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Scattering Matrix Synthesis Via Reactance Extraction

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Abstract—This paper considers the problem of multiport passive network synthesis for a rational bounded real scattering matrix $S(p)$ using state-space ideas. The technique is to extract reactances from a network synthesizing the prescribed $S(p)$ to yield a resistive coupling network that may contain transformers, resistors, and if there is no reciprocity constraint, gyrators. Here a minimum number of reactive elements are always sufficient to give a synthesis, reciprocal and nonreciprocal.

The synthesis method relies on an algebraic criterion for a class of rational matrices termed "discrete bounded real" that solves the passive synthesis problem. Starting with an arbitrary coordinate basis, a state-space description is found for a discrete bounded real matrix derived from the given bounded real $S(p)$. A coordinate transformation of the state-space is found from the solution of an algebraic equation, which yields almost immediately a passive synthesis for $S(p)$. When a synthesis for a symmetric bounded real $S(p)$ is required to use no gyrators, a further state-space basis transformation is found that preserves passivity and simultaneously inserts reciprocity.

The contributions of this paper, to scattering matrix synthesis, are twofold: first, to solve the nonreciprocal synthesis problem including the mimimal reactive element equivalence problem via calculations that need not rely on application of any classical ideas, and second, to present for the first time a reciprocal synthesis of a passive symmetric scattering matrix with a minimum number of reactances using state-space ideas.

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I. Introduction

In many recent papers [1]–[7], [19], [20], the problem of multiport passive network synthesis using state-space ideas has been considered. The synthesis of a rational bounded real scattering function or matrix is discussed in [1], while [2]–[7] discuss the synthesis of a rational positive real impedance function or matrix, and [19] and [20] consider syntheses of networks with two-variable and several-variable matrices, respectively.

In Youla and Tissi's paper [1], a new approach is considered for the synthesis of a passive n-port network with a rational bounded real scattering matrix. Here a network N synthesizing a rational bounded real $S(p)$ is regarded as a cascade connection of two networks N_1 and N_2, where N_1 is memoryless and may contain transformers, resistors, and if there is no reciprocity constraint, gyrators; N_2 is simply inductors and capacitors uncoupled from each other (see Fig. 1). Hence, we call the synthesis method a reactance extraction synthesis. The number of reactive elements used is always a minimum, being the same as the degree [8] of $S(p)$. The synthesis procedure of [1] is not at all simple; the process of bordering a certain matrix into a paraunitary matrix using the theory of Oono and Yasuura [9] is necessary.

In this paper, the same technique of reactance extraction synthesis is adopted. The method involves a more direct approach to solving the problem, because here an algebraic criterion for discrete bounded real rational matrices (to be defined later) is developed, which finds immediate application to the passive synthesis problem. Both the nonreciprocal and the reciprocal synthesis use a minimal number of reactive elements.

The state-space approach to network synthesis seems to have originated with [1]. As remarked above, the technique of [1] for synthesizing a passive scattering matrix relied on application of some classical ideas, going beyond even the notion of a matrix spectral factorization. No procedure could be given for reciprocal synthesis. In [2] and [3], the impedance synthesis problem was considered, and the extent to which non-state-space ideas was used was small. A certain matrix required for a coordinate basis change needed to be calculated, and one procedure suggested for this was spectral factorization. Since that time though, other calculation procedures have been studied, relying on the solution of a quadratic matrix inequality [10]. Reference [10] coupled with [2] and [3] then constitutes a complete purely state-space solution (including computational procedures) of the synthesis problem for passive impedances, and even solves the equivalence problem. Neither [2] nor [3] solves the *reciprocal synthesis problem*. Reference [4] provided a very good statement of various ways of how the reciprocal synthesis problem may have been attacked, but the first nonclassical solutions of the reciprocal synthesis problem are given in [5]–[7] all for impedances. Reference [5] describes a synthesis using a minimal number of reactances, and some of the ideas of this reference proved of great help in developing the reciprocal synthesis section of this paper. Reference [6] outlines two techniques, both nonminimal in the number of reactive elements, while one of these is described in detail in [7]. The synthesis of [7] is a state-space parallel of the classical Bayard synthesis, and uses a minimal number of resistors.

The contributions of this paper, to scattering matrix synthesis, are really twofold: first, to reduce the nonreciprocal problem to calculations independent of any classical notions while using minimal number of reactances, and second, to present for *the first time* a *reciprocal* synthesis of a passive symmetric scattering matrix using a minimum number of reactances via state-space techniques.

II. Review of State-Space Description for Rational Matrices

In [11], it is shown that any rational $n \times n$ matrix $W(p)$ with $W(\infty)$ finite possesses a decomposition of the form

$$W(p) = J + H'(pI - F)^{-1}G \tag{1}$$

where F, G, H, and J are real constant matrices (the prime denotes transpose). Any quadruple $\{F, G, H, J\}$ for which this formula holds is termed a realization of $W(p)$, and a minimal realization if F has minimal dimension. Several important properties of minimal realizations are 1) if $\{F, G, H, J\}$ is one minimal realization of $W(p)$, others are given by $\{TFT^{-1}, TG, (T^{-1})'H, J\}$ for arbitrary nonsingular T; 2) if $\{F, G, H, J\}$ is minimal, the realization is completely controllable and observable, and rank $[G, FG, \cdots, F^{k-1}G] = \text{rank}\,[H, F'H, \cdots, (F')^{k-1}H] = k$ where F is $k \times k$; 3) the minimal dimension k of F is the McMillan degree of the matrix $W(p)$, see [12], which is also the minimal number of reactive elements used in any passive synthesis of a bounded real (or positive real) $W(p)$.

III. Lumped Passive Scattering Synthesis Problem

It is well known that any $n \times n$ scattering matrix $S(p)$ that characterizes a linear, time-invariant, lumped, finite, passive n-port network is a real rational bounded real matrix, i.e., it satisfies the properties [8],

$$S(p) \text{ has all elements analytic in } \operatorname{Re} p \geq 0 \tag{2a}$$

$$I_n - S'^*(p)S(p) \geq 0 \text{ in } \operatorname{Re} p > 0 \tag{2b}$$

where the superscript * denotes conjugate and $A \geq 0$ (> 0) means that A is a nonnegative definite (positive definite) Hermitian matrix.

The bounded real properties in (2) can alternatively be characterized by the following algebraic criterion [13].

Lemma 1: Let $S(p)$ be a matrix of rational functions such that $S(\infty)$ is finite, and let $\{\hat{F}, \hat{G}, \hat{H}, \hat{J}\}$ be a minimal realization of $S(p)$. Suppose that all entries of $S(p)$ are analytic in the half plane, $\operatorname{Re}[p] \geq 0$. Then $S(p)$ is bounded real if, and only if, there exist real matrices P, $\hat{L}$, and $\hat{W}_0$ with P positive definite symmetric such that

$$P\hat{F} + \hat{F}'P = -\hat{H}\hat{H}' - \hat{L}\hat{L}' \tag{3a}$$

$$-P\hat{G} = \hat{H}\hat{J} + \hat{L}\hat{W}_0 \tag{3b}$$

$$I - \hat{J}'\hat{J} = \hat{W}_0'\hat{W}_0. \tag{3c}$$

The synthesis problem can be stated briefly as follows. Suppose given an $n \times n$ bounded real matrix $S(p)$ (perhaps symmetric) together with n positive numbers [14] r_1,

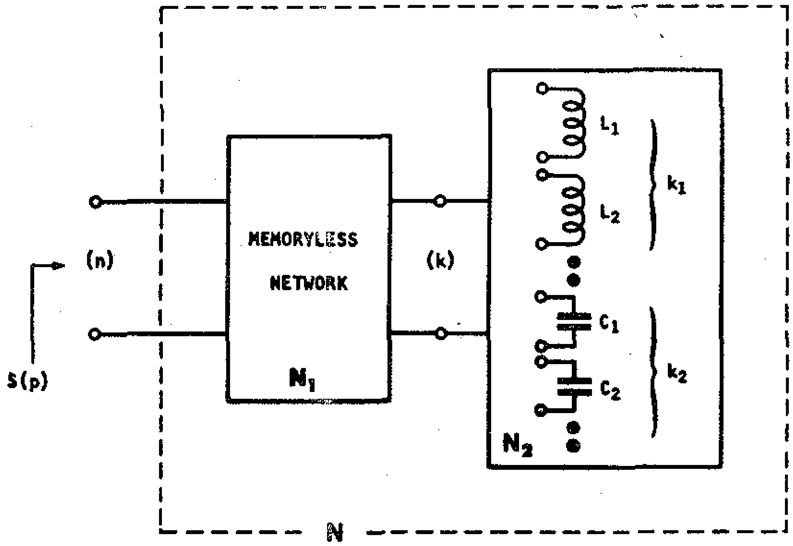

Fig. 1. Reactance extraction to obtain a resistive coupling network.

$r_2, \cdots, r_n$, where r_i is the ith normalization number associated with port i, realize $S(p)$ as the scattering description of an n-port passive network normalized to r_i at port i. Let a network N that synthesizes the prescribed $S(p)$ consist of an interconnection of two passive subnetworks N_1 and N_2 as shown in Fig. 1.

For the moment only the reciprocal case shall be considered. Suppose N_2 consists of k_1 inductors and k_2 capacitors (where $k_1 + k_2 = k$ is the degree of $S(p)$) whose values are $L_1, L_2, \cdots, L_{k1}, C_1, C_2, \cdots, C_{k2}$. If we assume that N_1 is described by a scattering matrix S_a, then because of the reciprocity, no energy-storing elements, and the passivity of N_1, S_a is symmetric, constant, and

$$I_{n+k} - S_a^2 \geq 0. \tag{4}$$

Of course, S_a is normalized to the given set of positive numbers r_i, $i = 1, \cdots, n$, at the input n-ports. The normalization numbers for the output k-ports of N_1 may be arbitrarily chosen (a different set of numbers results in a different matrix S_a for N_1). A useful choice is

$$\begin{aligned} r_{n+l} &= L_l \qquad l = 1, 2, \cdots, k_1 \\ &= \frac{1}{C_l} \qquad l = k_1 + 1, \cdots, k. \end{aligned} \tag{5}$$

With the above choice of normalization, N_2 is described by a lossless scattering matrix

$$B(p) = \frac{p-1}{p+1} \cdot \Sigma$$

where

$$\Sigma = [I_{k1} \dotplus (-1)I_{k2}]. \tag{6}$$

If the resulting S_a having the above normalization numbers is partitioned as:

$$S_a = \begin{array}{cc} \begin{bmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{bmatrix} & \begin{array}{c} (n) \\ (k) \end{array} \\ \begin{array}{cc} (n) & (k) \end{array} & \end{array} \tag{7}$$

where $S_{11} = S_{11}'$ and $S_{22} = S_{22}'$, then the interconnection of N_1 and N_2 results in N having a scattering matrix $S(p)$ given by

$$S(p) = S_{11} + S_{12}(sI_k - \Sigma S_{22})^{-1} \Sigma S_{12}'$$

with

$$s = \frac{p+1}{p-1} \quad \text{or} \quad p = \frac{s+1}{s-1}. \tag{8}$$

Therefore, if we regard (8) as a change of variables, $S(p)$ may naturally be regarded as a function of the complex variable s, i.e.,

$$S(p) = S\left(\frac{s+1}{s-1}\right) = W(s)$$

then

$$W(s) = S_{11} + S_{12}(sI_k - \Sigma S_{22})^{-1} \Sigma S_{12}'. \tag{9}$$

Thus, to solve the reciprocal passive synthesis problem, we must decompose $S(p)$ (after a change of variable of (8)) in the form (9) such that S_a as given by (7) satisfies the condition (4) and the symmetry requirement that $S_a = S_a'$. To solve the equivalence problem of reciprocal passive synthesis, we must find all possible decomposition of $S(p)$ such that the stated properties of the decomposition hold.

Evidently $W(s)$ of (9) always has $W(\infty)$ finite. Examination of (9) reveals that one possible realization for $W(s)$ in the sense of (1) in Section II, is given by

$$\{F, G, H, J\} = \{\Sigma S_{22}, \Sigma S_{12}', S_{12}', S_{11}\},$$

thus

$$S_a = [I_n \dotplus \Sigma]M \qquad M = \begin{bmatrix} J & H' \\ G & F \end{bmatrix}. \tag{10}$$

Therefore, the symmetry condition and the passivity condition (4) of S_a in terms of M are

$$I_{n+k} - M'M \geq 0 \tag{11}$$

and

$$(I_n \dotplus \Sigma)M = M'(I_n \dotplus \Sigma). \tag{12}$$

In conclusion, the passive reciprocal synthesis problem is solved if we can derive a minimal realization $\{F, G, H, J\}$ for $W(s)$ such that (11) and (12) hold. Further, any passive reciprocal synthesis is such that the associated S_a and M satisfy (11) and (12) and define a particular minimal realization of $W(s)$.

In the case of a nonreciprocal passive synthesis where gyrators are allowed, all reactances may be assumed inductive (or capacitive) [8]. The real constant scattering matrix S_a for N_1 is generalized to

$$S_a = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \tag{13}$$

with only one constraint that it satisfies the passivity condition of (4). Then instead of (9), $W(s)$ now becomes

$$W(s) = S_{11} + S_{12}(sI_k - S_{22})^{-1}S_{21}. \tag{9a}$$

Thus, one possible realization for $W(s)$ in the sense of (1), is

$$\{F, G, H, J\} = \{S_{22}, S_{21}, S_{12}', S_{11}\}$$

or

$$S_a = M$$

where M is defined in (10). Therefore, the passivity condition of (4) in terms of M is that of (11).

Hence, the passive nonreciprocal synthesis problem is solved if we derive a minimal realization $\{F, G, H, J\}$ for $W(s)$ such that (11) holds. The equivalence problem (with minimal number of reactive elements) is solved if we can derive all minimal realizations such that (11) holds.

Now it is easy to verify that the above bilinear transformation (8) maps the imaginary axis of the complex p plane onto the boundary of the unit circle on the complex s plane with the left-half p plane corresponding to the interior of the unit circle on the s plane. Consequently, as the bounded real property of the matrix $S(p)$ is defined in terms of certain properties holding in the right-half complex p plane, the bounded real nature of $W(s)$ will be defined in terms of certain properties holding outside the unit circle of the complex s plane. It follows from (2) that the real rational matrix $W(s)$ satisfies the conditions:

$$\text{entries of } W(s) \text{ are analytic in } |s| \geq 1, \tag{14a}$$

$$I_n - W'^*(s)W(s) \geq 0 \text{ in } |s| > 1. \tag{14b}$$

Any real rational matrix $W(s)$ satisfying the above conditions is termed "discrete bounded real." In analogy to the passive impedance synthesis procedures [2]–[6], we shall examine the discrete bounded real property of $W(s)$ in terms of the matrices F, G, H, J of a minimal realization of $W(s)$; a particular state-space coordinate transformation T will be obtained, such that the realization $\{TFT^{-1}, TG, (T^{-1})'H, J\}$ has

$$M_1 = \begin{bmatrix} J & H'T^{-1} \\ TG & TFT^{-1} \end{bmatrix} = \begin{bmatrix} J & H_1' \\ G_1 & F_1 \end{bmatrix} \tag{15a}$$

satisfying

$$I - M_1'M_1 \geq 0. \tag{15b}$$

Thus, the passive nonreciprocal synthesis problem will be solved.

IV. Algebraic Criterion for Discrete Bounded Real Matrices

The following lemma gives an interpretation in algebraic terms of the discrete bounded real conditions given in (14a) and (14b).

Lemma 2: Let $W(s)$ be a rational matrix with $W(\infty)$ finite and let $\{F, G, H, J\}$ be a minimal realization of $W(s)$. Suppose all singularities of elements of $W(s)$ lie inside the unit circle of the complex s plane. Then $W(s)$ is discrete bounded real if, and only if, there exist real matrices Q, L, and W_0 with Q positive definite symmetric such that

$$F'QF - Q = -HH' - LL' \tag{16a}$$

$$-F'QG = HJ + LW_0 \tag{16b}$$

$$I_n - J'J = W_0'W_0 + G'QG. \tag{16c}$$

Proof—Necessity: A similar idea to that given in [15] is used here. Consider the bilinear transformation $s = (p+1)/(p-1)$ of (8) mapping the unit circle in the complex s plane into the left-half complex p plane, the matrix $W(s)$ is transformed into

$$W(s) = W\left(\frac{p+1}{p-1}\right) = S(p) = \hat{J} + \hat{H}'(pI - \hat{F})^{-1}\hat{G}$$

where

$$\begin{aligned} \hat{F} &= (F - I_k)^{-1}(F + I_k) \\ \hat{G} &= \sqrt{2}\,(F - I_k)^{-1}G \\ \hat{H} &= -\sqrt{2}\,(F' - I_k)^{-1}H \\ \hat{J} &= J - H'(F - I_k)^{-1}G. \end{aligned} \tag{17}$$

It is easy to see that the above quantities are well defined, for $W(s)$ is analytic on $|s| = 1$, hence, F has no eigenvalue of unity and consequently, $(F - I_k)^{-1}$ exists. Also it is clear that (17) defines a minimal realization of $S(p)$ and $S(p)$ is bounded real if, and only if, $W(s)$ is discrete bounded real. Recalling Lemma 1, there exist real matrices $P = P' > 0$, $\hat{L}$, and $\hat{W}_0$ such that

$$\begin{aligned} P(F - I_k)^{-1}(F + I_k) + (F' + I_k)(F' - I_k)^{-1}P \\ = -2(F' - I_k)^{-1}HH'(F - I_k)^{-1} - \hat{L}\hat{L}' \end{aligned} \tag{18a}$$

$$\begin{aligned} -\sqrt{2}\,P(F - I_k)^{-1}G = -\sqrt{2}\,(F' - I_k)^{-1}HJ \\ + \sqrt{2}(F' - I_k)^{-1}HH'(F - I_k)^{-1}G + \hat{L}\hat{W}_0 \end{aligned} \tag{18b}$$

$$I_n - [J - H'(F - I_k)^{-1}G]' \cdot [J - H'(F - I_k)^{-1}G] = \hat{W}_0'\hat{W}_0 \quad (18c)$$

on substituting (17) for $\hat{F}$, $\hat{G}$, $\hat{H}$, and $\hat{J}$. Define matrices Q, L, and W_0 by

$$Q = P \quad (19a)$$

$$\sqrt{2}\, L = -(F' - I_k)\hat{L} \quad (19b)$$

$$W_0 = \hat{W}_0 + L'(F - I_k)^{-1}G. \quad (19c)$$

Upon some manipulation, (18) reduces to (16)

Sufficiency: Here we need only to verify (14b). With the aid of (16a), it can be shown that

$$(s^*I_k - F')Q(sI_k - F) + (s^*I_k - F')QF + F'Q(sI_k - F) = (|s|^2 - 1)Q + HH' + LL'.$$

On premultiplying by $G'(s^*I_k - F')^{-1}$, postmultiplying by $(sI_k - F)^{-1}G$ and using (16b) and (16c), the above equation after some manipulation, becomes

$$\begin{aligned} I_n - [J' + G'(s^*I_k - F')^{-1}H][J + H'(sI_k - F)^{-1}G] &= [W_0' + G'(s^*I_k - F')^{-1}L][W_0 + L'(sI_k - F)^{-1}G] \\ &\quad + (|s|^2 - 1)G'(s^*I_k - F')^{-1}Q(sI_k - F)^{-1}G. \end{aligned}$$

The right-hand side is clearly nonnegative definite Hermitian in $|s| > 1$; the left-hand side is, of course, $I_n - W'^*(s)W(s)$. Hence, the lemma is proved.

One might ask how Q can be determined. In fact several methods are available. One procedure is by performing a spectral factorization (see [3]); or, one can compute Q *algebraically*. For the latter approach, it appears easier to actually calculate P of the bounded real lemma instead and note from (19a) that $Q = P$. As shown in [10], *all* solutions P are obtained as the solution of the quadratic matrix inequality

$$P\hat{F} + \hat{F}'P + (P\hat{G} + \hat{H}\hat{J}) \cdot (I - \hat{J}'\hat{J})^{-1}(P\hat{G} + \hat{H}\hat{J})' + \hat{H}\hat{H}' \le 0.$$

Discussion of how to compute all solutions of this inequality starting from one special solution satisfying the equality is given in [10]. A solution satisfying the equality of the above quadratic matrix equation may be found using the method of [17].

There is, however, one problem here, that of requiring $(I - \hat{J}'\hat{J})$ to be nonsingular. It turns out that this apparent difficulty can always be overcome by extracting suitable transformers, gyrators, along with a minimum number of capacitors and inductors in a similar fashion as that outlined in [7]. The calculations are simple.

V. Passive Synthesis Procedure

With Lemma 2 in hand, we now turn back to the synthesis problem. Recall that, if $\{F, G, H, J\}$ is a minimal realization for $W(s)$, the problem of finding a passive network synthesizing $W(s)$ or $S(p)$ (omitting consideration of reciprocity for the moment) reduces to finding a nonsingular matrix T such that M_1 in (15a) satisfies (15b).

Lemma 2 gives the necessary and sufficient conditions satisfied by F, G, H, J for $W(s)$ to be discrete bounded real. The lemma guarantees the existence of a symmetric positive definite matrix Q. For such a matrix, the symmetric positive definiteness guarantees the existence of a nonsingular easily computable T satisfying

$$T'T = Q. \quad (20)$$

Theorem 1: Let $\{F, G, H, J\}$ be a minimal realization of a discrete bounded real $W(s)$ with $W(\infty)$ finite, and let Q be a symmetric positive definite matrix, which together with certain matrices L and W_0 satisfies (16). Suppose T is any nonsingular matrix satisfying (20); then M_1 in (15a) satisfies (15b).

Proof: Define $F_1 = TFT^{-1}$, $G_1 = TG$, $H_1' = H'T^{-1}$, $L_1' = L'T^{-1}$. Then (16) becomes

$$F_1'F_1 - I_k = -H_1H_1' - L_1L_1' \quad (21a)$$

$$-F_1'G_1 = H_1J + L_1W_0 \quad (21b)$$

$$I_n - J'J = W_0'W_0 + G_1'G_1. \quad (21c)$$

By direct calculation

$$\begin{aligned} I_{n+k} - M_1'M_1 &= \begin{bmatrix} I_n - J'J - G_1'G_1 & -(J'H_1' + G_1'F_1) \\ -(H_1J + F_1'G_1) & I_k - H_1H_1' - F_1'F_1 \end{bmatrix} \\ &= \begin{bmatrix} W_0' \\ L_1 \end{bmatrix} [W_0 \quad L_1'] \end{aligned} \quad (21)$$

using (21).

Evidently $I_{n+k} - M_1'M_1$ is nonnegative definite, or equivalently (15b) holds.

It follows from Theorem 1 that S_a, hence $W(s)$ or $S(p)$, is realizable using passive elements.

Clearly for a given minimal realization F, G, H, J of $W(s)$, each solution Q of (16) generates a matrix M_1 with the passivity property. Moreover, it is easy to see, as we show below, that any matrix M_1 with the passivity property defines a Q. Consequently the equivalence problem, i.e., the task of finding essentially all passive syntheses of minimal state-space dimension is the same as the problem of finding all Q satisfying (16). As remarked, this problem is discussed in [10].

To see that with every M_1 there must be associated a Q, simply observe that with every M_1 there is associated a T. Define Q by (20) and W_0 and L_1 by any factorization of $I - M_1'M_1$ of the form given above. Also define L by $L = T'L_1$. Then the relations (16) are readily verified.

VI. Reciprocal Passive Synthesis

To solve the reciprocal passive synthesis problem, the following results are necessary; the first (Lemma 3), in fact, gives the algebraic characterization of the symmetry property of a real rational matrix.

Lemma 3: (See [1], [4].) Let $W(s)$ be any real syn metric rational matrix with $W(\infty)$ finite and let $\{F, G, H, J\}$ be any minimal realization for $W(s)$. Then there exists a real symmetric nonsingular matrix A, uniquely determined by the realization $\{F, G, H, J\}$, such that

$$AF = F'A; \quad AG = H. \tag{22}$$

Moreover, A may be expressed as ([16], pp. 86 and 106)

$$A = T'\Sigma T \tag{23}$$

where T is a real nonsingular matrix and $\Sigma = [I_{k1} \dotplus (-1)I_{k2}]$ is unique to within permutation of the diagonal entries.

Lemma 4: Let C be a real matrix with the property that C is similar to a real symmetric matrix D; then $(I - D'D)$ is positive definite (nonnegative definite) if $(I - C'C)$ is positive definite (nonnegative definite), but not necessarily conversely.

The proof of the lemma is straightforward, and is omitted.

For the reciprocal passive synthesis, we start by generating a minimal realization $\{F_1, G_1, H_1, J\}$ for $W(s)$, which satisfies (21) in Theorem 1. Lemma 3 then guarantees the existence of a unique symmetric nonsingular matrix A (actually readily computable) with

$$AF_1 = F_1'A; \qquad AG_1 = H_1. \tag{24}$$

The symmetric A can always be represented as

$$A = BU = UB \tag{25}$$

where B is symmetric positive definite and U is symmetric and orthogonal (see [18], p. 204).[1] Because U is symmetric, it has real eigenvalues, and because it is orthogonal, it has eigenvalues with unity modulus. Hence, the eigenvalues must be $+1$ or -1. Therefore, U may be written as

$$U = V\Sigma V' \tag{26}$$

where V is orthogonal and $\Sigma = [I_{k1} \dotplus (-1)I_{k2}]$.

Theorem 2: Let $\{F_1, G_1, H_1, J\}$ be a minimal realization for a discrete bounded real $W(s)$ with $W(\infty)$ finite, which has the property that (21) is satisfied. Let A be a real symmetric nonsingular matrix satisfying (24). With B, Σ, and V as defined in (25) and (26), a nonsingular matrix given by

$$T_1 = V'B^{1/2} \tag{27}$$

then defines a minimal realization $\{T_1F_1T_1^{-1},\ T_1G_1,\ (T_1^{-1})'H_1, J\}$ and a matrix

$$\mathbf{M} = \begin{bmatrix} J & H_1'T_1^{-1} \\ T_1G_1 & T_1F_1T_1^{-1} \end{bmatrix}$$

such that

$$(I_n \dotplus \Sigma)\mathbf{M} = \mathbf{M}'(I_n \dotplus \Sigma) \tag{28}$$

and

$$I_{n+k} - \mathbf{M}'\mathbf{M} \geq 0. \tag{29}$$

Proof of Reciprocity Condition of (28): It is not hard to see that to prove (28) we need only show, after substituting T_1 from (27), that

$$\Sigma V'B^{1/2}G_1 = V'B^{-1/2}H_1 \tag{30a}$$

and

$$\Sigma V'B^{1/2}F_1B^{-1/2}V = V'B^{-1/2}F_1'B^{1/2}V\Sigma. \tag{30b}$$

First we note from (25) and (26) that

$$BV\Sigma V' = V\Sigma V'B.$$

It follows, by noting $\Sigma^2 = I$ and multiplying on the left by $\Sigma V'$ and on the right by $V\Sigma$, that $\Sigma V'BV = V'BV\Sigma$. Because $V'BV$ commutes with Σ and simultaneously is a symmetric matrix, $V'BV$ must be of the form

$$V'BV = \begin{array}{cc} & \begin{array}{cc} (k_1) & (k_2) \end{array} \\ & \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{array}{c} (k_1) \\ (k_2) \end{array} \end{array} \tag{31}$$

with Λ_1 and Λ_2 being symmetric positive definite matrices.

As a result of (31),

$$\Sigma V'B^{1/2}V = V'B^{1/2}V\Sigma \tag{32}$$

where

$$V'B^{1/2}V = \begin{bmatrix} \Lambda_1^{1/2} & 0 \\ 0 & \Lambda_2^{1/2} \end{bmatrix}. \tag{33}$$

From (24), on substituting (25) and (26), we have

$$BV\Sigma V'G_1 = H_1.$$

Multiplying on the left by $V'B^{-1/2}$ yields

$$V'B^{1/2}V\Sigma V'G_1 = V'B^{-1/2}H_1.$$

Hence using (32),

$$\Sigma V'B^{1/2}G_1 = V'B^{-1/2}H_1. \tag{30a}$$

From (24), using (25) and (26), we have

$$BV\Sigma V'F_1 = F_1'V\Sigma V'B.$$

Multiplying on the left by $V'B^{-1/2}$ and the right by $B^{-1/2}V$ yields

$$V'B^{1/2}V\Sigma V'F_1B^{-1/2}V = V'B^{-1/2}F_1'V\Sigma V'B^{1/2}V,$$

hence,

$$\Sigma V'B^{1/2}F_1B^{-1/2}V = V'B^{-1/2}F_1'B^{1/2}V\Sigma \tag{30b}$$

on using (32). Therefore (28) is proved.

Proof of Passivity Condition of (29): Since $\mathbf{M} = (I_n \dotplus V'B^{1/2})M_1(I_n \dotplus B^{-1/2}V)$ where

$$M_1 = \begin{bmatrix} J & H_1' \\ G_1 & F_1 \end{bmatrix}$$

has $I_{n+k} - M_1'M_1 \geq 0$, $\mathbf{M}$ can be written as

[1] The proof of [18] can be extended simply to establish the above results that rely on the fact that $A = A'$.

$$\mathbb{M} = (I_n \dotplus V'B^{1/2}V)(I_n \dotplus V')M_1(I_n \dotplus V)(I_n \dotplus V'B^{-1/2}V)$$
$$= (I_n \dotplus V'B^{1/2}V)M_2(I_n \dotplus V'B^{-1/2}V)$$

where $M_2 = (I_n \dotplus V')M_1(I_n \dotplus V)$ and, obviously, $I_{n+k} - M_2'M_2 \geq 0$ because $I_{n+k} - M_1'M_1 \geq 0$ and $(I_n \dotplus V')$ is orthogonal. Now $V'B^{1/2}V$ has the special form shown in (33), and therefore,

$$\mathbb{M} = (I_n \dotplus \Lambda_1^{1/2} \dotplus \Lambda_2^{1/2})M_2(I_n \dotplus \Lambda_1^{-1/2} \dotplus \Lambda_2^{-1/2})$$
$$= \Lambda_0 M_2 \Lambda_0^{-1} \tag{34}$$

with

$$\Lambda_0 = (I_n \dotplus \Lambda_1^{1/2} \dotplus \Lambda_2^{1/2}). \tag{35}$$

The problem then is to prove that $I_{n+k} - \mathbb{M}'\mathbb{M} \geq 0$ from the condition that $I_{n+k} - M_2'M_2 \geq 0$.

On multiplying on the left of (34) by $(I_n \dotplus \Sigma)$ and noting that $(I_n \dotplus \Sigma)$ commutes with Λ_0, we have

$$(I_n \dotplus \Sigma)\mathbb{M} = \Lambda_0(I_n \dotplus \Sigma)M_2\Lambda_0^{-1}. \tag{36}$$

Now

$$I_{n+k} - M_2'M_2 = I_{n+k} - M_2'(I_n \dotplus \Sigma)(I_n \dotplus \Sigma)M_2 \geq 0$$

and (36) says that $(I_n \dotplus \Sigma)M_2$ is similar to $(I_n \dotplus \Sigma)\mathbb{M}$, which has been shown to be symmetric. It follows from Lemma 4, on identifying $(I_n \dotplus \Sigma)M_2$ with C and $(I_n \dotplus \Sigma)\mathbb{M}$ with D, that $I_{n+k} - \mathbb{M}'(I_n \dotplus \Sigma)(I_n \dotplus \Sigma)\mathbb{M} \geq 0$. Hence,

$$I_{n+k} - \mathbb{M}'\mathbb{M} \geq 0$$

proving (29). This completes the proof of Theorem 2.

With Theorems 1 and 2 in hand, the synthesis problem is solved. Unfortunately, space limitations preclude the presentation of an example.

VII. Conclusion

In classical scattering matrix synthesis methods, for example [8], the Belevitch synthesis and the Oono–Yasuura synthesis, the key idea is to extract all resistive components to obtain a lossless coupling network. With this approach, a synthesis that uses a minimal number of resistors is always possible.

In this paper, the approach makes use of the results of system theory on minimum state-space realizations. The technique is to extract reactances to yield a resistive coupling network. Here a minimum number of reactive elements is always sufficient to give a synthesis. Furthermore, in the nonreciprocal case where gyrators are allowed, one can find Q, L, and W_0 in (16) such that L has a minimum number of rows; in this case the synthesis achieved uses the minimal number of resistors also.

We conclude by mentioning an open problem. Though the minimal reactive element equivalence problem, as viewed in state-space terms, is essentially solved for nonreciprocal networks (by an algorithm presented elsewhere [10] for obtaining all solutions of a quadratic matrix inequality), there still remains the equivalence problem for reciprocal networks. To be sure, one can examine all networks (reciprocal and nonreciprocal) equivalent to a prescribed reciprocal network and cast out the nonreciprocal networks; but a more satisfactory "solution" would be desirable.

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