*Stability Properties of Kalman–Bucy Filters**

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BSTRACT: *Conditions are found guaranteeing asymptotic stability, but not necessarily niform asymptotic stability, of Kalman–Bucy filters. Relaxation of a uniform complete ntrollability condition proves the key to generating the results.*

Introduction

The earliest results dealing with the stability of the Kalman–Bucy filter ppear in (1); they were derived by taking the dual of known results for the near regulator problem. While the results are of course valid, they are icomplete, in the sense that there are filtering problems which are not ally the dual of any meaningful regulator problem—an example is given Section II. Further, some filters are derived which, though asymptotically able, are not exponentially asymptotically stable. (An example is given.) ince the stability theory of (1) is restricted to providing (sufficient) conitions for exponential asymptotic stability, the asymptotic stability of lters with the nonexponential asymptotic stability property cannot be tablished by theorems of (1). Our aim here is to present a set of more mplete stability results than (1), which cope with the situations mentioned. The main idea in extending the theory of (1) is not to insist on the uniform omplete controllability of a certain pair of quantities, nor even their comlete controllability; a modified form of complete controllability condition, owever, does guarantee stability results. We consider continuous time sults in Section III, and discrete time results in Section IV.

Example

Consider the one-dimensional autonomous system

$$\dot{x} = 0, \tag{1}$$

$$z = x + v. \tag{2}$$

he system state is x, with $x(0)$ a gaussian random variable of mean x_0 and riance $p_0 > 0$. The variable z is a noisy measurement of x, with v repreenting a zero mean gaussian white noise process of covariance $r\delta(t-\tau)$.

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The process v and random variable $x(0)$ are independent. The optimal filt is defined for $t \geqslant 0$ as

$$\dot{\hat{x}} = pr^{-1}(z-\hat{x}), \quad \hat{x}(0) = x_0, \tag{3}$$

where

$$\dot{p} = -\frac{p^2}{r}, \quad p(0) = p_0. \tag{4}$$

It is easily found that

$$p(0) = \frac{1}{tr^{-1}+p_0^{-1}}$$

so that the optimal filter is

$$\dot{\hat{x}} = \frac{1}{t+rp_0^{-1}}(z-\hat{x}). \tag{5}$$

The following points should be noted. First, the optimal filter is evidentl asymptotically stable, but not exponentially asymptotically stable. Second the dual optimal control problem would require minimization of a perfor mance index of the form

$$\int_{-\infty}^{0} ru^2\,dt + p_0 x^2(0).$$

In view of the lower limit of $-\infty$ on the integral, this is not the sort of problem which optimal control theory purports to solve. Third, the system (1) is no uniformly completely controllable or even completely controllable.

III. Continuous-time Results

We first review the well-known solution to the filtering problem (1). We are given a plant with input noise $u(\cdot)$ and output noise $v(\cdot)$:

$$\dot{x} = F(t)x + G(t)u,$$

$$z = H'(t)x + v.$$

The random processes $u(\cdot)$ and $v(\cdot)$ are gaussian, independent, of zero mean and

$$E[u(t)u'(\tau)] = Q(t)\delta(t-\tau), \quad E[v(t)v'(\tau)] = R(t)\delta(t-\tau)$$

with $R(t)$ positive definite. The initial time is t_0, and $x(t_0)$ is a gaussian random variable of mean x_0 and variance P_0, with $x(t_0)$ independent of $u(\cdot)$ and $v(\cdot)$. The optimum filter yielding a minimum variance estimate $\hat{x}(t)$ of $x(t)$ is

$$\dot{\hat{x}} = F(t)\hat{x} + P(t)H(t)R^{-1}(t)H'(t)(z-\hat{x}),$$

where

$$\dot{P} = PF' + FP - PHR^{-1}H'P + GQG', \quad P(t_0) = P_0. \tag{10}$$

The matrix $P(t)$ exists and is nonnegative definite for all $t \geqslant t_0$.

We need the following preliminary result.

Lemma 3.1. Let $\Phi(\cdot,\cdot)$ be the transition matrix of $\dot{y} = Fy$ and let $P(\cdot)$ be defined as above. Then the nullspace of $P(t_1)$ and the nullspace of

$$W(t_1, t_0) = \Phi(t_1, t_0) P_0 \Phi'(t_1, t_0) + \int_{t_0}^{t_1} \Phi(t_1, t) G(t) Q(t) G'(t) \Phi'(t_1, t)\, dt \tag{11}$$

are the same; in particular, both matrices are either simultaneously singular or simultaneously nonsingular.

Remark. The matrix $W(t_1, t_0)$ is a generalized form of controllability matrix associated with (6)–(8).

Proof of lemma: Suppose $P(t_1)$ is singular, and that $P(t_1)a = 0$ for some $a \neq 0$. Let $\Phi_f(\cdot,\cdot)$ be the transition matrix associated with

$$\dot{y} = (F - PHR^{-1}H')y \tag{12}$$

and define

$$S(t) = \Phi_f(t_1, t) P(t) \Phi_f'(t_1, t). \tag{13}$$

Observing that

$$\frac{d}{dt} \Phi_f(t_1, t) = -\Phi_f(t_1, t)(F - PHR^{-1}H')$$

it follows that

$$\begin{aligned} \dot{S}(t) &= -\Phi_f(t_1, t)[(F - PHR^{-1}H')P - \dot{P} + P(F' - HR^{-1}H'P)]\Phi_f'(t_1, t) \\ &= \Phi_f(t_1, t)[PHR^{-1}H'P + GQG']\Phi_f'(t_1, t). \end{aligned} \tag{14}$$

From (13), we see that $S(t) \geqslant 0$ and thus $a'S(t)a \geqslant 0$ for all t in $[t_0, t_1]$. From (14) we have that $\dot{S}(t) \geqslant 0$ for all t and thus $a'\dot{S}(t)a \geqslant 0$ for all t in $[t_0, t_1]$. By the definition of a as being in the nullspace of $P(t_1)$, we have $a'S(t_1)a = 0$. Therefore $a'S(t)a = 0$ for all t in $[t_0, t_1]$; by the nonnegativity of $S(t)$ and the formula (13), we have

$$P(t)\Phi_f'(t_1, t)a = 0, \quad t \in [t_0, t_1]. \tag{15}$$

Also, since $S(t)a = 0$ for $t \in [t_0, t_1]$, we have $\dot{S}(t)a = 0$ for $t_0 \leqslant t \leqslant t_1$, and so (14) yields

$$Q(t)G'(t)\Phi_f'(t_1, t)a = 0, \quad t \in [t_0, t_1]. \tag{16}$$

Now by the definition of $\Phi_f(\cdot,\cdot)$,

$$\frac{d}{dt}\Phi_f'(t_1, t)a = -(F' - HR^{-1}H'P)\Phi_f'(t_1, t)a = -F'\Phi_f'(t_1, t)a$$

by (15). This implies that

$$\Phi_f'(t_1, t)a = \Phi'(t_1, t)a$$

and so (15) and (16) yield

$$P(t)\Phi'(t_1, t)a = 0, \quad Q(t)G'(t)\Phi'(t_1, t)a = 0. \tag{17}$$

It follows simply from the definition (11) that a is in the nullspace of $W(t_1, t_0)$.

For the converse, suppose that a is in the nullspace of $W(t_1, t_0)$. It follows that

$$P_0 \Phi'(t_1, t_0) a = 0, \quad Q(t) G'(t) \Phi'(t_1, t) a = 0. \tag{18}$$

Now postmultiply the differential equation for P, viz. (10), by $\Phi'(t_1, t) a$ this yields

$$\dot{P}\Phi'(t_1, t) a - PF' \Phi'(t_1, t) a = (F - PHR^{-1} H') P\Phi'(t_1, t) a + GQG' \Phi'(t_1, t) a.$$

Using the second of (18) and the definition of $\Phi(\cdot, \cdot)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} [P(t) \Phi'(t_1, t) a] = (F - PHR^{-1} H') [P(t) \Phi'(t_1, t) a]$$

and using the first of (18), we obtain

$$P(t) \Phi'(t_1, t) a = 0.$$

On setting $t = t_1$, the equation $P(t_1) a = 0$ is recovered, as required. This proves the lemma.

The lemma provides a necessary and sufficient condition for the invertibility of $P(t_1)$ which we shall use in deriving a stability result. Notice that if $P(t_1)$ is invertible, $P(t)$ is invertible for all $t \geqslant t_1$. In particular if P_0 is nonsingular, $P(t)$ is nonsingular for all t.

The stability result we are seeking will be proved using a Lyapunov function

$$V(y, t) = y'(t) P^{-1}(t) y(t) \tag{19}$$

for the system (12), which is the homogeneous equation associated with the filter (9). For $V(y, t)$ to be a valid candidate Lyapunov function, it is necessary that $P^{-1}(t) \geqslant \alpha_1 I > 0$ for all t, or that $P(t) \leqslant \alpha_2 I$ for all t, where α_1 and α_2 are positive constants. We now note conditions for such bounds to exist, established in (2).

We define the matrices

$$M(t_1, t_2) = \int_{t_1}^{t_2} \Phi(t_2, t) G(t) Q(t) G'(t) \Phi'(t_2, t) \, \mathrm{d}t \tag{20}$$

and

$$N(t_1, t_2) = \int_{t_1}^{t_2} \Phi'(t, t_2) H(t) R^{-1}(t) H'(t) \Phi(t, t_2) \, \mathrm{d}t. \tag{21}$$

If $N^{-1}(t - \sigma, t)$ exists for all $t \geqslant t_0 + \sigma$, then (2) shows that

$$P(t) \leqslant N^{-1}(t - \sigma, t) + M(t - \sigma, t). \tag{22}$$

By assuming that F is bounded [which ensures that $\| \Phi(t, \tau) \| \leqslant \alpha_3 \exp \alpha_4 (t - \tau)$ for constants α_3 and α_4] and that G and Q are bounded, it follows that $M(t - \sigma, t)$ is bounded. By assuming that $R^{-1}(t)$ is bounded and that $[F(t), H(t) R^{-1}(t)]$ is uniformly completely observable, it follows that there exists a σ such that $N^{-1}(t - \sigma, t)$ is bounded. In summary, we have:

Lemma 3.2. If F, G, Q and R^{-1} are bounded, and if $[F, H]$ is uniformly completely observable, then $P(t)$ is bounded.

Now we can state the stability result.

Theorem 3.1. With quantities as defined earlier, suppose that F, G, H, Q and R^{-1} are bounded, that $[F, HR^{-\frac{1}{2}}]$ is uniformly completely observable, and that $W(t_0, t_1)$ is nonsingular for some t_1. Then the optimal filter is asymptotically stable.

Remark. Notice that the assumption that H is bounded is new.

Proof: By Lemma 3.1, $P^{-1}(t)$ exists for all $t \geq t_1$, and by Lemma 3.2, $P^{-1}(t) \geq \alpha_1 I > 0$ for some positive constant α_1. As a Lyapunov function for

$$\dot{y} = (F - PHR^{-1}H')y \tag{12}$$

we adopt (for $t \geq t_1$)

$$V(y, t) = y' P^{-1} y. \tag{19}$$

Differentiation yields

$$\dot{V} = -y'(HR^{-1}H' + P^{-1}GQG'P^{-1})y. \tag{23}$$

The uniform complete observability of $[F(t), H(t)R^{-\frac{1}{2}}(t)]$ implies uniform complete observability of $[F - PHR^{-1}H', HR^{-\frac{1}{2}}]$ by a theorem of (3). The fact that

$$\dot{V} \leq -y'HR^{-1}H'y$$

and the fact that $[F - PHR^{-1}H', HR^{-\frac{1}{2}}]$ is uniformly completely observable guarantee asymptotic stability of (12) by the main theorem of (4).

The stability results of (1) and (2) require $[F, GQ^{\frac{1}{2}}]$ to be uniformly completely controllable. This ensures existence of an upper bound on P^{-1}, which is sufficient to guarantee that the asymptotic stability is uniform; it is exponential also because $F - PHR^{-1}H'$ is bounded. The uniform complete controllability condition is here replaced by one demanding nonsingularity of (11) for some t_1.

For the example of Section II, we see that the conditions of the theorem are satisfied; in particular, $W(t_0, t_1)$ is nonsingular for all t_1 since P_0 is nonsingular. Notice that $[F, GQ^{\frac{1}{2}}]$ is not uniformly completely controllable. A Lyapunov function is provided by $(tr^{-1} + p_0^{-})y^2$, which is unbounded. Thus exponential asymptotic stability cannot be concluded from this Lyapunov function, and the system does not in fact have the exponential asymptotic stability property.

One might well ask what happens if $W(t_0, t_1)$ is singular for all t_1. We can answer this question in rough terms. The matrix $W(t_0, t_1)$ is in fact $E[x(t_1)x'(t_1)]$, the covariance of the plant state at time t_1, so that if this is singular for all t_1, we know exactly one or more linear functionals of the state vector, and there is no need to estimate them. By a change of coordinate basis, we can obtain $x(t)$ in the form

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

where $x_2(t)$ is exactly known for all t and $E[x_1(t)x_1'(t)]$ is nonsingular for $t \geq t_1$. We then proceed merely to estimate $x_1(t)$, and we are guaranteed that the matrix $W(t_0, t_1)$ associated only with x_1 will be nonsingular. The filter estimating $x_1(t)$ can then be asymptotically stable.

Although as we have just argued, it is possible to get around the difficulty associated with singular $W(t_0, t_1)$, it is more difficult to get around the requirement for uniform complete observability. If states which are not observable are asymptotically stable, the situation may be satisfactory, but if not, then trouble should be expected. For example, consider the plant

$$\dot{x} = 0,$$

$$z = \frac{1}{t+1}x + v$$

with $p_0 = 1$, $E[v(t)v'(\tau)] = \delta(t-\tau)$. The uniform complete observability property is lacking. The associated Riccati equation is

$$\dot{p} = -\frac{p^2}{(t+1)^2}$$

which yields

$$p(t) = \frac{t+1}{2t+1}.$$

The homogeneous equation associated with the filter is

$$\dot{y} = -\frac{1}{(t+1)(2t+1)}y$$

with solution

$$y(t) = \frac{t+1}{2t+1}y(0).$$

Clearly, there is no asymptotic stability property here.

IV. Discrete-time Systems

The results for discrete-time systems are indeed very similar. As the plant, we take

$$x(k+1) = F(k)x(k) + G(k)u(k+1), \tag{24}$$

$$z(k) = H'(k)x(k) + v(k). \tag{25}$$

Here $u(\cdot)$ and $v(\cdot)$ are independent gaussian white noise processes, with zero mean and covariances

$$E(u(k)u'(l)] = Q(k)\delta(k-l), \quad E[v(k)v'(l)] = R(k)\delta(k-l). \tag{26}$$

To derive stability results, it appears necessary to assume that $F(k)$ and $R(k)$ are invertible for each k.

The *initial state* $x(k_0)$ of (24) is assumed to be a gaussian random variable at mean x_0 and covariance P_0, and is also assumed to be independent of the $w(\cdot)$ and $v(\cdot)$ processes. The optimal filter is defined by

$$\left.\begin{aligned} \hat{x}(k) &= F(k-1)\hat{x}(k-1) + K(k)\,[z(k) - H'(k)\,F(k-1)\,\hat{x}(k-1)], \\ \hat{x}(k_0) &= x_0, \end{aligned}\right\} \quad (27)$$

where

$$K(k) = \hat{P}(k)\,H(k)\,[H'(k)\,\hat{P}(k)\,H(k) + R(k)]^{-1}, \quad (28)$$

$$\left.\begin{aligned} \hat{P}(k) &= F(k-1)\,P(k-1)\,F'(k-1) + G(k-1)\,Q(k)\,G'(k-1), \\ P(k) &= \hat{P}(k) - K(k)\,H'(k)\,\hat{P}(k), \\ P(k_0) &= P_0. \end{aligned}\right\}. \quad (29)$$

The analog of Lemma 3.1, essentially proved in [(5), see p. 238], is as follows:

Lemma 4.1. Let $\Phi(\cdot,\cdot)$ be the transition matrix of $x(k+1) = F(k)\,x(k)$. ith quantities as defined above and with $R(k)$ *nonsingular for all* k, the llspace of $P(k_1)$ and the nullspace of

$$W(k_0, k_1) = \Phi(k_1, k_0)\,P_0\,\Phi'(k_1, k_0) + \sum_{k=k_0+1}^{k_1} \Phi(k_1, k)\,G(k-1)\,Q(k)\,G'(k-1)\,\Phi'(k_1, k) \quad (30)$$

e the same.

We refer the reader to (5) for a proof. The point of this lemma is to provide nditions which guarantee invertibility of $P(k_1)$. Notice that without the sumption that $F(k)$ is invertible for all k, so that $\Phi(k,l)$ is invertible for all and l, the invertibility of $P(k_1)$ does not imply invertibility of $P(k)$ for $\geqslant k_1$, because, as may be seen from (30), $W(k_0,k)$ for $k \geqslant k_1$ may have rank ss than $W(k_0,k_1)$ if $\Phi(\cdot,\cdot)$ is not always invertible.

To ensure invertibility of $P(k)$ for all $k \geqslant k_1$ if $P(k_1)$ is invertible is the main eason for our demanding invertibility of the $F(k)$. The assumption that $R(k)$ invertible is required to make Lemma 4.1. go through.

We seek a stability result for

$$y(t) = [F(k-1) - K(k)\,H'(k)\,F(k-1)]\,y(k-1) \quad (31)$$

which is the homogeneous equation associated with the optimal filter) hrough use of a Lyapunov function

$$V[y(k), k] = y'(k)\,P^{-1}(k)\,y(k). \quad (32)$$

An overbound for $P(k)$ can be achieved as in the continuous case. See (5) 234, where a variant of the following result is proven:

Lemma 4.2. If F, F^{-1}, G, Q and R^{-1} are bounded and if $[F, HR^{-\frac{1}{2}}]$ is niformly completely observable, then $P(k)$ is bounded.

This bound on $P(k)$ is necessary if (32) is to be a satisfactory Lyapunov unction. Using arguments as in (5), p. 241, we can show that

$$V[y(k), k] - V[y(k-1), k-1] \leqslant -y'(k)\,H(k)\,R^{-1}(k)\,H'(k)\,y(k) \quad (33)$$

and the asymptotic stability of (31) can be deduced from the unif complete observability of $[F, HR^{-\frac{1}{2}}]$ and an assumption that H is bound. Summarizing:

Theorem 4.1. With quantities as defined earlier, suppose that F, F^{-1}, G, Q and R^{-1} are bounded, that $[F, HR^{-\frac{1}{2}}]$ is uniformly completely observa and that $W(k_0, k_1)$ is nonsingular for some k_1. Then the optimal filte asymptotically stable.

The remarks which we have made for the continuous-time problem apply, *mutatis mutandis*, to the discrete-time problem. As an exam illustrating application of the theorem, consider the plant

$$x(k+1) = x(k),$$

$$z(k) = x(k) + v(k)$$

with $x(0)$ gaussian and of mean x_0 and variance $P_0 > 0$, $v(k)$ a white gaussi zero mean process with

$$E[v(k)\,v(l)] = \delta(k-l)$$

and $v(\cdot)$ and $x(k_0)$ independent. Clearly, the conditions of the theorem a satisfied, with $W(k_0, k_1)$ nonsingular for all k_1 because P_0 is nonsingular.

The optimal filter can be computed to be

$$\hat{x}(k) = \frac{1}{k+1}[z(k) - \hat{x}(k-1)] + \hat{x}(k-1) = \frac{k}{k+1}\hat{x}(k-1) + \frac{1}{k+1}z(k).$$

Though asymptotically stable, this filter is not exponentially asymptotical stable.

V. Conclusions

The main results are to be found in Theorems 3.1 and 4.2, which yie conditions for non-exponential asymptotic stability of optimal filter Uniform observability is almost essential. At first glance, some modifi form of complete controllability is also needed, though one can argue wi the aid of a coordinate basis change that this requirement can be dispens with after some modification to the filter.

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·ɢʜ Energy Physics, Vol. III, edited by E. H. S. Burhop. 380 pages, diagrams, illustr., 6 × 9 in. New York, London, Academic Press, 1969. Price, $19.50.

Volume III of the four-volume series "High Energy Physics" contains three complementary articles, as follows: B. H. Bransden on the *K meson–nucleon interaction*, E. H. S. Burhop on *mesonic atoms* and C. Rubbia on *weak interaction physics*.

In Bransden's article the phenomenology of the KN and $\bar{K}$N interaction processes is discussed first, including an account of the Y* states appearing as resonances in the latter interaction and a phenomenological treatment of the multiple scattering effects within the K-deuterium and $\bar{K}$-deuterium systems. An account of the dispersion theoretic approach to KN and $\bar{K}$N interactions is given, from which some evidence on the π interaction and on the KNY coupling constants is derived.

Burhop's article is concerned with the atomic, molecular and nuclear information obtained from observations on mesonic atoms. For mesons in Bohr orbits, the effects of the nuclear shape and the nuclear excitations on the energy and intensity of their X-ray emission lines are discussed, as well as the effects of the strong interactions responsible for their nuclear capture reactions. However, the information obtained about elementary particle interactions from these studies is discussed rather little in this article, being left in part to the article by Rubbia. Burhop also includes a detailed discussion about the lightest muonic atoms (pμ and dμ) and molecules (such as pμp and pμd, for example). These interesting systems undergo quite complicated scattering, transfer and nuclear reaction processes, and the discussion given could well have been expanded; in fact, this fascinating topic really deserves a separate article by itself. Muonium (the system μ^+e^-) is also discussed in considerable detail, including a section about its chemistry.

Rubbia's article gives a concise presentation of the major items of information we have concerning the weak interactions, as viewed by the phenomenologist. The discussion given is sketchy but provides a good summary outline of this topic, with adequate references. The information obtained about the muon capture interaction from the study of muonic atoms and molecules is discussed here, complementing the approach in the previous article. Special attention is given to the evidence on CPT and CP symmetries, the CP-violating decay interactions observed for the neutral K-mesons being discussed in considerably more detail than are the weak interactions better known to us.

These articles must be regarded as reviews of the state of our knowledge on these topics, rather than as textbook material. They are not expository in character, but aim rather to coordinate the work reported in the literature, giving fairly complete references. This reviewer feels that such review articles should give rather more critical guidance to the reader than is generally the case here. It is puzzling that reference should be made equally to papers which are good and reliable as to papers which could well be forgotten. Early, unreliable data is often presented together with recent accurate work, sometimes giving the impression of a significant discrepancy. An example is given by the data on π^- capture in ^{3}He (see p. 192); "the results obtained are not in agreement" gives the reader the impression of controversy, whereas there is every reason to accept the result obtained with high statistics in a helium bubble chamber, which simply supersedes the data obtained six years earlier.

The reviewer noted many misprints and mis-spellings of names, at the rate of about one per page, which should have been picked up in proofreading. For example, in Eq. (III-59) on p. 36 one a has no superscript and there appears an undefined quantity A^1_2 (presumably a misprint for A^1_1). Again, Pusterla is spelt both

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