*On the Stability of Fixed-lag Smoothing Algorithms**

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ABSTRACT: *The stability of fixed-lag smoothers for linear system state estimation is considered. The main result proved is that the smoother is unstable when the usual set of conditions hold which guarantee asymptotic stability of the optimum filter.*

I. Introduction

This paper is concerned with an examination of the stability of the fixed-lag smoothing algorithms obtained by Rauch (1), Meditch (2) and others (3–6). It is shown that these algorithms are generally unstable. Roughly speaking, this instability arises because formulation of the smoothed estimate demands processing of measurement signals by an operator which is the adjoint of the operator required to generate a filtered estimate. Since the operator generating the filtered estimate is generally stable in the appropriate sense, its adjoint will be unstable. [The interpretation of the smoother in terms of a causal filter and its adjoint is due to Kailath and Frost (4).]

As far as we are aware, the only discussion of possible stability difficulties with the linear smoother algorithms occurs in Ref. (5). Of course, the fact that the optimal fixed-lag smoother is generally unstable implies the futility of attempting to use the smoother over all but short time intervals.

An outline of the material to be covered is as follows. In Section II, we review the smoothing algorithms, and in Section III we illustrate their application to a simple example which exhibits the instability property. The main result is established in Section IV for the case when the smoother is time invariant, and in the Appendix for the case where the smoother is time varying. For the time-invariant case, the discrete time result is given.

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II. Fixed-lag Smoothing—Review

Kailath and Frost (4) obtain the smoothing algorithms in a direct manner, and their results will be quoted for the continuous time problem. It is demonstrated in (4) that the results are equivalent to the earlier results of Meditch summarized in (2).

The general problem considered is the following. One is given the set of observations

$$z(s) = H'(s)\,x(s) + w(s), \quad t_0 \leqslant s < t+\Delta, \quad \Delta > 0. \tag{1}$$

Here the superscript prime denotes matrix transposition, and second order statistics are given for the $w(\cdot)$ and $x(\cdot)$ processes, assumed to be gaussian, as follows.*

$$\overline{w(t) = 0}, \quad \overline{w(t)\,w'(s)} = R(t)\,\delta(t-s) \tag{2}$$

for all t and s and for $R(t)$ a positive definite symmetric matrix, and

$$\overline{x(t)} = 0, \quad \overline{x(t)\,x'(t)} < \infty, \quad \overline{x(t)\,w'(s)} = 0 \tag{3}$$

for all t and s with $s > t$.

The problem is to compute that estimate of $x(t)$ denoted by $\hat{x}(t\,|\,t+\Delta)$, which minimizes the error covariance

$$\begin{aligned}\Sigma(t\,|\,t+\Delta) &\triangleq \overline{[x(t) - \hat{x}(t\,|\,t+\Delta)]\,[x(t) - \hat{x}(t\,|\,t+\Delta)]'} \\ &= \overline{\tilde{x}(t\,|\,t+\Delta)\,\tilde{x}'(t\,|\,t+\Delta)},\end{aligned} \tag{4}$$

where the superscript tilde denotes the error or difference between the original state vector and its smoothed estimate. A fixed lag smoother is one for which t is a running variable; measurements arrive continuously, thus with $t+\Delta$ as the present time, the fixed lag smoother at time $t+\Delta$ produces an estimate of $x(t)$ using the measurements available up to $t+\Delta$.

For the stated problem, the smoothed estimate, $\hat{x}(t\,|\,t+\Delta)$ is given, in terms of the filtered estimate and its covariance, by the equation†

$$\hat{x}(t\,|\,t+\Delta) = \hat{x}(t\,|\,t) + \int_t^{t+\Delta} P(t,s)\,H(s)\,R^{-1}(s)\,\nu(s)\,ds, \tag{5}$$

where $\hat{x}(t\,|\,t)$ is, as the notation implies, the minimum variance filtered estimate,

$$\nu(s) = z(s) - H'(s)\,\hat{x}(s\,|\,s) \tag{6}$$

with $\nu(\cdot)$ known as the innovations process, and

$$P(t,s) = \overline{\tilde{x}(t\,|\,t)\,\tilde{x}'(s\,|\,s)}, \tag{7}$$

which is the covariance function of the error in the filtered estimate.

* A superscript bar denotes the expected value operation.

† We are indebted to a reviewer for pointing out that Eq. (5) offers the possibility of achieving smoothing without instability, by mechanizing the equation on a computer. Instability arises, as we shall see, when one attempts to mechanize *dynamic* equations for the smoother, rather than the *nondynamic* Eq. (5). Mechanization directly of (5) will, however, put a greater computational load on the smoother.

The covariance of the error in the smoothed estimate, $\sum(t|t+\Delta)$, is related to the covariance of error in the filtered estimate, $P(t,s)$, and is given by

$$\sum(t|t+\Delta) = P(t,t) - \int_t^{t+\Delta} P(t,s)\,H(s)\,R^{-1}(s)\,H'(s)\,P(s,t)\,\mathrm{d}s. \tag{8}$$

Equations (5) and (8) are general results, in that there has been no condition imposed on the structure of the $x(\cdot)$ process. Also these serve only to relate the smoothing problem to the filtering problem and not to solve the former. But now suppose that $x(\cdot)$ is generated by the signal model

$$\dot{x}(t) = F(t)\,x(t) + G(t)\,u(t), \tag{9}$$

where $u(\cdot)$ is a gaussian process with

$$\overline{u(t)} = 0, \quad \overline{u(t)\,u'(s)} = Q(t)\,\delta(t-s), \quad \overline{u(t)\,w'(s)} = 0 \tag{10}$$

for all t and s, and $x(t_0)$ is a gaussian random variable with

$$\overline{x(t_0)} = 0, \quad \overline{x(t_0)\,x'(t_0)} = P_0, \quad \overline{x(t_0)\,u'(s)} = 0. \tag{11}$$

From (5) it follows that the smoothed and filtered estimates may be related by

$$\begin{aligned}\dot{\hat{x}}(t|t+\Delta) = F(t)\,\hat{x}(t|t+\Delta) + G(t)\,Q(t)\,G'(t)\,P^{-1}(t,t)\,[\hat{x}(t|t+\Delta) - \hat{x}(t|t)]\\ + P(t,t)\,\phi'(t+\Delta,t)\,H(t+\Delta)\,R^{-1}(t+\Delta)\,\nu(t+\Delta)\end{aligned} \tag{12}$$

with the error variance satisfying

$$\begin{aligned}\dot{\sum}(t|t+\Delta) = [F(t) + G(t)\,Q(t)\,G'(t)\,P^{-1}(t,t)]\sum(t|t+\Delta) - G(t)\,Q(t)\,G'(t)\\ + \sum(t|t+\Delta)\,[F(t) + G(t)\,Q(t)\,G'(t)\,P^{-1}(t,t)]'\\ - P(t,t)\,\phi'(t+\Delta,t)\,H(t+\Delta)\,R^{-1}(t+\Delta)\\ \times H'(t+\Delta)\,\phi(t+\Delta,t)\,P(t,t).\end{aligned} \tag{13}$$

The quantity $\phi(\cdot,\cdot)$ occurring in these equations is the transition matrix of $F(t) - P(t,t)\,H(t)\,R^{-1}(t)\,H'(t)$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t,s) = [F(t) - P(t,t)\,H(t)\,R^{-1}(t)\,H'(t)]\,\phi(t,s), \quad \phi(s,s) = I. \tag{14}$$

We shall note subsequently, conditions for the existence of the inverse $P^{-1}(t,t)$ occurring in (12).

Equations (12) through (14) constitute the main results of (4) for the continuous-time smoothing problem. In order to implement these equations, the minimum variance filtered estimate $\hat{x}(t|t)$ and its error covariance function $P(t,t)$ are required. As is well known, these quantities are given by the set of equations:

$$\dot{\hat{x}}(t|t) = F(t)\,\hat{x}(t|t) + P(t,t)\,H(t)\,R^{-1}(t)\,\nu(t), \tag{15}$$

$$\begin{aligned}\dot{P}(t,t) = F(t)\,P(t,t) + P(t,t)\,F'(t) + G(t)\,Q(t)\,G'(t)\\ - P(t,t)\,H(t)\,R^{-1}(t)\,H'(t)\,P(t,t).\end{aligned} \tag{16}$$

The stability of Eq. (12) is what concerns us here. The appropriate homogeneous equation is

$$\dot{z}(t) = [F(t) + G(t)Q(t)G'(t)P^{-1}(t,t)]z(t). \tag{17}$$

Despite the fact that any physical implementation of (12) must contain delay lines [as reference to, for example, (2) will show] so that the physical implementation of (12) will not be via a finite dimensional system, it is still true that the homogeneous Eq. (17) determines the stability of (12). If (17) represents an unstable system, then the inevitable small errors arising in any physical realization of (12) will build up and overpower the true estimate.

III. Example

This example illustrates the calculations required in computing the smoother and illustrates the instability of Eq. (12).

Let the signal process $x(\cdot)$ be given by the solution to the differential equation

$$\dot{x}(t) = -ax(t) + u(t)$$

with

$$\overline{u(t)\,u'(s)} = q\delta(t-s), \quad \overline{x^2(t_0)} = p_0, \quad \overline{u(t)\,x(t_0)} = 0,$$

for some constant $a \geqslant 0$. The process $x(\cdot)$ is observed over the interval $t_0 \leqslant s < t+\Delta$, via the relationship

$$z(s) = hx(s) + w(s),$$

where

$$\overline{w(t)\,w(s)} = \delta(t-s)$$

and the processes $u(\cdot)$ and $w(\cdot)$ are assumed uncorrelated. We are required to find the recursive smoothed estimate, $\hat{x}(t|t+\Delta)$ for a positive constant Δ.

For this example, Eq. (16) becomes

$$\dot{p}(t,t) = -2ap(t,t) + q - h^2p^2(t,t), \quad p(t_0,t_0) = p_0,$$

which can be solved to yield

$$p(t,t) = (1/h^2)[-a + \gamma\tanh(\gamma t + \theta)],$$

where $\gamma = \sqrt{(a^2+qh^2)} > 0$, and $\theta = -\gamma t_0 + \tanh^{-1}(p_0h^2 + a/\gamma)$. The function $p^{-1}(t,t)$ is

$$p^{-1}(t,t) = (1/q)[a + \gamma\tanh(\gamma t + \alpha)],$$

where $\alpha = -\gamma t_0 + \tanh^{-1}(qp_0 - a/\gamma)$.

Now, without writing down the smoothing equations, stability can be checked. Equation (17) becomes, for this example,

$$\dot{z}(t) = [\gamma\tanh(\gamma t + \alpha)]z(t).$$

This is easily seen to represent an unstable system.

To simplify the ensuing analysis, let $t_0 \to -\infty$ to obtain the steady-state result

$$\dot{\hat{x}}(t|t+\Delta) = \gamma\hat{x}(t|t+\Delta) - (a+\gamma)\,\hat{x}(t|t) - (\gamma - a)\exp(-\gamma\Delta)\,\hat{x}(t+\Delta|t+\Delta) + (1/h)(\gamma - a)\exp(-\gamma\Delta)\,z(t+\Delta),$$

which can be used to obtain the transfer function yielding $\hat{x}(t|t+\Delta)$ from $z(t+\Delta)$. This transfer function, denoted by $H_0(s)$, is given by

$$H_0(s) = \frac{(\gamma - a)}{h}\left[\frac{\exp(-s\Delta)}{s+\gamma} + \frac{s+a}{(s-\gamma)(s+\gamma)}(\exp(-\gamma\Delta) - \exp(-s\Delta))\right].$$

This transfer function is also that resulting from the classical Wiener theory where the filter is obtained from a spectral factorization of the power spectral density of the $z(\cdot)$ process. The first term

$$\frac{(\gamma - a)}{h}\frac{\exp(-s\Delta)}{s+\gamma}$$

is the transfer function yielding the filtered estimate delayed by Δ seconds and is the only term remaining if $\Delta \equiv 0$. The second term can be interpreted as a product of two transfer functions. The first term of the product is

$$\frac{s+a}{s+\gamma}$$

and yields the innovations $\nu(\cdot)$ from the measurements $z(\cdot)$, while the second term, viz.

$$\frac{\exp(-\gamma\Delta) - \exp(-s\Delta)}{s-\gamma},$$

yields from $\nu(\cdot)$ the correcting quantity which must be added to the filtered estimate of $x(t)$ to obtain a smoothed estimate. This second term of the product has impulse response

$$\begin{array}{ll} e^{\gamma(t-\Delta)}, & 0 \leqslant t \leqslant \Delta, \\ 0, & \text{elsewhere,} \end{array}$$

and although nominally this is an impulse response that will map bounded inputs into bounded outputs, it seems that any attempted physical implementation of the impulse responses (with other than an approximating impulse response) will be unstable. Figure 1 shows two theoretically possible implementations, neither of which will work in practice.

IV. Main Result

In this section, we establish the instability of the smoother under appropriate conditions, where it is assumed that instability of the smoother is equivalent to instability of

$$\dot{z}(t) = [F(t) + G(t)\,Q(t)\,G'(t)\,P^{-1}(t,t)]\,z(t). \tag{17}$$

In the Appendix, we prove the result for the nonstationary case; for the moment we restrict attention to the stationary case, i.e. F, G, H, Q and R are constant with $t_0 = -\infty$. We introduce the following assumption:

Assumption 1. In terms of the notation given earlier, the pair $[F, H]$ is completely observable, and with D any matrix such that $DD' = Q$, the pair $[F, GD]$ is completely controllable.

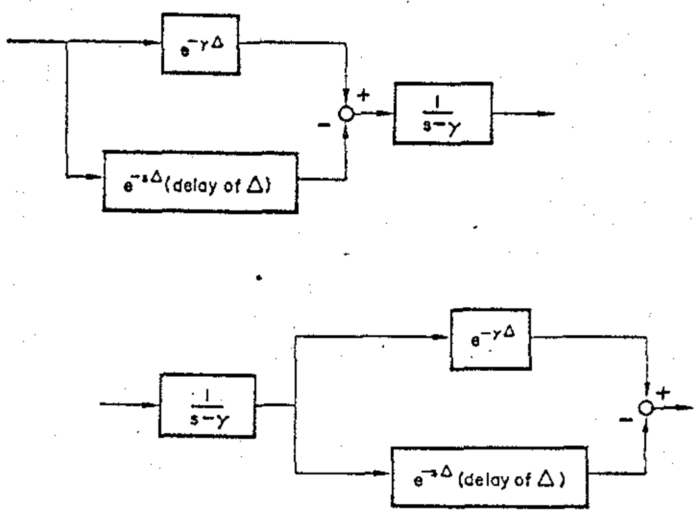

FIG. 1. Two unstable example realizations.

As is known, (11), these conditions guarantee existence of the optimal filter, its exponential asymptotic stability and existence of $P^{-1}(t,t)$, which is constant for all t. We denote $P(t,t)$ in this case by Π.

Theorem I. Under the above Assumption 1 and with the notation of Section II, Eq. (17), and therefore the optimal smoother, is exponentially unstable.

Observe that the theorem claims the smoother is unstable precisely when the filter is asymptotically stable.

Proof of Theorem I. We exhibit a suitable Lyapunov function so that a minor modification of the instability theorem of Cetaev (7) can be invoked. For this purpose, consider the Lyapunov function

$$V(z) = z'(t)\,\Pi^{-1} z(t). \tag{18}$$

Note that the assumption guarantees positive definiteness of Π, see (11), and thus of Π^{-1}. The matrix, Π being $P(t,t)$ for the stationary case, satisfies (16), i.e.

$$F\Pi + \Pi F' + GQG' - \Pi H R^{-1} H' \Pi = 0$$

and so

$$\Pi^{-1} F + F'\Pi^{-1} + \Pi^{-1} GQG' \Pi^{-1} - HR^{-1}H' = 0. \tag{19}$$

Now we compute $\dot{V}(z)$ using (17) and (19). We have

$$\dot{V}(z) = z'(t)[\Pi^{-1}(F+GQG'\Pi^{-1})+(F+GQG'\Pi^{-1})'\Pi^{-1}]z(t)$$
$$= z'(t)[\Pi^{-1}GQG'\Pi^{-1}+HR^{-1}H']z(t). \tag{20}$$

Clearly, $\dot{V}(z)$ is nonnegative, but not necessarily positive definite. Nevertheless, instability follows, by analogy with a well-known stability result, if we can prove that $\dot{V}(z)$ cannot be identically zero along a nonzero trajectory. We prove this by contradiction. Thus, suppose $\dot{V}(z) \equiv 0$, but $z(t_0) \neq 0$. Equation (20) implies

$$QG'\Pi^{-1}z(t) \equiv 0, \quad H'z(t) \equiv 0, \tag{21}$$

and using the first of these equations in (17), it follows that

$$\dot{z}(t) = Fz(t).$$

This equation and the second of relations (21) imply that $z(t_0) = 0$ because $[F, H]$ is completely observable. This is a contradiction and hence (17) is unstable.

Discrete-time Results

The reader is referred to (2) for the pertinent smoothing algorithms. It may be noted that the discrete-time smoother is a finite dimensional system. Stability is still determined by the homogeneous part of the algorithm just as in the continuous-time case. The appropriate Lyapunov function is given by

$$V(z) = z'(k)\,\Pi^{-1}z(k), \tag{22}$$
$$z(k+1) = (\phi+GQG'(\phi')^{-1}\,\Pi^{-1})\,z(k) \tag{23}$$

and Π is the stationary error covariance for the filtered estimate. Computing the forward difference of $V(z)$:

$$V[z(k+1)]-V[z(k)] = z'(k)[\Pi^{-1}\phi^{-1}GQG'(\phi')^{-1}\Pi^{-1}+\phi_1'HR^{-1}H'\phi_1]z(k), \tag{24}$$

where $\phi_1 = \phi+GQG'(\phi')^{-1}\Pi^{-1}$. It follows, using arguments similar to those in the continuous case, that this difference cannot be identically zero along a non-zero trajectory of (23). It can then be concluded that the system (23) is unstable.

Time-varying Continuous-time Result

Assumption 1 is replaced by

Assumption 2. The matrices F, G, H, Q, R and R^{-1} are bounded, the pair $[F, H]$ is uniformly completely observable, and with D any matrix such that $DD' = Q$, the pair $[F, GD]$ is uniformly completely controllable.

These conditions serve to guarantee [see (10) and (11)] that the optimal filter exists even if $t_0 = -\infty$, that it is exponentially asymptotically stable, that $P(t,t)$ is positive definite, guaranteeing existence of $P^{-1}(t,t)$ for all $t \geqslant t_0+\delta_0$, where δ_0 is a constant independent of t_0, and that $P(t,t)$ and $P^{-1}(t,t)$ are bounded.

Alternatively, we may replace Assumption 2 by

Assumption 3. The matrices F, G, H, Q and R^{-1} are bounded, the pair $[F, H]$ is uniformly completely observable, and the matrix $W(t_1, t_0)$ defined by

$$W(t_1,t_0) = \phi(t_1,t_0)\,P_0\,\phi'(t_1,t_0) + \int_{t_0}^{t_1} \phi(t_1,t)\,G(t)\,Q(t)\,G'(t)\,\phi'(t_1,t)\,\mathrm{d}t$$

is nonsingular for some t_1.

These conditions guarantee (12) that the optimal filter is asymptotically stable, that $P^{-1}(t,t)$ exists for $t > t_1$ and that $P(t,t)$ is bounded. The main result is as follows:

Theorem II. Under Assumption 2 (Assumption 3) and with the notation of Section II, Eq. (17), and therefore the optimal fixed-lag smoother, is exponentially unstable (is unstable).

This theorem is proven in the Appendix. Note that, as is shown for the stationary case, the conditions implying asymptotic stability of the filter imply instability of the smoother.

V. Concluding Remarks

Since in nearly all cases the optimal filter is asymptotically stable, it follows that the optimum smoother is unstable. Thus, even if smoothing is carried out over a limited time interval, numerical errors can be expected, while the fixed-lag smoother cannot be expected to operate successfully over an ever-increasing time interval. Since smoothing, if correctly done, offers well understood advantages over filtering, it is to be hoped that suboptimal smoothers can be developed which are free of the computational problems of the optimal smoother. For such suboptimal smoothers to be worth while, they would have to offer performance superior to that of the regular filter. The actual checking of their performance in a given situation could involve great computational effort, which might represent a drawback to their use.

Appendix. Proof of Theorem II

We establish the proof by a generalization of the proof in Section III to time-varying Lyapunov functions. Prior to the development of the proof we shall state several useful inequalities.

Preliminaries

(1) *Uniform complete controllability and observability.* As stated in (9) and (10), the system $[F, GD]$ is uniformly completely controllable if, for some $\delta > 0$, any two of the following three conditions hold for all t (any two imply the third)*

$$0 < \alpha_1(\delta)\,I \leqslant W(t, t+\delta) \leqslant \alpha_2(\delta)\,I, \tag{A.1}$$

$$0 < \alpha_3(\delta)\,I \leqslant \phi(t+\delta, t)\,W(t, t+\delta)\,\phi'(t+\delta, t) \leqslant \alpha_4(\delta)\,I, \tag{A.2}$$

$$\|\phi(t,\tau)\| \leqslant \alpha_5(|t-\tau|) \quad \text{for all } t, \tau, \tag{A.3}$$

* If A and B are symmetric matrices, $A > B$ ($A \geqslant B$) means $A - B$ is positive (non-negative) definite.

where

$$W(t, t+\delta) = \int_t^{t+\Delta} \phi(t,\lambda)\, GQG'\, \phi'(t,\lambda)\, d\lambda. \tag{A.4}$$

Uniform complete observability of $[F, H]$ is defined in a dual manner (10) to the above in terms of the matrix

$$M(t, t+\delta) = \int_t^{t+\Delta} \phi'(\lambda, t)\, H(\lambda)\, H'(\lambda)\, \phi(\lambda, t)\, d\lambda \tag{A.5}$$

and the first inequality becomes

$$0 < \beta_1(\delta)\, I \leqslant M(t, t+\delta) \leqslant \beta_2(\delta)\, I. \tag{A.6}$$

(2) *Vector Schwarz inequality.* For a, c vectors and B a matrix, with

$$a = \int_{t_0}^{t_1} B(t)\, c(t)\, dt$$

we have the inequality

$$\| a \|^2 \leqslant \int_{t_0}^{t_1} \|B(t)\|^2\, \mathrm{d}t \int_{t_0}^{t_1} \| c(t) \|^2\, \mathrm{d}t, \tag{A.7}$$

(3)
$$\| a+b \|^2 \geqslant \| a \|^2 + \| b \|^2 - 2\, \| a \| \cdot \| b \| \tag{A.8}$$

for vectors a, b.

(4)
$$\| Aa \|^2 \leqslant [\mathrm{tr}\,(A'A)]\, \| a \|^2 \tag{A.9}$$

for any matrix A and vector a.

We now proceed by exhibiting a suitable Lyapunov function, $V(z, t)$, and proving that the change in $V(\cdot, \cdot)$ along a length δ of trajectory is strictly increasing away from the origin.

Let the candidate Lyapunov function be given by

$$V(z, t) = z'(t)\, P^{-1}(t, t)\, z(t) \tag{A.10}$$

with $P(t, t)$ given by Eq. (16). Note that under Assumption 2, $P(t, t)$ and $P^{-1}(t, t)$ are bounded, and that under Assumption 3, $P(t, t)$ is bounded. Using Eq. (17) and the relationship

$$\frac{\mathrm{d}}{\mathrm{d}t}[P^{-1}(t, t)] = -P^{-1}(t, t)\, \dot{P}(t, t)\, P^{-1}(t, t)$$

the rate of change of $V(z, t)$ along a trajectory is given by

$$\dot{V}(z, t) = z'(t)\, [H(t)\, R^{-1}(t)\, H'(t) + P^{-1}(t, t)\, G(t)\, Q(t)\, G'(t)\, P^{-1}(t, t)]\, z(t). \tag{A.11}$$

Clearly, $\dot{V}(z, t)$ is nonnegative definite but not positive definite. However, instability follows if we can prove that $V(z, t)$ is strictly increasing along an incremental length of trajectory, δ. [Note that Eqs. (A.10) and (A.11) establish the lack of exponential stability; see, for example, Brockett (8), p. 202.] We prove that $V(z, t)$ is strictly increasing by proving the following claim by contradiction.

Lemma. If δ is the interval length used in the definition of uniform complete observability, then for arbitrary t_2 and $z(t_2)$, (with $t_2 > t_1$ in case Assumption 3 holds),

$$\int_{t_2}^{t_2+\delta} \dot{V}(z, t) \geqslant \alpha_6\, z'(t_2)\, z(t_2) \tag{A.12}$$

for some constant α_6 independent of t_2 and $z(t_2)$.

Proof of this lemma will establish Theorem II.

Suppose that the claim of the lemma is false. Then given an arbitrary $\varepsilon > 0$, there exists a particular t_2 and $z(t_2)$ such that

$$\int_{t_2}^{t_2+\delta} \dot{V}(z,t)\,\mathrm{d}t \leqslant \varepsilon z'(t_2)\, z(t_2), \tag{A.13}$$

which implies from (A.11) that

$$\int_{t_2}^{t_2+\delta} z'(t)\, P^{-1}(t,t)\, G(t)\, Q(t)\, G'(t)\, P^{-1}(t,t)\, z(t)\,\mathrm{d}t \leqslant \varepsilon z'(t_2)\, z(t_2). \tag{A.14}$$

Now from (A.11),

$$\begin{aligned}\int_{t_2}^{t_2+\delta} \dot{V}(z,t)\,\mathrm{d}t &\geqslant \int_{t_2}^{t_2+\delta} z'(t)\, H(t)\, R^{-1}(t)\, H'(t)\, z(t)\,\mathrm{d}t \\ &\geqslant \alpha_7 \int_{t_2}^{t_2+\delta} z'(t)\, H(t)\, H'(t)\, z(t)\,\mathrm{d}t \quad \text{for some positive constant } \alpha_7 \\ &\geqslant \alpha_7 \int_{t_2}^{t_2+\delta} \| H'(t)\, \phi(t,t_2)\,[z(t_2)+\lambda(t)]\, \|^2\,\mathrm{d}t,\end{aligned} \tag{A.15}$$

where $\phi(\cdot,\cdot)$ and $\lambda(\cdot)$ are defined as follows: $\phi(\cdot,\cdot)$ is the transition matrix of $\dot{z} = Fz$; with $\hat{\phi}(\cdot,\cdot)$ the transition matrix of (17),

$$\lambda(t) = \int_{t_2}^{t} \phi(t,\tau)\, G(\tau)\, Q(\tau)\, G'(\tau)\, P^{-1}(\tau,\tau)\, \hat{\phi}(\tau,t_2)\,\mathrm{d}\tau z(t_2). \tag{A.16}$$

(Note that the definitions of $\lambda(\cdot)$, $\phi(\cdot,\cdot)$ and $\hat{\phi}(\cdot,\cdot)$ guarantee that the solution of (17) satisfies

$$z(t) = \phi(t,t_2)\,[z(t_2)+\lambda(t)]$$

which makes the result in (A.15) valid.)

We can obtain an upper bound on $\lambda(t)$ in (A.13) by using (A.7) and $Q = DD'$:

$$\|\lambda(t)\|^2 \leqslant \int_{t_2}^{t} \|\phi(t,\tau)\, G(\tau)\, D(\tau)\|^2\,\mathrm{d}\tau \int_{t_2}^{t} \| D'(\tau)\, G(\tau)\, P^{-1}(\tau,\tau)\, z(\tau)\|^2\,\mathrm{d}\tau,$$

which by (A.14) and the bounds on F, G and Q becomes

$$\|\lambda(t)\|^2 \leqslant \alpha_8(\delta)\, \varepsilon z'(t_2)\, z(t_2) \tag{A.17}$$

for a positive constant $\alpha_8(\delta)$.

Now use (A.7) and (A.8) in (A.15) to obtain

$$\begin{aligned}\int_{t_2}^{t_2+\delta} \dot{V}(z,t)\,\mathrm{d}t \geqslant{}& \alpha_7 \int_{t_2}^{t_2+\delta} \| H'(t)\, \phi(t,t_2)\, z(t_2)\|^2\,\mathrm{d}t \\ &+ \alpha_7 \int_{t_2}^{t_2+\delta} \| H'(t)\, \phi(t,t_2)\, \lambda(t)\|^2\,\mathrm{d}t \\ &- 2\alpha_7 \left[\int_{t_2}^{t_2+\delta} \| H'(t)\, \phi(t,t_2)\, z(t_2)\|^2\,\mathrm{d}t\right]^{\frac{1}{2}} \left[\int_{t_2}^{t_2+\delta} \| H'(t)\, \phi(t,t_2)\, \lambda(t)\|^2\,\mathrm{d}t\right]^{\frac{1}{2}} \\ \geqslant{}& \alpha_7\{\beta_1(\delta)\, \| z(t_2)\|^2 - 2[\beta_2(\delta)\, \| z(t_2)\|^2]^{\frac{1}{2}}\, [n\beta_2(\delta)\, \alpha_8(\delta)\, \varepsilon\, \| z(t_2)\|^2]^{\frac{1}{2}}\}\end{aligned}$$

by applying (A.5), (A.6), (A.9) and (A.17). Finally, by collecting terms, the inequality becomes

$$\int_{t_2}^{t_2+\delta} \dot{V}(z,t)\,\mathrm{d}t \geqslant \alpha_7[\beta_1(\delta) - 2\sqrt{(n)}\, \beta_2(\delta)\, \sqrt{(\alpha_8(\delta))}\, \sqrt{(\varepsilon)}]\, \| z(t_2)\|^2. \tag{A.18}$$

Equation (A.18) contradicts (A.13) if ε is sufficiently small, and the result is thus proven.

Under Assumption 2, $V(z, t)$ satisfies the inequality

$$\alpha_9 \|z\|^2 \leqslant V(z, t) \leqslant \alpha_{10} \|z\|^2$$

for positive constants α_9 and α_{10}, and this inequality with (A.12) establishes exponential instability. Under Assumption 2, $V(z, t)$ satisfies

$$\alpha_9 \|z\|^2 \leqslant V(z, t)$$

and instability without necessarily exponential instability follows.

The discrete-time results should follow analogously.

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