = -\frac{\delta\alpha'}{\lambda_0}x_0 - \frac{\delta\gamma'}{\lambda_0}y_0,$$

which is to say that $x_0(\cdot)$, $y_0(\cdot)$ is a solution pair of (6). Suppose that $y_0(0) = w$, while we know from the form of $x_0(t)$ in (10) that $x_0(0) = 0$. Plainly $w \neq 0$; for otherwise $x_0(t)$ and $y_0(t)$ would be zero for all t, and as a consequence so would ϕ_0. Now

$$\begin{bmatrix} x_0(T_0) \\ y_0(T_0) \end{bmatrix} = \Phi(T_0, 0; \lambda_0)\begin{bmatrix} 0 \\ w \end{bmatrix}$$

and the definition of $y_0(t)$ in (10) ensures that $y_0(T_0) = 0$, i.e., $\Phi_{22}(T_0, 0; \lambda_0)w = 0$, as required.

The converse is straightforward. With $\Phi_{22}(T_0, 0; \lambda_0)$ singular, let w be a nonzero vector in the null space and define $x_0(\cdot)$ and $y_0(\cdot)$ by

$$\begin{bmatrix} x_0(t) \\ y_0(t) \end{bmatrix} = \Phi(t, 0; \lambda_0)\begin{bmatrix} 0 \\ w \end{bmatrix}.$$

Then define $\phi_0(\cdot)$ by (9). The constraints $x_0(0) = 0$ and $y_0(T_0) = 0$ then readily yield $x_0(t) = \int_0^t \beta(s)\phi_0(s)\,ds$ and $y_0(t) = \int_t^{T_0} \delta(s)\phi_0(s)\,ds$. When these expressions are inserted into (9), the eigenvalue equation, with $\phi_0(\cdot)$ as eigenfunction, is recovered. This completes the proof.

Contained within the above proof is the constructive procedure for an eigenfunction $\phi_0(\cdot)$ associated with eigenlength T_0 and eigenvalue λ_0.

COROLLARY 1. *With definitions as before, let T_0 be an eigenlength associated with an eigenvalue λ_0. Then for any nonzero vector w in the null space of $\Phi_{22}(T_0, 0; \lambda_0)$, there is an associated eigenfunction $\phi_0(\cdot)$ given by*

$$\phi_0(t) = \frac{1}{\lambda_0}[\alpha'(t) \quad \gamma'(t)]\Phi(t, 0; \lambda_0)\begin{bmatrix} 0 \\ w \end{bmatrix}$$

(11)

$$= \frac{1}{\lambda_0}[\alpha'(t)\Phi_{12}(t, 0; \lambda_0) + \gamma'(t)\Phi_{22}(t, 0; \lambda_0)]w.$$

A number of points are worthy of note. First, the problem of establishing the eigenlengths associated with a fixed eigenvalue is in theory straightforward: equation (7) is integrated while $\Phi_{22}(t, 0; \lambda)$ is tested for singularity. Those values of t for which the matrix is singular are, of course, the eigenlengths. To find the smallest eigenlength, one can form a Riccati equation with the property that the escape time of the solution of the equation is the same as the smallest eigenlength. This procedure has been suggested in [4], [5] and is discussed briefly in the next section.

Second, the problem of determining the various eigenvalues associated with a fixed eigenlength apparently requires the repeated solution of (7) for different values of λ, to permit examination of the resulting $\Phi_{22}(T, 0; \lambda)$.

Third, in the standard Fredholm theory, the eigenvalues for a fixed eigenlength are the negative reciprocals of solutions of an equation obtained by setting the Fredholm determinant equal to zero [12]. The multiplicity of a zero of the Fredholm determinant is an upper bound on (but not necessarily equal to) the number of associated eigenfunctions. Here we have a stronger result: linearly independent vectors $w_1, w_2, \cdots, w_r$ in the nullspace of $\Phi(T_0, 0; \lambda_0)$ generate linearly independent eigenfunctions $\phi_{01}(\cdot), \phi_{02}(\cdot), \cdots, \phi_{0r}(\cdot)$. To see this, suppose the contrary, i.e., suppose there exist constants $a_1, \cdots, a_r$, not all zero, such that $\sum_{i=1}^r a_i\phi_{0i} = 0$ identically in t. Evidently $\sum_{i=1}^r a_iw_i$ is in the null space of $\Phi(T_0, 0; \lambda_0)$ and the associated eigenfunction, being $\sum_{i=1}^r a_i\phi_{0i}$, is zero. Now in the proof of Theorem 1, we showed that the vector functions $x_0(\cdot)$ and $y_0(\cdot)$ defined by

$$\begin{bmatrix} x_0(t) \\ y_0(t) \end{bmatrix} = \Phi(t, 0; \lambda_0)\begin{bmatrix} 0 \\ \sum_{i=1}^r a_iw_i \end{bmatrix}$$

satisfy $x_0(t) = \int_0^t \beta(s)[\sum_{i=1}^r a_i\phi_{0i}(s)]\,ds = 0$ and $y_0(t) = \int_t^{T_0} \delta(s)[\sum_{i=1}^r a_i\phi_0(s)]\,ds = 0$.

But also

$$\begin{bmatrix} 0 \\ \sum_{i=1}^{r} a_i w_i \end{bmatrix} = \Phi(0, t; \lambda_0) \begin{bmatrix} x_0(t) \\ y_0(t) \end{bmatrix}$$

with the left side nonzero on account of the independence of the w_i. Hence the contradiction.

We now turn to some simple examples, chiefly to check that our method also easily yields some well-known results. We should stress that the main value of the method lies in its computational aspects, viz. solution via reduction to an initial value differential equation, which is often more suited than the original equation for solution on a digital computer.

Example 1. *Volterra kernels.* Such kernels are zero for $s > t$ and therefore we take $\gamma \equiv \delta \equiv 0$ in (4). Referring now to (7), we see that

$$(d/dt)\Phi_{22}(t, s; \lambda) = 0, \qquad \Phi_{22}(s, s; \lambda) = I$$

from which it is evident that $\Phi_{22}(t, 0; \lambda)$, being identically equal to the unit matrix, will never be singular for any t or λ. Accordingly, as is well known, there are no eigenvalues.

Example 2. *Degenerate* (*Goursat*) *kernels.* In this case, we have $\alpha \equiv \gamma$, $\beta \equiv \delta$ so that $K(t, s)$ is

$$K(t, s) = \alpha'(t)\beta(s).$$

Equation (7) then yields

$$\text{(12)} \qquad \dot{\Phi}_{12} = \frac{\beta\alpha'}{\lambda}[\Phi_{12} + \Phi_{22}], \qquad \dot{\Phi}_{22} = -\frac{\beta\alpha'}{\lambda}[\Phi_{12} + \Phi_{22}].$$

By addition, we have

$$\dot{\Phi}_{12} + \dot{\Phi}_{22} = 0$$

so that applying the boundary condition of (7) yields

$$\Phi_{12} + \Phi_{22} = I.$$

Therefore from (12) we obtain

$$\dot{\Phi}_{22} = -\beta\alpha'/\lambda.$$

Consequently, eigenvalues and eigenlengths are defined by the singularities of

$$\Phi_{22}(t, 0; \lambda) = I - \frac{1}{\lambda}\int_0^t \beta(\tau)\alpha'(\tau)\, d\tau.$$

We should remark that nonsymmetric (Goursat) kernels may have no eigenvalues. Volterra kernels already provide one example. Another simple example is $K(t, s) = \sin t \cos s$; following the above procedure leads to

$$\Phi_{22}(t, 0; \lambda) = 1 - (1/\lambda)\sin^2 t.$$

If $t = 2\pi$, there are clearly no eigenvalues.

2.1. Simultaneous eigenvalues. We now turn to a study of (2). In some Gaussian discrimination and signal selection problems, the problem often arises [13], [14] of solving (2) when the $K_i(t, s)$ are covariances of two Gaussian processes. In [15], a state-variable method of solution, following [6], is described, *assuming knowledge of linear finite-dimensional dynamical systems generating* K_1 *and* K_2 (in the sense that the system outputs have covariances K_1 and K_2 when the system inputs are Gaussian white noise). Here, we dispense with this sometimes restrictive assumption, and require only the knowledge of α_i and β_i (since covariances are symmetric) in (5). The theory is immediately extendable to the case when K_1 and K_2 are neither nonnegative definite nor self-adjoint, but simply have the form of (5).

Since the proof is so similar to that of Theorem 1, we only state the main result.

Consider the differential equation

$$
(13)\quad \frac{d}{dt}\Phi(t,s;\lambda) = \left[\begin{array}{cc|cc}
-\dfrac{\beta_1\alpha_1'}{1-\lambda} & \dfrac{\lambda\beta_1\alpha_2'}{1-\lambda} & \dfrac{\beta_1\beta_1'}{1-\lambda} & \dfrac{\lambda\beta_1\beta_2'}{1-\lambda} \\[2ex]
-\dfrac{\beta_2\alpha_1'}{1-\lambda} & \dfrac{\lambda\beta_1\alpha_2'}{1-\lambda} & -\dfrac{\beta_2\beta_1'}{1-\lambda} & \dfrac{\lambda\beta_2\beta_2'}{1-\lambda} \\[2ex]
\hline
\dfrac{\alpha_1\alpha_1'}{1-\lambda} & -\dfrac{\lambda\alpha_2\alpha_2'}{1-\lambda} & \dfrac{\alpha_1\beta_1'}{1-\lambda} & -\dfrac{\lambda\alpha_1\beta_2'}{1-\lambda} \\[2ex]
\dfrac{\alpha_2\alpha_1'}{1-\lambda} & -\dfrac{\lambda\alpha_2\alpha_2'}{1-\lambda} & \dfrac{\alpha_2\beta_1'}{1-\lambda} & \dfrac{\lambda\alpha_2\beta_2'}{1-\lambda}
\end{array}\right]\Phi(t,s;\lambda)
$$

and partition Φ in the same manner as the first matrix on the right side of (13):

$$
\Phi(t,s;\lambda) = \begin{bmatrix} \Phi_{11}(t,s;\lambda) & \Phi_{12}(t,s;\lambda) \\ \Phi_{21}(t,s;\lambda) & \Phi_{22}(t,s;\lambda) \end{bmatrix},
$$

Then we have the following theorem.

THEOREM 2. *With the above definitions, T is an eigenlength associated with eigenvalue $\lambda_0 \neq 1$ if and only if $\Phi_{22}(T, 0; \lambda_0)$ is singular. Then, with w any nonzero vector in the null space of $\Phi_{22}(T, 0; \lambda_0)$, an associated eigenfunction $\phi_0(\cdot)$ is given by*

$$
(14)\qquad \phi_0(t) = \frac{1}{1-\lambda}[-\alpha_1'(t) \quad \lambda\alpha_2'(t)\colon -\beta_1'(t) \quad \lambda\beta_2'(t)]\begin{bmatrix} \Phi_{12}(t,0;\lambda_0)w \\ \Phi_{22}(t,0;\lambda_0)w \end{bmatrix}.
$$

3. Riccati equations and Fredholm resolvents. As we noted in the Introduction, the relation between Riccati equations and Fredholm resolvents has been studied in [2], [3], [6] for symmetric kernels. Here, following [4], [5], [7], we shall pursue the nonsymmetric case in some detail.

We shall consider the Fredholm equation

$$
(15)\qquad f(t) + \int_0^T K(t,s)f(s)\,ds = g(t), \qquad 0 \leqq t \leqq T,
$$

where

$$K(t, s) = \begin{cases} \alpha'(t)\beta(s), & t \geqq s, \\ \gamma'(t)\delta(s), & s \geqq t, \end{cases}$$

and the entries of the vectors $\alpha(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$, $\delta(\cdot)$ are comprised of continuous functions.

3.1. The Riccati equation associated with a kernel. In [5], Kalaba et al. show that the solution of (15), for all t and T in some range $0 \leqq t, T \leqq T_1$, can be reduced to an initial value problem requiring the solution of various differential equations, including the Riccati equation

$$dP/dt = [\beta + P\delta][\gamma' + \alpha' P]. \tag{16}$$

However, though the whole solution procedure in [4], [5] hinges on (16) having a solution, no proof is given to show that existence of a solution to (15) implies existence of a solution to (16). This is the subject of the next result.

THEOREM 3. *With $K(\cdot, \cdot)$, $\alpha(\cdot)$, $\beta(\cdot)$, $r(\cdot)$ and $\delta(\cdot)$ as in* § 1, (15) *possesses a solution for all* $g(\cdot)$ *and all T in the interval* $[0, T_1]$ *if and only if* (16) *possesses a solution in the interval* $[0, T_1]$.

Proof.[2] It is well known from the theory of integral equations that (15) possesses a solution for all $g(\cdot)$ and all T in $[0, T_1]$ if and only if 1 is not an eigenvalue of $K(\cdot, \cdot)$ for any eigenlength in $[0, T_1]$. This is equivalent to insisting, in the notation of the previous section, that $\Phi_{22}(t, 0; 1)$ is nonsingular for all $t \in [0, T_1]$. Using the well-known connection between the differential equations (6) and (16) as discussed in, for example, [16], it is noted that the solution of (16) is given explicitly by

$$P(t) = \Phi_{12}(t, 0; 1)\Phi_{22}^{-1}(t, 0; 1) \tag{17}$$

and evidently nonsingularity of Φ_{22} is sufficient to guarantee the finite nature of $P(t)$. This nonsingularity is also necessary, provided we can show that if $\Phi_{22}(T, 0; 1)$ is singular for some T, the null space of $\Phi_{22}(T, 0; 1)$ is not contained in the null space of $\Phi_{12}(T, 0; 1)$. To see this is impossible suppose there exists a constant vector w in both null spaces. Then by Corollary 1 there is an associated eigenfunction $\phi_0(t)$ which is of course not identically zero; but by direct calculation, the associated $x_0(T)$ and $y_0(T)$ are both zero, implying $x_0(t)$ and $y_0(t)$ are zero for all t and thus $\phi_0(t)$ is zero for all t, which is a contradiction.

It is interesting to connect the above result with those of [7]; in this reference, one starts with a Riccati equation

$$\dot{P} = A + BP + PC + PDP, \qquad P(0) = F \tag{18}$$

and from this derives, after several calculations, a certain integral operator. Then it is shown that the existence of a solution to (18) on $[0, T]$ is necessary and sufficient for the existence of a resolvent kernel for the operator on every $[0, T_1]$, $T_1 \leqq T$. In contrast to [7], which constructs a map of Riccati equations into operators, we have constructed a map of operators into Riccati equations.

[2] Another proof will be provided by Theorem 4 below.

3.2. Relations between the Riccati equations and the resolvent kernel. As the results of [7] might suggest, we can construct P from the resolvent kernel of $K(\cdot,\cdot)$, and construct the resolvent kernel for $K(\cdot,\cdot)$ quite simply from the solution of P of (16). The details follow, the main results being presented as theorem statements.

THEOREM 4. *Given the kernel* (4), *defined on* $[0, T]$, *suppose that for all* $T_1 \leqq T$, *the resolvent kernel* $Q(t, s; T_1)$ *exists, satisfying*

$$Q(t,s;T_1) = K(t,s) + \int_0^{T_1} K(t,\tau)Q(\tau,s;T_1)\,d\tau \tag{19a}$$

and

$$Q(t,s;T_1) = K(t,s) + \int_0^{T_1} Q(t,\tau;T_1)K(\tau,s)\,d\tau. \tag{19b}$$

Then the matrix $P(\cdot)$ *defined by* (16) *is also defined by*

$$\begin{aligned} &P(0) = 0,\\ &\dot{P}(T_1) = \left\{\beta(T_1) + \int_0^{T_1} Q(t,T_1;T_1)\beta(t)\,dt\right\}\left\{\gamma(T_1) + \int_0^{T_1} Q(T_1,t;T_1)\gamma(t)\,dt\right\}'. \end{aligned} \tag{20}$$

Proof.[3] It is sufficient to show that

$$P(T_1)\delta(T_1) = \int_0^{T_1} Q(t,T_1;T_1)\beta(t)\,dt, \tag{21a}$$

$$P(T_1)\alpha(T_1) = \int_0^{T_1} Q(T_1,t;T_1)\gamma(t)\,dt. \tag{21b}$$

For then $P(T_1)$, as defined by (20), satisfies the differential equation (16). But the continuity of $\alpha, \beta, \gamma, \delta$ guarantees uniqueness of the solution, and thus the proof is complete.

We shall prove merely (21a); the proof of (21b) follows similarly. The proof depends on an interesting identity discovered independently by Siegert [18] and Bellman [19] for self-adjoint kernels, and in more general form by Krein [20], [21]. This identity (in integral, rather than differential form) is as follows:

$$Q(t,s;T_1) = Q_+(t,s) + Q_-(t,s) + \int_0^{T_1} Q_-(t,\tau)Q_+(\tau,s)\,d\tau, \tag{22}$$

$0 \leqq t, s \leqq T_1$, where $Q_+(t,s) = Q(t,s;t)l(t-s)$ and $Q_-(t,s) = Q(t,s;s)l(s-t)$.

Adopt the further notation $Q_1 \circ Q_2$ for kernels Q_1, Q_2 to denote the integral $\int_0^{T_1} Q_1(t,\tau)Q_2(\tau,s)\,d\tau$, the notation $Q_1 \cdot \phi$ for a kernel Q_1 and a function ϕ to denote the integral $\int_0^{T_1} Q_1(t,s)\phi(s)\,ds$ and the notation $\langle\phi_1,\phi_2\rangle$ for vector functions ϕ_1 and ϕ_2 to denote $\int_0^{T_1} \phi_1'(t)\phi_2(t)\,dt$.

[3] This proof follows an idea used by Geesey (cf. [2],[17]) for symmetric kernels.

Turning to the proof of (21a) (which we have to deduce from (20)), we note the following sequence of equalities:

$$
\begin{aligned}
P(T_1) &= \langle [(I + Q_-^a)\cdot\beta]', [(I + Q_+)\cdot\gamma]'\rangle \\
&= \langle \beta', [(I + Q_-)\circ(I + Q_+)\cdot\gamma]'\rangle \\
&= \langle \beta', [(I + Q_- + Q_+ + Q_-\circ Q_+)\cdot\gamma]'\rangle \\
&= \langle \beta', [(I + Q)\cdot\gamma]'\rangle .
\end{aligned}
\tag{23}
$$

The first equality follows from (20) and the definition of Q_- and Q_+, the second and third by straightforward manipulation, and the fourth using (22).

Now multiply (23) on the right by $\delta(T_1)$ to obtain

$$
P(T_1)\delta(T_1) = \left\langle \beta', \left[\int_0^{T_1} Q(t, s; T_1)\gamma'(s)\delta(T_1)\, ds + \gamma'(t)\delta(T_1) \right] \right\rangle. \tag{24}
$$

In (19b), replace s by T_1 and τ by s, and recognize that $K(s, T_1) = \gamma'(s)\delta(T_1)$, since $s < T_1$. Accordingly, (24) becomes

$$
\begin{aligned}
P(T_1)\delta(T_1) &= \langle \beta'(t), Q(t, T_1; T_1)\rangle \\
&= \int_0^{T_1} Q(t, T_1; T_1)\beta(t)\, dt
\end{aligned}
\tag{21a}
$$

as required.

Theorem 4 demonstrates one half of the story, viz. obtaining the Riccati equation solution from the resolvent kernel. The derivation of the resolvent kernel from the Riccati equation follows from the results of [4], [5], which implicitly contain a formula for $Q(t, T_1; T_1)$.

THEOREM 5. *The identity* (22) *expresses* $Q(t, s; T_1)$ *in terms of* $Q(t, s; s)$ *and* $Q(t, s; t)$. *Formulas for* $Q(t, s; s)$ *and* $Q(t, s; t)$ *are as follows*:

$$
Q(t, s; s) = [\alpha'(t)P(t) + \gamma'(t)]\Psi(t, s)\delta(s), \tag{25}
$$

where

$$
\frac{d}{dt}\Psi(t, s) = -\delta(t)[\alpha'(t)P(t) + \gamma'(t)]\Psi(t, s), \qquad \Psi(s, s) = I \tag{26}
$$

and

$$
Q(t, s; t) = [\delta'(s)P'(s) + \beta'(s)]\hat{\Psi}(s, t)\alpha(t), \tag{27}
$$

where

$$
\frac{d}{ds}\hat{\Psi}(s, t) = -\alpha(s)[\delta'(s)P(s) + \beta'(s)]\hat{\Psi}(s, t), \qquad \hat{\Psi}(t, t) = I. \tag{28}
$$

Proof. As remarked above, (25) and (26) are to be found in [4], [5]. We deduce here (27) and (28), by using (25) and (26). From the resolvent kernel definition (19b),

$$
Q(t, s; t) = K(t, s) + \int_0^t Q(t, \tau; t)K(\tau, s)\, d\tau,
$$

or, with $S(s, t) = K(t, s)$,

$$Q(t, s; t) = S(s, t) + \int_0^t S(s, \tau)Q(t, \tau; t)\, dt. \tag{29}$$

Now noting from (19a) that $Q(s, t; t)$ satisfies

$$Q(s, t; t) = K(s, t) + \int_0^t K(s, \tau)Q(\tau, t; t)\, d\tau, \tag{30}$$

it is evident that $Q(t, \cdot; t)$ bears the same relationship to $S(\cdot, \cdot)$ as $Q(\cdot, t; t)$ bears to $K(\cdot, \cdot)$. In particular, the calculation of $Q(t, \cdot; t)$ can be achieved in the same manner as the calculation of $Q(\cdot, t; t)$. Equations (27) and (28) follow easily, on observing that the Riccati equation associated with $S(\cdot, \cdot)$ has solution $P'(t)$.

3.3. Further connection with the work of Schumitzky. In contrast to Schumitzky [7], who can associate with any Riccati equation an integral kernel, our theory will only associate an integral kernel with a Riccati equation of the form (16). But for equations of this form, one would expect the two theories to coincide. They do of course, but after a fashion. Our theory associates the kernel (4) with (16). Schumitzky's theory associates a quite different kernel, call it $\hat{K}(t, s)$. It is however possible to show, with calculations that are not particularly illuminating, that

$$I - K = [I + w_+] \circ [I - \hat{K}] \circ [I + w_-], \tag{31}$$

where w_+ and w_- are causal and anticausal respectively. This means that $[I - K]^{-1}$ and $[I - \hat{K}]^{-1}$ exist or do not exist together, and statements regarding the existence of resolvent kernels for $\hat{K}$, as in [7], may be replaced by statements regarding the existence of resolvent kernels for K, as here.

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