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Impedance Matrices and Their State-Space Descriptions

In this correspondence, we consider a property characterizing state-space descriptions of positive real matrices. The property was proved for positive real functions in [1], conjectured for positive real matrices in [2], and proved for positive real matrices in [3]. It is as follows.

Lemma

Let $Z(s)$ be an $n \times n$ matrix of rational functions of a complex variable s, with each entry of $Z(\infty)$ finite. Let F, G, H, J be a minimal realization of $Z(s)$, i.e.,

$$Z(s) = J + H'(sI - F)^{-1}G \tag{1}$$

with $[F, G]$ completely controllable, and $[F, H]$ completely observable. Then $Z(s)$ is positive real if and only if there exist $P = P'$, positive definite, and matrices L and W such that

$$\begin{aligned} PF + F'P &= -LL' \\ PG &= H - LW \\ J + J' &= W'W. \end{aligned} \tag{2}$$

This lemma has proved important in discussion of a diversity of topics, including network synthesis [4]. The proof that $Z(s)$ is positive real if (2) holds for appropriate P, L, and W is relatively straightforward, but the converse is difficult. We present an argument on physical grounds establishing the converse, which builds on the well-known result of network theory that there exists a network of passive R, L, C, transformer, and gyrator elements synthesizing $Z(s)$ with a minimal number of reactive elements [5].

A proof also using this result has also been obtained by Layton [6]. This proof proceeds quite differently and in outline is as follows. After replacement of all capacitors by inductor-loaded gyrators, the network synthesizing $Z(s)$ is viewed as a nondynamic network (i.e., one containing only resistors, transformers, and gyrators) terminated in inductors. An impedance representation of this nondynamic network is exhibited, and related to the F, G, H, and J matrices. The passivity of the nondynamic network is then shown to imply algebraic relations among F, G, H, and J that can, after manipulation, be reduced to (2).

Suppose N is a network synthesizing $Z(s)$, with $Z(s)$ described by

$$\begin{aligned} \dot{x} &= Fx + Gu \\ y &= H'x + Ju. \end{aligned} \tag{3}$$

In these equations, u corresponds to the port current vector, y to the port voltage vector, but x need *not* correspond directly to capacitor voltages and inductor currents.

Suppose at time t, $\boldsymbol{N}$ is excited. The energy stored in $\boldsymbol{N}$ is $\frac{1}{2}x_1'x_1$, where x_1 is a vector whose entries are proportional to capacitor voltages and inductor currents. Now there exists a state-space description for $\boldsymbol{N}$ with x_1 as state vector, and the dimension of x_1 is the number of reactive elements in $\boldsymbol{N}$, which is the degree [5] of $Z(s)$, which in turn is the dimension of x in view of the controllability and observability assumptions. Again by these last two assumptions, it follows that there exists a nonsingular T such that $x = Tx_1$, and therefore, with $P = T'T$ the energy stored in $\boldsymbol{N} = \frac{1}{2}x'Px$ where $P = P'$ is positive definite.

Now consider the voltages existing across each resistor in $\boldsymbol{N}$. At time t, these must be linear functions of the state $x(t)$ and the input current $u(t)$. Therefore, with $v_R(t)$ denoting the vector of such voltages,

$$v_R = Mx + Nu$$

for some matrices M and N. If $\boldsymbol{G}$ is a diagonal matrix whose entries are all the conductances of the network, the rate of energy dissipation in $\boldsymbol{N} = (Mx + Nu)'\boldsymbol{G}(Mx + Nu)$. Now from the fundamental equation

power input = rate of increase of stored energy + rate of energy dissipation

we have

$$(H'x + Ju)u = \frac{d}{dt}[\tfrac{1}{2}x'Px] + (Mx + Nu)'\boldsymbol{G}(Mx + Nu).$$

Computing the derivative with the aid of (3), we obtain

$$\begin{aligned} u'J'u + x'Hu &= \tfrac{1}{2}x'(PF + F'P)x + x'PGu \\ &\quad + x'M'\boldsymbol{G}Mx + 2x'M'\boldsymbol{G}Nu + u'N'\boldsymbol{G}Nu \end{aligned}$$

or

$$\begin{aligned} u'[\tfrac{1}{2}(J + J') - N'\boldsymbol{G}N]u + x'(H - PG - 2M'\boldsymbol{G}N)u \\ + x'[-\tfrac{1}{2}(PF + F'P) - M'\boldsymbol{G}M]x = 0. \end{aligned}$$

Now this equation holds for all x and u, and so

$$\begin{aligned} J + J' &= 2N'\boldsymbol{G}N \\ H - PG &= 2M'\boldsymbol{G}N \\ PF + F'P &= -2M'\boldsymbol{G}M. \end{aligned}$$

By setting $W = \sqrt{2}\boldsymbol{G}^{1/2}N$ and $L = \sqrt{2}M'\boldsymbol{G}^{1/2}$, the desired (2) are recovered.

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