Technical Notes and Correspondence

A Note on Transmission Zeros of a Transfer Function Matrix

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Abstract—**A new proof is provided to show that the numerator polynomials of the Smith-McMillan form of a rational transfer function matrix are the same as the invariant factors of a map $sI-A_t$, with A_t a map determined from the system state equations.**

I. Introduction

Let $W(s)$ be an $m\times p$ real rational transfer function matrix with $W(\infty)=0$ and $\{A,B,C\}$ a minimal realization of $W(s)$, with A of dimension $n\times n$. We shall show that the numerator polynomials of the Smith–McMillan canonical form associated with $W(s)$ are identical with the invariant factors of a map $sI-A_t$ where[1] A_t is a certain map constructed from A, B, and C. This result has been established via a more extended argument by Moore and Silverman [1], which has a number of ideas in common with a report of Bengtsson [2] yet to appear in the open literature. The result also appears in a forthcoming paper by Corfmat and Morse [3] which builds on ideas hinted at in [4]. Finally, the zeros of a system are studied using the Morse ideas in Dickinson's doctoral dissertation [5], while we are given to understand that a doctoral dissertation of Sebakhy also establishes the result.

II. Smith–McMillan Form

Recall the definition of the Smith–McMillan form [6], [7]. Let $\phi(s)$ be the monic least common denominator of entries of $W(s)$, and let the polynomial matrix $\phi(s)W(s)$ have Smith decomposition [8]

$$\phi(s)W(s)=U_L(s)T(s)U_R(s). \tag{1}$$

This means that $U_L(s)$, $U_R(s)$ are square polynomial matrices which are unimodular, i.e., have constant determinant, and $T(s)$ is an $n\times p$ polynomial matrix whose entries are all zero, save possibly those on the diagonal

$$T(s)=\operatorname{diag}[t_1(s),t_2(s),\cdots,t_k(s)]. \tag{2}$$

Here, $k=\min(m,p)$ and $t_i(s)$ is the ratio of the greatest common divisor (gcd) of all $i\times i$ minors of $\phi(s)W(s)$ to the gcd of all $(i-1)\times(i-1)$ minors of $\phi(s)W(s)$. (By convention, zero dimension minors are set equal to 1.) All nonzero $t_i(s)$ are monic; if $t_i(s)$ is nonzero, so is $t_j(s)$ for $j<i$, and $t_{i-1}(s)$ divides $t_i(s)$. The set of $t_i(s)$ are termed the invariant factors of $T(s)$.

Now with $t_i(s)/\phi(s)$ expressed as a ratio $\epsilon_i(s)/\psi_i(s)$ of two monic polynomials with no common factor, the Smith–McMillan form associated with $W(s)$ is the matrix

$$T(s)/\phi(s)=\mathcal{E}(s)\Psi^{-1}(s)=\operatorname{diag}\left[\frac{\epsilon_1(s)}{\psi_1(s)},\frac{\epsilon_2(s)}{\psi_2(s)},\cdots,\frac{\epsilon_k(s)}{\psi_k(s)}\right]. \tag{3}$$

In case $\epsilon_i(s)$ is zero, we take $\psi_i(s)=1$. The divisibility properties of the $t_i(s)$ imply that $\epsilon_{i-1}(s)$ divides $\epsilon_i(s)$, provided that $\epsilon_i(s)$ is nonzero, and $\psi_i(s)$ divides $\psi_{i-1}(s)$. A little thought shows that $\psi_1(s)=\phi(s)$. The matrices $\mathcal{E}(s)$ and $\Psi(s)$ of dimensions $m\times p$ and $p\times p$ have obvious definitions as diagonal matrices.

As shown in [7, ch. 3, theorem 4.1], the polynomials $\epsilon_i(s)$ and $\psi_i(s)$ have a straightforward characterization in terms of A, B, and C: let the Smith form of $(sI-A)$ be S_A. Then if $n\geqslant k$,

$$S_A=\begin{bmatrix} I_{n-k} & 0\\ 0 & \operatorname{diag}[\psi_k(s),\psi_{k-1}(s),\cdots,\psi_1(s)]\end{bmatrix} \tag{4a}$$

while if $n\leqslant k$,

$$S_A=\operatorname{diag}[\psi_k(s),\psi_{k-1}(s),\cdots,\psi_{k-n+1}(s)] \tag{4b}$$

and $\psi_1(s),\cdots,\psi_{k-n}(s)$ are all unity. Also, let the Smith form of

$$M(s)=\begin{bmatrix} -sI+A & B\\ C & 0\end{bmatrix} \tag{5}$$

be S_M. Then

$$S_M=\begin{bmatrix} I_n & 0\\ 0 & \operatorname{diag}[\epsilon_1(s),\epsilon_2(s),\cdots,\epsilon_k(s)]\end{bmatrix}. \tag{6}$$

This latter result will be used below.

III. The Map A_t

We shall now indicate how the map A_t is defined. Let $\mathcal{X}$ be the state space in which resides the state vector x appearing in the equation

$$\dot{x}=Ax+Bu\quad y=Cx. \tag{7}$$

Let $\mathcal{U}$, $\mathcal{Y}$ be defined similarly.

An (A,B)-invariant subspace $\mathcal{V}\subset\mathcal{X}$ is a subspace with the property [9]

$$A\mathcal{V}\subset\mathcal{V}+\mathcal{B} \tag{8}$$

where $\mathcal{B}$ is the range of the operator defined by B, i.e., the set $B\mathcal{U}$. Equivalently [9], $\mathcal{V}$ is (A,B)-invariant if and only if for some F,

$$(A+BF)\mathcal{V}\subset\mathcal{V}. \tag{9}$$

Using the characterization (8), it is not hard to prove a crucial property: that there is a largest (A,B)-invariant subspace contained in $\mathcal{N}(C)$, the null-space of C. Henceforth, let $\bar{\mathcal{V}}$ denote this particular (A,B)-invariant subspace. Then

$$(A+B\bar{F})\bar{\mathcal{V}}\subset\bar{\mathcal{V}},\qquad \text{for some } \bar{F} \tag{10a}$$

$$C\bar{v}=0,\qquad \text{for all } \bar{v}\in\bar{\mathcal{V}} \tag{10b}$$

$$A\mathcal{V}\subset\mathcal{V}+B \text{ and } Cv=0,\qquad \text{for all } v\in\mathcal{V} \text{ implies } \mathcal{V}\subset\bar{\mathcal{V}}. \tag{10c}$$

Define $\{A|\mathcal{B}\}$ as the space spanned by $\mathcal{B}+A\mathcal{B}+\cdots+A^{n-1}\mathcal{B}$ where A is $n\times n$. A *controllability subspace* $\mathcal{R}$ is a subspace of $\mathcal{X}$ with the defining property [9]

$$\mathcal{R}=\{A+BF|\mathcal{B}\cap\mathcal{R}\} \tag{11}$$

for some F. An important result is that there is a largest controllability subspace contained in $\mathcal{N}(C)$ defined by

$$\bar{\mathcal{R}}=\{A+B\bar{F}|\mathcal{B}\cap\bar{\mathcal{V}}\} \tag{12}$$

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[1] The subscript t refers to *transmission zeros.*

while it is, of course, also true that

$$\bar{\mathcal{R}}=\{A+B\bar{F}|\mathcal{B}\cap\bar{\mathcal{R}}\}. \tag{13}$$

From (12) we see that

$$\bar{\mathcal{R}}\subset\{A+B\bar{F}|\bar{\mathcal{V}}\}=\bar{\mathcal{V}}+(A+B\bar{F})\bar{\mathcal{V}}+\cdots=\bar{\mathcal{V}}$$

using (10a). This means that $\bar{\mathcal{V}}$ is isomorphic to the direct sum of $\bar{\mathcal{R}}$ and $\bar{\mathcal{V}}/\bar{\mathcal{R}}$.

Now suppose that the state-space $\mathcal{X}$ is regarded as the direct sum $\bar{\mathcal{R}}\oplus\bar{\mathcal{S}}\oplus\mathcal{W}$ with, $\mathcal{S}$ and $\mathcal{W}$ isomorphic to $\bar{\mathcal{V}}/\bar{\mathcal{R}}$ and $\mathcal{X}/\bar{\mathcal{V}}$, respectively, and $\bar{\mathcal{R}}\oplus\bar{\mathcal{S}}=\bar{\mathcal{V}}$, and with the coordinate basis so chosen that vectors of the form $[x_1'\ 0\ 0]'$, $[0\ x_2'\ 0]'$, and $[0\ 0\ x_3']'$ lie in $\bar{\mathcal{R}}$, $\mathcal{S}$, and $\mathcal{W}$, respectively, where the x_i are given suitable dimensions.

$$A+B\bar{F}=\begin{bmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\0&0&A_{33}\end{bmatrix}$$

and (10b) implies that C has the form $[0\ 0\ C_3]$. From (13), it is evident that $\bar{\mathcal{R}}$ is $A+B\bar{F}$ invariant, so that $A_{21}=0$, i.e.,

$$A+B\bar{F}=\begin{bmatrix}A_{11}&A_{12}&A_{13}\\0&A_{22}&A_{23}\\0&0&A_{33}\end{bmatrix}. \tag{14}$$

The map A_t of [1] corresponds to the matrix A_{22}.

IV. Constraints on B in the Special Coordinate Basis

For some square nonsingular matrix G, it is evident that we can write

$$BG=\begin{bmatrix}B_{11}&B_{21}\\B_{12}&B_{22}\\0&B_{23}\end{bmatrix}$$

where the rank of B_{23} equals the numbers of its columns. Evidently,

$$\mathcal{B}\cap\bar{\mathcal{V}}=\left\{\begin{bmatrix}B_{11}\\B_{12}\\0\end{bmatrix}u_1, u_1 \text{ arbitrary, save for its dimension}\right\}.$$

Equation (12), however, makes clear that $\mathcal{B}\cap\bar{\mathcal{V}}\subset\bar{\mathcal{R}}$, so that we must have $B_{12}=0$. Consequently,

$$\begin{bmatrix}\mathcal{B}_{11}\\0\\0\end{bmatrix}=\mathcal{B}\cap\bar{\mathcal{R}}$$

and by (14),

$$\{A+B\bar{F}|\mathcal{B}\cap\bar{\mathcal{R}}\}=\begin{bmatrix}\{A_{11}|\mathcal{B}_{11}\}\\0\\0\end{bmatrix}.$$

Since by (13) this is the same as $\bar{\mathcal{R}}$, we conclude that $[A_{11},B_{11}]$ is a completely controllable pair.

V. Main Result

The matrix A_{22} and the polynomials $\epsilon_i(s)$ are connected in the following way.

Theorem: The invariant factors of the matrix $M(s)$ in (5) consist of a number equal to 1, together with the invariant factors of $sI-A_{22}$.

Notice that by (6), this means that the invariant factors of $sI-A_{22}$ agree with the set $\epsilon_l(s)$, $\epsilon_{l+1}(s),\cdots,\epsilon_k(s)$ for some l for which $\epsilon_1(s),\epsilon_2(s),\cdots,\epsilon_{l-1}(s)$ must be 1.

Proof: The invariant factors of a matrix are preserved under pre- and post-multiplication by unimodular matrices [8]; therefore, the invariant factors of $M(s)$ are the same as those of

$$M(s)\begin{bmatrix}I&0\\F&G\end{bmatrix}=\begin{bmatrix}-sI+(A+BF)&BG\\C&0\end{bmatrix}=\begin{bmatrix}-sI+A_{11}&A_{12}&A_{13}&B_{11}&B_{21}\\0&-sI+A_{22}&A_{23}&0&B_{22}\\0&0&-sI+A_{33}&0&B_{23}\\0&0&C_3&0&0\end{bmatrix}.$$

In turn, the invariant factors are the same as those of

$$N(s)=\left[\begin{array}{cc|c|cc}-sI+A_{11}&B_{11}&A_{12}&A_{13}&B_{21}\\\hline 0&0&-sI+A_{22}&A_{23}&B_{22}\\\hline 0&0&0&-sI+A_{33}&B_{23}\\0&0&0&C_3&0\end{array}\right].$$

Now the invariant factors of $[-sI+A_{11}\ \ B_{11}]$ are all unity, since $[A_{11},B_{11}]$ is completely controllable. (This is easy to prove, but is also a known result, e.g., see [7, ch. 2, theorem 6.2].) Below, we show that the invariant factors of

$$P(s)=\begin{bmatrix}-sI+A_{33}&B_{23}\\C_3&0\end{bmatrix} \tag{15}$$

are also all unity. From the block structure of $N(s)$ and the characterization of invariant factors via ratios of gcd's of minors, it is then evident that the invariant factors of $M(s)$ are as claimed.

To show that the invariant factors of $P(s)$ are all unity, it is enough to show that for no value of s is the rank of $P(s)$ less than the number of columns of $P(s)$. (This is equivalent to showing that for any value of s, there exists a maximum size minor which is nonzero; this implies that the greatest common divisor of all maximum size minors will be nonzero for all s, or equivalently, must be 1.) Suppose to the contrary, i.e., there exists s_i, v, and w with v and w not both zero, for which

$$\begin{bmatrix}-s_iI+A_{33}&B_{23}\\C_3&0\end{bmatrix}\begin{bmatrix}v\\w\end{bmatrix}=0. \tag{16}$$

Observe that if $v=0$, this equation yields $B_{23}w=0$ and therefore $w=0$, since B_{23} has full column rank. Hence, $v\neq0$. We can now demonstrate a contradiction, to the effect that the subspace of $\mathcal{X}$ by

$$\hat{\mathcal{V}}=sp\left\{\bar{\mathcal{V}},\ \begin{bmatrix}0\\0\\v\end{bmatrix}\right\}$$

is an (A,B)-invariant subspace contained in $\mathcal{N}(C)$. (Since $\bar{\mathcal{V}}$ was chosen as the largest such subspace, and $[0\ \ 0\ \ v']'\notin\bar{\mathcal{V}}$, the contradiction is clear.) We now have, using (16) and the known forms for $A+BF$ and B,

$$(A+BF)\begin{bmatrix}0\\0\\v\end{bmatrix}+\begin{bmatrix}B_{11}&B_{21}\\0&B_{22}\\0&B_{23}\end{bmatrix}\begin{bmatrix}0\\w\end{bmatrix}=\begin{bmatrix}A_{13}v\\A_{23}v\\0\end{bmatrix}+\begin{bmatrix}0\\0\\s_iv\end{bmatrix}+\begin{bmatrix}B_{21}w\\B_{22}w\\0\end{bmatrix}$$

or

$$A\begin{bmatrix}0\\0\\v\end{bmatrix}\in\hat{\mathcal{V}}+\mathcal{B}.$$

Combining this with the fact that $A\bar{\mathcal{V}}\subset\bar{\mathcal{V}}+\mathcal{B}\subset\hat{\mathcal{V}}+\mathcal{B}$, it follows that $A\hat{\mathcal{V}}\subset\hat{\mathcal{V}}+B$. Also, $v\in\mathcal{N}(C_3)$ so that $[0\ \ 0\ \ v']'\in\mathcal{N}(C)$. Thus, $\hat{\mathcal{V}}$ is indeed an (A,B)-invariant subspace contained in $\mathcal{N}(C)$, and the desired contradiction is established.

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Smoothing of Signals with Bounded Error

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***Abstract*—Samples of a signal are observed with bounded errors. In this correspondence a simple and direct solution to the problem of constructing a smooth curve based on these observations is given. The smoothness of a curve is considered proportional to the energy in the second derivative of the signal.**

Introduction

In many situations one observes samples of a signal and one knows that in measurement a certain amount of error has occured. These samples are then used to construct an approximation or an estimate of the original signal. This process has been termed "Smoothing." If there were no errors we would have interpolation. There is no unique method of optimal reconstruction. If we assume that the signal is a random process and the errors are random variables, then one can use statistical methods of smoothing [1]. In this correspondence we assume that the error is bounded and we know this bound. The original signal was smooth and hence we want a smooth reconstruction. We consider smoothness proportional to the energy in the second derivative of the signal. By attempting to minimize this energy we tacitly assume that the signal we reconstruct is twice differentiable. This is a very reasonable criterion. Since if $a>b$, we know that $\cos bt$ is smoother than $\cos at$, also the energy in the second derivative of $\cos at$ is more than that in $\cos bt$.

The notation used in this correspondence is as follows. I will be the closed interval (a,b); $D^n\phi(x)$ will stand for $d^n\phi(x)/dx$; (v,w) will be the inner product $\int_a^b v(x)w(x)dx$, and $\|v\|^2$ will be the norm derived from it, in other words, equal to (v,v).

Let us define the following.

$PC^2(I) \equiv$ a set of real functions ϕ defined on I such that a) $D\phi$ is continuously differentiable, b) there exists α_i, $0 \leqslant i \leqslant s+1$ with $a=\alpha_0<\alpha_1<\cdots<\alpha_{s+1}=b$ such that in each open interval (α_i,α_{i+1}), $D^2\phi$ is continuously differentiable, and c) $\Sigma_{i=0}^{s}\int_{\alpha_i}^{\alpha_{i+1}}|D^2\phi(x)|^2dx<\infty$, that is, the L^2-norm of $D^2\phi$ is finite.

Next we give a precise formulation to the problem solved in this correspondence.

Problem: The observed signal belongs to the set H, where

$$H=\{w\in PC^2(I);|w(x_i)-f_i|\leqslant \xi_i, i=0,1,\cdots,n\}. \tag{1}$$

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Our observations are f_i, $i=0,1,\cdots,n$ and the error bounds are ξ_i, $i=0,1,\cdots,n$.

Find a function $w\in H$, which minimizes $\|D^2v\|$ over all $v\in H$.

A more general version of this problem was solved by Laurent [2]. Here we have a problem of optimization in infinite-dimensional spaces. In this correspondence we give a clever method of reducing this optimization to a finite-dimensional space. This is done by characterizing the finite-dimensional space in which the optimal solution has to lie. With this reduction the problem becomes one of quadratic programming under inequality constraints. We have used Rosen's [3] algorithm for its solution.

Optimal Solution

We define a set

$$M(\Delta)=\{p(x)\in C^2(I);p(x) \text{ is a cubic polynomial in each interval } (x_i,x_{i+1}), i=0,1,\cdots,n-1\}. \tag{2}$$

It is easy to verify that $M(\Delta)$ is a $(n+3)$-dimensional subspace of $PC^2(I)$. Next we define a subspace of $M(\Delta)$, which we will denote by $Nat\,(\Delta)$. The elements of $Nat(\Delta)$ are called cubic natural splines.

$$Nat(\Delta)=\{p(x)\in M(\Delta);D^2p(x_0)=D^2p(x_n)=0\}. \tag{3}$$

If we are given $p(x_i)=f_i$, $i=0,1,\cdots,n$ then this defines a unique $p\in Nat(\Delta)$. We give below a method of calculating this. This method was discussed by Greville [5]. Let us define

$$D^2p(x_i)=s_i, \qquad i=0,1,\cdots,n \tag{4}$$

and

$$s=\text{col}(s_1,s_2,\cdots,s_{n-1}). \tag{5}$$

If we know s, then values of s_i, f_i, s_{i+1}, and f_{i+1} define a unique cubic polynomial in the interval (x_i,x_{i+1}). Note that since we are constructing $p\in Nat\ \Delta$, $s_0=s_n=0$. The vector s can be calculated from the equation

$$As=Bf \tag{6}$$

where, for $i,j=1,2,\cdots,(n-1)$,

$$A_{ij}=\begin{cases} x_i-x_{i-1}, & \text{if } j=i-1\\ 2(x_{i+1}-x_{i-1}), & \text{if } i=j\\ x_{i+1}-x_i, & \text{if } j=i+1\\ 0, & \text{otherwise}\end{cases}$$

and, for $i=1,\cdots,(n-1)$, $j=0,1,\cdots,n$,

$$B_{ij}=\begin{cases} \dfrac{6}{x_i-x_{i-1}}, & \text{if } j=i-1\\ -6\left(\dfrac{1}{x_{i+1}-x_i}+\dfrac{1}{x_i-x_{i-1}}\right), & \text{if } j=i\\ \dfrac{6}{(x_{i+1}-x_i)}, & \text{if } j=i+1\\ 0, & \text{otherwise}\end{cases} \tag{7}$$

and

$$f=\text{col}(f_0,f_1,\cdots,f_n). \tag{8}$$

It is very fast to calculate s from (13) as matrix A is triple-diagonal. The spline obtained in this way is called the interpolating spline.

Let us define two sets $\tilde{C}$ and $\tilde{C}(w)$ by

$$\tilde{C}=H\cap Nat(\Delta) = \{w\in Nat\Delta;|w(x_i)-f_i|\leqslant\xi_i, i=0,1,\cdots,n\} \tag{9}$$

$$\tilde{C}(w)=\{v\in Nat\Delta;|v(x_i)-g_i|\leqslant\xi_i, i=0,1,\cdots,n\} \tag{10}$$