Passive reciprocal state-space synthesis using a minimum number of resistors

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Abstract

The paper presents a state-space synthesis of multiport passive reciprocal networks using a minimum number of resistors. The method relies on a Gauss-factorisation procedure similar to Bayard's classical synthesis procedure. After a brief review of state-space formulation of the synthesis problem, a construction of a state-space realisation for a prescribed symmetric impedance is developed with the aid of a Gauss factorisation. The realisation constructed is nonminimal in general, but it is passive and used a minimal number of resistors. The constraint on the Gauss factorisation imposed by symmetry is then examined and interpreted in state-space terms to give a reciprocal synthesis.

1 Introduction

Problems of network synthesis have historically been of interest to the filter designer, and, despite the advent of RC active filters, there still remain many situations where passive filters are preferred, or indeed must be employed. But, despite the enormous amount known about passive filters, much still remains to be discovered. For example, it is known that two ways of synthetising a lowpass filter are to use a ladder arrangement of LC elements or a cascade of lattices. One way offers sensitivity problems in the passband, the other in the stopband. The question arises as to what compromises might be achievable with other synthesis arrangements, and what these synthesis arrangements might be.

It seems likely that the answer to questions such as this will arise when much more is known about the theory of equivalent networks, i.e. the theory explaining the relation between two (or more) networks possessing the same terminal behaviour. It also seems that the most promising tool for achieving further results in the theory of equivalent networks is the state-space description of networks, where connections between two or more differing equations describing networks with the same terminal behaviour are well understood. The reason the number of results derived via state-space ideas is still fairly limited is that techniques for translating state-space equation descriptions of networks into network synthesis (i.e. lists of components with an interconnection scheme) are few in number at present. Consequently, a search for new state-space syntheses procedures appears justified on the grounds that fallout in practical applications may result.

The problem of synthetising a prescribed rational positive real impedance function or matrix using state-space ideas has been considered in References 1–3, while Reference 4 discusses the synthesis of a bounded real scattering function or matrix using state-space techniques. Both Reference 1, which is a generalisation of Reference 2, and Reference 3 present network realisations which use a minimal number of reactive elements and resistors, and both methods will handle nonreciprocal synthesis problems.

When a synthesis of a symmetric impedance matrix is required to be reciprocal, i.e. to use no gyrators, both methods (as also the method in Reference 4) raise difficulties. Indeed, the very nature of the methods—forcing the synthetising network to have a minimum number of resistors and reactive elements—rules out the possibility of always deriving a reciprocal network without modification of the method. As Reference 5 points out, it is not in general the case that a symmetric positive real impedance matrix may be synthetised by a network that simultaneously is reciprocal, has the minimal number of resistors and has the minimal number of reactive elements. Hitherto available state-space syntheses have dropped the first requirement; here we shall drop the third.

Notwithstanding the above remarks, there are further reasons why the methods of References 1 and 3 tend to rule out reciprocal syntheses. In Reference 1, a lossless network is determined such that termination of some of the ports of this network in unit resistors leads to the prescribed impedance being observed at the remaining ports; the lossless network upon examination proves to have an impedance matrix $Z_L(s)$ with $Z_L(\infty)$ almost always skew. In Reference 3, a memoryless network is determined such that termination of some of the ports of this network in unit inductances leads to the prescribed impedance being observed at the remaining ports. Then there are never any capacitors present in the synthetising network, a constraint clearly often incompatible with reciprocity.

The difficulties with the methods of References 1 and 3 can be partly resolved by permitting the lossless and memoryless networks, respectively, to be described by hybrid matrices, rather than impedance matrices. But this does not resolve the basic difficulty, i.e. that it is not always possible to have reciprocity together with resistive and reactive-element minimality.

In the literature describing what may be termed classical synthesis procedures, reciprocal minimal-resistive syntheses may be found. For a good review, see Reference 5. One of the better known of these techniques is that due to Bayard,[6] which can be made to depend on a Gauss-factorisation procedure discussed extensively in Reference 5. It is Bayard's procedure essentially that will be described here in state-space terms.

It may well be asked what the advantages are of carrying out in state-space terms a synthesis akin to a classical procedure. Computationally, there is not perhaps a great deal to choose. But the state-space technique does offer greater scope for extension to problems such as the equivalent network problem, as discussed earlier.

In passing from a state-space realisation of $Z(s)$ to a reciprocal synthesis, as we do here, we have in part been guided by the ideas of Layton,[7] who, though unable to give a solution of the reciprocal-synthesis problem, gave a number of partial results.

The plan of the paper is as follows. In Section 2, we review the notions of the description of an impedance by state-space means, and the Gauss factorisation. Section 3 indicates straightforward preliminary simplifications for the synthesis problem, and Section 4 shows how, with the aid of a Gauss factorisation, to construct a state-space realisation for $Z(s)$ that eventually leads to a reciprocal synthesis. The realisation obtained is nonminimal, and results from using the Gauss factorisation. Section 5 is the core of the paper; here is examined the constraint in the Gauss factorisation imposed by reciprocity; an interpretation of the constraint is given in state-space terms, which yields a reciprocal synthesis. Sections 6 and 7 are devoted to examples and concluding remarks, respectively.

2 Review

In a state-space representation of an impedance $Z(s)$, the port current i becomes the input u and the port voltage v the output y of a set of state-space equations; thus

$$\dot{x} = Fx + Gu \quad \ldots \ldots \ldots \quad (1a)$$

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$$y = H'x + Ju \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

with $u \equiv i$ and $y \equiv v$. Of course, $Z(s) = J + H'(sI - F)^{-1}G$. Any quadruple $\{F, G, H, J\}$ for which this formula holds is termed a realisation of $Z(s)$, and a minimal realisation if F has minimal dimensions. We recall several key properties[8] of realisations: (a) If $\{F, G, H, J\}$ is one realisation of $Z(s)$, others are provided by $\{TFT^{-1}, TG, (T^{-1})'H, J\}$, where T ranges over the set of nonsingular matrices; (b) if $\{F, G, H, J\}$ is mimimal, all other minimal realisations are of the form $\{TFT^{-1}, TG, (T^{-1})' H, J\}$ for some nonsingular T; (c) if $\{F, G, H, J\}$ is minimal, the realisation is completely observable and controllable, and rank $[G, FG, \ldots F^{p-1}G] =$ rank $[H, F'H, \ldots (F')^{p-1}H] = p$, where F is $p \times p$.

If $Z(s)$ is rational positive real, the matrix $Z(s) + Z'(-s)$ is nonnegative definite for almost all $s = j\omega$, with ω real. Further, there exists an infinity of rational matrices $W(s)$ so that

$$Z(s) + Z'(-s) = W'(-s)W(s) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

There are many results covering the factorisation of eqn. 2; among these we note the so-called Gauss factorisation:[5]

Lemma 1: Let $Z(s)$ be a rational $n \times n$ positive real matrix. Suppose no element of $Z(s)$ possesses a purely imaginary pole, and that $Z(s) + Z'(-s)$ has rank r almost everywhere. Then there exists an $r \times r$ diagonal matrix N_1 of real Hurwitz polynomials, and an $n \times r$ matrix N_2 of real polynomials, so that

$$Z(s) + Z'(-s) = N_2(-s)[N_1(-s)]^{-1}[N_1(s)]^{-1}N_2'(s) \quad . \quad . \quad . \quad . \quad (3)$$

Moreover, if $Z(s) = Z'(s)$, N_1 and N_2 exist as above, with, also, $N_2(s) = N_2(-s)$.

The proof of this lemma given in Reference 5 is essentially constructive; i.e. N_1 and N_2 are found which satisfy eqn. 3. The factor $W(s) = [N_1(s)]^{-1}N_2'(s)$ so constructed will be used in Section 4 in the development of a state-space synthesis of $Z(s)$. Note that there is no claim that N_1 and N_2 are unique; indeed, they are not so, and neither is $[N_1(s)]^{-1}N_2'(s)$.

Frequently in network-synthesis problems, solutions $W(s)$ of eqn. 2 are required with the property that the $W(s)$ have minimal degree and constant ranks in Re $[s] > 0$. Reference 5 shows that the construction of such $W(s)$ may be a good deal more difficult computationally than the construction of the Gauss factorisation $W(s)$.

3 Preliminary simplifications for synthesis problem

In this Section, we show how the problem of synthetising an arbitrary positive real $Z(s)$ with a minimal number of resistors may be reduced to the problem of synthetising, with a minimal number of resistors, a positive real $\hat{Z}(s)$ so that (a) no element of $\hat{Z}(s)$ possesses a purely imaginary pole (b) $\hat{Z}(\infty)$ is nonsingular (c) normal rank $[\hat{Z}(s) + \hat{Z}'(-s)] =$ normal rank $[Z(s) + Z'(-s)]$ and (d) $\hat{Z}(s)$ is symmetrical if $Z(s)$ is symmetrical.

We shall first reduce the problem of synthetising $Z(s)$ to the problem of synthetising $Z_n(s)$ with $Z_n(s)$ satisfying all but (a) above.

If $Z(\infty)$ is nonsingular, no reduction is required. But suppose $Z(\infty)$ is singular. Then $Z(s)$ may be singular or nonsingular (almost everywhere).

In the former case, there exists a constant matrix T (not square) so that $Z(s) = T'Z_2(s)T$, where $Z_2(s)$ is a nonsingular positive real matrix.[5] Moreover, normal rank $[Z_2(s) + Z_2'(-s)] =$ normal rank $[Z(s) + Z'(-s)]$, and $Z_2(s)$ is symmetrical if $Z(s)$ is symmetrical. If $Z_2(s)$ is synthetisable, so is $Z(s)$, with the aid of a multiport transformer connected to the input ports of a network synthetising $Z_2(s)$. Similar remarks apply to an admittance $Y(s)$ which is singular; i.e. the synthesis of such an admittance is equivalent to the synthesis of an admittance which is nonsingular almost everywhere. Moreover, the positive real property, normal rank $[Y(s) + Y'(-s)]$, and symmetry if present, are all preserved in replacing the singular admittance by a nonsingular one.

Turning to the second case, evidently $Y_1(s) = [Z(s)]^{-1}$ exists and is positive real; but, because $Z(\infty)$ is singular, $Y_1(s)$ must have some elements with a pole at infinity. Then, as is well known,[5] we may write $Y_2(s) = Y_1(s) - sC$, where $Y_2(\infty)$ is finite, $Y_2(s)$ is positive real, C is nonnegative definite and the degree[5] of $Y_2(s)$ is less than the degree of $Y_1(s)$. Now the problem of synthetising $Y_2(s)$ is equivalent to the problem of synthetising $Y_1(s)$, and thus $Z(s)$: parallel connection of transformer-coupled capacitors to a synthesis of $Y_2(s)$ clearly yields a synthesis of $Y_1(s)$. Note that $Y_2(s) + Y_2'(-s) = Y_1(s) + Y_1'(-s) = Y_1(s)[Z(s) + Z'(-s)]Y_1'(-s)$, and thus normal rank $[Y_2(s) + Y_2'(-s)] =$ normal rank $[Z(s) + Z'(-s)]$. Finally, note that $Y_1(s)$, and thus $Y_2(s)$ are symmetrical if $Z(s)$ is symmetrical.

Now if $Y_2(s)$ is singular, proceed by finding a nonsingular $Y_3(s)$ and a constant matrix U so that $Y_2(s) = U'Y_3(s)U$. If $Y_2(s)$ is nonsingular, or if it is singular and $Y_3(s)$ has been found, examine $Y_2(\infty)$ [or $Y_3(\infty)$]. If this matrix is singular, proceed by inversion to obtain $Z_2(s) = [Y_2(s)]^{-1}$ (or $[Y_3(s)]^{-1}$), and then $Z_3(s) = Z_2(s) - sL$ etc. Eventually, one must arrive at $Z_n(s)$ such that $Z_n(\infty)$ is nonsingular, or else the successive degree reductions lead to $Y_n(s)$ or $Z_n(s)$ being constant (and thus immediately synthetisable). Disregarding this second case, we observe that, by transformer coupling and pole extractions at infinity, $Z_n(s)$ has $Z_n(\infty)$ nonsingular, normal rank $[Z_n(s) + Z_n'(-s)] =$ normal rank $[Z(s) + Z'(-s)]$ and $Z_n(s)$ is symmetrical if $Z(s)$ is symmetrical. A synthesis of $Z(s)$ follows from a synthesis of $Z_n(s)$ by addition of suitable transformers, capacitors and inductors.

To obtain $\hat{Z}(s)$, a well known device is used. Using a partial-fraction expansion or other technique, $Z_n(s)$ is written as the sum of two positive real impedances, one with elements possessing purely imaginary poles—call this $Z_L(s)$—and one with elements possessing no purely imaginary poles together with $Z_n(\infty)$—this being $\hat{Z}(s)$. The matrix $Z_L(s)$ is, of course, lossless positive real; synthesis procedures are well known; Reference 5 gives the classical approaches and Reference 9 gives the state-space approach.

Because $Z_n(\infty)$ is nonsingular, $\hat{Z}(\infty) = Z_n(\infty)$ is nonsingular. A standard property of lossless positive real matrices yields $Z_L(s) + Z_L'(-s) = 0$ for all s, and so $\hat{Z}(s) + \hat{Z}'(-s) = Z_n(s) + Z_n'(-s)$ and has normal rank equal to the normal rank of $Z(s) + Z'(-s)$. Finally, symmetry of $Z_n(s)$ is readily found to guarantee symmetry of $\hat{Z}(s)$ and $Z_L(s)$.

The fact that $\hat{Z}(s) + \hat{Z}'(-s)$ and $Z(s) + Z'(-s)$ have the same normal rank has a well known physical significance:[5] the minimal number of resistors required for synthetising either impedance is the same, being precisely this rank. Since we know how to obtain a synthesis of $Z(s)$ from a synthesis of $\hat{Z}(s)$, it follows that a synthesis of $\hat{Z}(s)$ with a minimal number of resistors yields a synthesis of $Z(s)$ with a minimal number of resistors.

4 Construction of a state-space realisation for a prescribed $Z(s)$

In accordance with the results of Section 3, we shall consider, with no loss of generality, the problem of synthetising an $n \times n$ positive real matrix $Z(s)$ so that no element possesses a purely imaginary pole. The constraints, such as a requirement that $Z(\infty)$ be nonsingular, will not be imposed for the moment. Suppose that normal rank $[Z(s) + Z'(-s)] = r$; then, by use of the Gauss-factorisation procedure of Reference 5, we may assume the availability of a diagonal $r \times r$ matrix of Hurwitz polynomials $N_1(s)$ and an $n \times r$ matrix of polynomials $N_2(s)$, even if $Z(s)$ is symmetric, so that one solution $W(s)$ of

$$Z(s) + Z'(-s) = W'(-s)W(s) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is $\quad W(s) = [N_1(s)]^{-1}N_2'(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$

The construction of a realisation of $Z(s)$ that eventually yields a synthesis will be preceded by the construction of a realisation for $W'(s) = N_2(s)[N_1(s)]^{-1}$: we shall determine matrices F, G, L and W_0 so that

$$W'(s) = W_0' + G'(sI - F)^{-1}L \quad . \quad . \quad . \quad . \quad . \quad (5)$$

[Subsequently, F and G will appear in a realisation of $Z(s)$.] W_0 is determined immediately from eqn. 4 on setting $s = \infty$. [It is a guaranteed part of the factorisation procedure indicated in lemma 1 that $W_0 = W(\infty) < \infty$.] To determine F and L, write*

$$[N(s)]^{-1} = \mathrm{diag}\left\{\frac{1}{p_1(s)}, \frac{1}{p_2(s)}, \ldots, \frac{1}{p_r(s)}\right\}$$

where each $p_i(s)$ is a Hurwitz polynomial. Define $\hat{F}_i'$ to be the companion matrix with $p_i(s)$ as characteristic polynomial, i.e.

$$\hat{F}_i' = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & & 1 \\ -p_{ir_i} & \cdot & \cdot & \cdot & -p_{i1} \end{bmatrix}$$

in obvious notation, and define $\hat{l}_i = [0 \;\; 0 \ldots 0 \;\; 1]'$. It is readily verified that $[\hat{F}_i', \hat{l}_i]$ is completely controllable[8] for each i, and that, for each i,

$$(sI - \hat{F}_i')^{-1}\hat{l}_i = \frac{1}{p_i(s)}[1 \;\; s \ldots s^{r_i-1}]'$$

Now define $\hat{F}$ and $\hat{L}$ by

$$\hat{F}' = \begin{bmatrix} \hat{F}_1' & 0 & \ldots & 0 \\ 0 & \hat{F}_2' & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ 0 & & & \hat{F}_r' \end{bmatrix} \quad \text{and } \hat{L} = \begin{bmatrix} \hat{l}_1 & 0 & \ldots & 0 \\ 0 & \hat{l}_2 & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ 0 & & & \hat{l}_r \end{bmatrix}$$

The pair $[\hat{F}', \hat{L}]$ is completely controllable because each $[\hat{F}_i', \hat{l}_i]$ is completely controllable.

We now seek a matrix $\hat{G}$ so that $W'(s) = W_0' + \hat{G}'(sI - \hat{F}')^{-1}\hat{L}$. Subsequently we shall use $\hat{F}$, $\hat{G}$ and $\hat{L}$ to define F, G and L *satisfying eqn. 5. From eqn. 4, we observe that the* $(i-j)$th entry of $V(s) = W'(s)$ is

$$v_{ij}(s) = \frac{n_{2ij}(s)}{p_j(s)} = v_{0ij} + \frac{\hat{n}_{ij}(s)}{p_j(s)}$$

where v_{0ij} is $v_{ij}(\infty)$, and the degree of $\hat{n}_{ij}(s)$ is less than the degree of $p_j(s)$ with $\hat{n}_{ij}(s)$ the residue of the polynomial $n_{2ij}(s)$, modulo $p_j(s)$. Let $\hat{n}_{ij}'$ be the row vector obtained from the coefficients of $\hat{n}_{ij}(s)$, arranged in ascending powers of s. *Then, since* $(sI - \hat{F}')^{-1}\hat{L}$ *has the form*

$$\begin{bmatrix} \dfrac{1}{p_1(s)}\begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\ s^{r_1-1} \end{bmatrix} & 0 & \cdot & \cdot \\ 0 & \dfrac{1}{p_2(s)}\begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\ s^{r_2-1} \end{bmatrix} & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \end{bmatrix}$$

it follows that

$$\hat{G}' = \begin{bmatrix} \hat{n}_{11}' & \hat{n}_{12}' & \ldots & \hat{n}_{1r}' \\ \hat{n}_{21}' & \cdot & & \\ \cdot & & \cdot & \\ \cdot & & & \\ \hat{n}_{n1}' & & & \end{bmatrix}$$

In the construction procedure for $\hat{F}'$ and $\hat{L}$, if there exists a $p_i(s)$ such that $p_i(s)$ is a constant, the corresponding $\hat{F}_i'$ block must be of zero order; i.e. the rows and columns corresponding to the block $\hat{F}_i'$ in the $\hat{F}'$ matrix must be missing. Correspondingly, the rows in the $\hat{L}$ matrix occupied by the vector $\hat{l}_i$ will evanesce, but the associated single column of zeros must remain to give the right dimension; since, if $\hat{F}'$ is $p \times p$, $\hat{L}$ is $p \times r$, where r is the normal rank of $Z(s) + Z'(-s)$. Consequently, the number of columns in the matrix $\hat{L}$ is always r. The matrix $\hat{G}'$ will have $\hat{n}_{mi}'$ zero for all m, and the corresponding column will also evanesce because $\hat{G}'$ has p columns.

Matrices F, G and L are determined from $\hat{F}$, $\hat{G}$ and $\hat{L}$ as follows. For each i, define P_i as the symmetric positive solution of $P_i\hat{F}_i + \hat{F}_i'P_i = -\hat{l}_i\hat{l}_i'$. As shown in Reference 10, such a P_i exists because the characteristic polynomial of $\hat{F}_i$ is of the Hurwitz type and the pair $[\hat{F}_i', \hat{l}_i]$ is completely controllable. Let T_i be any square matrix so that $T_i'T_i = P_i$, and define $F_i = T_i\hat{F}_iT_i^{-1}$ and $l_i = (T_i')^{-1}\hat{l}_i$. Also, define F to be the direct sum of the F_i, T to be the direct sum of the T_i and

$$L = \begin{bmatrix} l_1 & 0 & \cdot & \cdot \\ 0 & l_2 & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ \cdot & & & l_r \end{bmatrix}$$

Then $F = T\hat{F}T^{-1}$, $L = (T')^{-1}\hat{L}$, and, since $F_i + F_i' = -l_il_i'$,

$$F + F' = -LL' \quad \ldots\ldots\ldots\ldots \quad (6)$$

Finally, define $G = T\hat{G}$.

Then $\{F, G, L, W_0\}$ is a realisation of $W(s)$, with the additional properties that eqn. 6 holds and $[F, L]$ is completely observable (the latter property follows from the complete observability of $[\hat{F}, \hat{L}]$, or, equivalently, from the complete controllability of $[\hat{F}', \hat{L}]$).

Define now

$$H = G + LW_0 \quad \ldots\ldots\ldots\ldots \quad (7)$$

Then, with F, G and H defined as above, and with $J = Z(\infty)$, $\{F, G, H, J\}$ is a realisation of $Z(s)$. To see this, observe that

$$\begin{aligned} W'(-s)W(s) = {} & W_0'W_0 + W_0'L'(sI - F)^{-1}G \\ & + G'(-sI - F')^{-1}LW_0 \\ & + G'(-sI - F')^{-1}LL'(sI - F)^{-1}G \end{aligned}$$

From eqn. 6, the right-hand side becomes

$$\begin{aligned} & W_0'W_0 + W_0'L'(sI - F)^{-1}G + G'(-sI - F')^{-1}LW_0 \\ & \quad + G'(-sI - F')^{-1}[-sI - F' + sI - F](sI - F)^{-1}G \\ & = W_0'W_0 + (G' + W_0'L')(sI - F)^{-1}G \\ & \quad + G'(-sI - F')^{-1}(LW_0 + G) \end{aligned}$$

From eqn. 7, this becomes

$$\begin{aligned} W'(-s)W(s) = {} & W_0'W_0 + H'(sI - F)^{-1}G \\ & + G'(-sI - F')^{-1}H, \end{aligned}$$

The above quantity is also $Z(s) + Z'(-s)$. The fact that F is a direct sum of matrices all of which have a Hurwitz characteristic polynomial then guarantees that $Z(s) = Z(\infty) + H'(sI - F)^{-1}G$, since every element of $Z(s)$ can only have left-halfplane poles.

The above procedure for the construction of a realisation of $W(s)$ and $Z(s)$ has nowhere made use of the symmetry of $Z(s)$. The procedure up to this point is therefore valid for symmetric and nonsymmetric $Z(s)$. The procedure yields a passive synthesis of an arbitrary positive real $Z(s)$ which, in general, is nonreciprocal, *in the following way.*

The matrix

$$Z_c = \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix}$$

may be regarded as the impedance matrix of a nondynamic network. When all but the first n ports are terminated in unit inductances, the impedance seen at the first n ports is readily found to be[3]

$$Z(s) = J + H'(sI - F)^{-1}G$$

It follows that, if $Z_c + Z_c'$ is nonnegative definite, implying that Z_c is synthetisable with passive elements, a passive synthesis of $Z(s)$ results. Indeed, the nonnegative definite con-

* $A = \mathrm{diag}\,\{a_1, a_2, \ldots, a_n\}$ means that A is a diagonal matrix with diagonal entries in order $a_1, a_2, \ldots a_n$

straint holds, for it is easy to check, using eqns. 6 and 7 and the constraint $W_0'W_0 = J + J'$, following from eqn. 2, that

$$Z_c + Z_c' = \begin{bmatrix} W_0' \\ -L \end{bmatrix} [W_0 \quad -L']$$

The synthesis is, in general, nonreciprocal, since $Z_c = Z_c'$ will not, in general, hold.

5 Reciprocal synthesis of a symmetric positive real $Z(s)$

As indicated in Section 3, there is no loss of generality in assuming that $Z(\infty)$ is nonsingular, and that no element of $Z(s)$ has a purely imaginary pole. Because $Z(s)$ is positive real, $Z(j\omega) + Z'(-j\omega)$ is nonnegative definite for all ω; from the symmetry of $Z(s)$, and letting $\omega \to \infty$, it follows that $Z(\infty)$ is nonnegative definite, and, by its nonsingularity, positive definite.

Since, then, normal rank $[Z(s)+Z'(-s)] >$ rank $[Z(\infty) + Z'(-\infty)] =$ rank $[2Z(\infty)]$, it follows that normal rank $[Z(s) + Z'(-s)]$ is n. Then $W(s)$, $N_1(s)$ and $N_2(s)$ as defined in Section 4 are $n \times n$ matrices. Further, $W_0'W_0 = 2J$, and thus W_0 is nonsingular.

We note the following result:

Lemma 2: With all quantities as defined in Section 4, $L'(sI - F)^{-1}L$ is symmetric, and

$$-H'(sI - F)^{-1}L = G'(sI - F')^{-1}L \quad . \quad . \quad . \quad . \quad (8)$$

Proof: Consideration of the definitions of F and L shows that $L'(sI - F)^{-1}L$ is a diagonal matrix, the ith diagonal entry being $l_i'(sI - F_i)^{-1}l_i$. Hence the desired symmetry is present.

To prove eqn. 8 we shall, in fact, prove that

$$W_0' - H'(sI - F)^{-1}L = W_0' + G'(sI - F')^{-1}L \quad . \quad (9)$$

Of course, the right-hand side of eqn. 9 is precisely $W'(s)$.

The ijth element of the right-hand side of eqn. 9 has already been written down as

$$\frac{n_{2ij}(s)}{p_j(s)} = v_{0ij} + \frac{\hat{n}_{ij}(s)}{p_j(s)} = v_{0ij} + \boldsymbol{n}_{ij}'(sI - F_j')^{-1}\, l_j$$

for a certain $\boldsymbol{n}_{ij}'$. [Note that $\boldsymbol{n}_{ij}'$ occupies that position in G which corresponds to the position in $\hat{G}$ occupied by $\hat{\boldsymbol{n}}_{ij}'$. Moreover, $\boldsymbol{n}_{ij}' = \hat{\boldsymbol{n}}_{ij}'T_j$.]

Evidently,*

$$\begin{aligned} n_{2ij}(s) &= v_{0ij} \det(sI - F_j') + \boldsymbol{n}_{ij}' \operatorname{adj}(sI - F_j')l_j \\ &= v_{0ij} \det(-sI - F_j') + \boldsymbol{n}_{ij}' \operatorname{adj}(-sI - F_j')l_j \end{aligned}$$

on using the even nature of $n_{2ij}(s)$. Now the highest power of s occurring in $n_{2ij}(s)$ is r_j, where F_j is $r_j \times r_j$, provided that v_{0ij} is nonzero. The nonsingularity of W_0 guarantees that, for any j, there is an i (say $i = I$) such that v_{0Ij} is nonzero. The evenness of $n_{2Ij}(s)$ then implies that r_j is even. The argument clearly holds for all j. From the evenness of r_j,

$$\begin{aligned} n_{2ij}(s) &= v_{0ij} \det(sI + F_j') - \boldsymbol{n}_{ij}' \operatorname{adj}(sI + F_j')l_j \\ &= \det(sI - F_j - l_jl_j')[v_{0ij} - \boldsymbol{n}_{ij}'(sI + F_j')^{-1}l_j'] \end{aligned}$$

on using $F_j + F_j' = -l_jl_j'$. Now $\det(sI - F_j - l_jl_j') = \det(sI - F_j)\det[I - (sI - F_j)^{-1}l_jl_j'] = \det(sI - F_j)[1 - l_j'(sI - F_j)^{-1}l_j]$. Hence

$$\begin{aligned} n_{2ij}(s) = \det(sI - F_j)[v_{0ij} \\ - \boldsymbol{n}_{ij}'(sI + F_j')^{-1}l_j][1 - l_j'(sI - F_j)^{-1}l_j] \end{aligned}$$

Now observe that $(sI + F_j')^{-1}l_jl_j'(sI - F_j)^{-1} = (sI + F_j')^{-1}[(sI - F_j) - (sI + F_j')](sI - F_j)^{-1} = (sI + F_j')^{-1} - (sI - F_j)^{-1}$. Hence

$$n_{2ij}(s) = \det(sI - F_j)[v_{0ij} - (v_{0ij}l_j' + \boldsymbol{n}_{ij}')(sI - F_j)^{-1}l_j]$$

and thus

$$\frac{n_{2ij}(s)}{p_j(s)} = v_{0ij} - (v_{0ij}l_j' + \boldsymbol{n}_{ij}')(sI - F_j)^{-1}l_j$$

* adj A stands for the adjoint of A, i.e. adj $A = |A|A^{-1}$

But the $(i - j)$th element of the left-hand side of eqn. 9 can readily be shown (using $H = G + LW_0$) to be

$$v_{0ij} - (\boldsymbol{n}_{ij}' + v_{0ij}l_j')(sI - F_j)^{-1}l_j$$

which proves lemma 2.

The idea behind the synthesis procedure is to replace the realisation $\{F, G, H, J\}$ of $Z(s)$ by a realisation $\{TFT', TG, TH, J\}$, where T is an orthogonal matrix, so that the new realisation possesses additional properties to those possessed by the realisation $\{F, G, H, J\}$. These additional properties make a reciprocal synthesis straightforward.

In order to generate T, we shall make use of lemma 2. First, we shall define a symmetric matrix P. Then we shall show that P satisfies a number of constraints, including a constraint that all its eigenvalues be $+1$ or -1. Then we shall take T as any orthogonal matrix so that $P = T'\Sigma T$, where Σ is a diagonal matrix with diagonal entries all $+1$ or -1.

Lemma 3: With F, G, H, L and W_0 as defined in Section 4, the equations

$$FP = PF' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10a)$$

$$PL = L \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10b)$$

define a unique matrix P. Moreover, P is symmetric, and the following equations also hold:

$$PH = -G \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11a)$$

$$PG = -H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11b)$$

$$PF = F'P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11c)$$

$$P = P^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11d)$$

Proof: We have, with F a $p \times p$ matrix,

$$\begin{aligned} &P[L \quad F'L \quad (F')^2L \quad \ldots \quad (F')^{p-1}L] \\ &\quad = [L \quad FL \quad F^2L \quad \ldots \quad (F)^{p-1}L] \quad . \quad . \quad . \quad (12) \end{aligned}$$

from eqn. 10. The matrix $[L\ F'L \ldots (F')^{p-1}L]$ has rank p, since $[F', L]$ is completely controllable. Thus P is uniquely defined.

We now observe that $[F, L]$ is completely controllable. The reasoning is as follows: because $[F', L]$ is completely controllable, $[F' - LK', L]$ is completely controllable for all K. Take $K = -L$; this yields the result that $[F' + LL', L]$ or $[-F, L]$ is completely controllable.

It then follows from eqn. 12 that P must be nonsingular. Consequently,

$$\begin{aligned} L'(sI - F)^{-1}L &= L'P(sI - P^{-1}FP)^{-1}P^{-1}L \\ &= L'P(sI - F')^{-1}L \end{aligned}$$

from eqn. 10, while

$$L'(sI - F)^{-1}L = L'(sI - F')^{-1}L$$

using lemma 2.

The complete controllability of $[F', L]$ implies that $L'P = L'$ or $P'L = L$. Eqn. 10a yields $FP' = P'F'$ on transposition, and evidently P and P' both satisfy $FX = XF'$ and $XL = L$. Since the solution of these equations is unique, it follows that $P = P'$.

Now consider eqn. 8 of lemma 2. We have

$$\begin{aligned} G'(sI - F')^{-1}L &= -H'(sI - F)^{-1}L \\ &= -H'P(sI - P^{-1}FP)^{-1}P^{-1}L \\ &= -H'P(sI - F')^{-1}L \end{aligned}$$

from which

$$PH = -G \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11a)$$

Since $H = G + LW_0$, we have, from eqn. 11a,

$$-G = PG + PLW_0 = PG + LW_0$$

or $\quad PG = -H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11b)$

Next, from $F + F' = - LL'$, we have

$$PF = - PF' - PLL'$$
$$= - FP - LL' \text{ from eqns. } 10a \text{ and } b$$
$$= - FP - LL'P \text{ from } P = P' \text{ and eqn. } 10b$$
$$= - (F + LL')P$$
$$= F'P \text{ from eqn. 6} \quad (11c)$$

From eqns. $10b$ and $11c$, we see that $P^{-1}L = L$ and $FP^{-1} = P^{-1}F'$, or, in other words, P^{-1} satisfies $FX = XF'$ and $XL = L$. Hence eqn. $11d$ holds, proving lemma 3.

Eqn. $11d$ implies that $P^2 = I$, and thus the eigenvalues of P must be $+1$ or -1. Since P is symmetric, there exists an orthogonal matrix T such that $P = T'\Sigma T$, with Σ consisting of $+1$ and -1 entries in the diagonal positions and zeros elsewhere.

Now define a realisation $\{F_1, G_1, H_1, J\}$ of $Z(s)$ by $F_1 = TFT'$, $G_1 = TG$ and $H_1 = TH$, and define the hybrid matrix

$$M = \begin{bmatrix} J & -H_1' \\ G_1 & -F_1 \end{bmatrix} \quad (13)$$

Lemma 4: With quantities as defined above, it is claimed that

(a) $M + M'$ is nonnegative definite with rank n

(b) With $\dotplus$ denoting a direct sum, $(I_n \dotplus \Sigma)M$ is symmetric

Proof:

(a)
$$M + M' = (I_n \dotplus T)\begin{bmatrix} 2J & -H' + G' \\ G - H & -F - F' \end{bmatrix}(I_n \dotplus T')$$
$$= (I_n \dotplus T)\begin{bmatrix} W_0'W_0 & -W_0'L' \\ -LW_0 & +LL' \end{bmatrix}(I_n \dotplus T')$$
$$= (I_n \dotplus T)\begin{bmatrix} W_0' \\ -L \end{bmatrix}[W_0 \quad -L'](I_n \dotplus T')$$

The rank condition is obviously fulfilled, since $[W_0 \quad -L']$ has rank no less than W_0, which has rank n.

(b) It is necessary and sufficient to show that $J = J'$, $\Sigma G_1 = - H_1$ and $\Sigma F_1 = F_1'\Sigma$. The first is immediate from the symmetry of $Z(s)$. From eqn. $11b$, $T'\Sigma TG = - H$ or $\Sigma(TG) = - TH$, which is $\Sigma G_1 = - H_1$. Finally, from eqn. $11c$, $T'\Sigma TF = F'T'\Sigma T$ or $\Sigma TFT' = TF'T'\Sigma$, i.e. $\Sigma F_1 = F_1'\Sigma$.

With lemma 4 in hand, the main work is completed.[7]

The matrix M is realisable as a hybrid matrix, the current-excited ports corresponding to those rows of $I_n \dotplus \Sigma$ which have $+1$ on the diagonal, and the voltage-excited ports corresponding to those rows of $I_n \dotplus \Sigma$ which have -1 on the diagonal. Part (a) of lemma 4 guarantees that the realisation can be achieved using passive components only, and part (b) that reciprocal components only need be used. (See Reference 7 for a full discussion and the following Section for examples.) The fact that $M + M'$ has rank n means that the synthesis of M can be achieved with n resistors, but no fewer.

A synthesis of $Z(s)$, of course, follows from that for M, by terminating current-excited ports other than the first n of the network synthetising M in unit inductances and voltage-excited ports in unit capacitances. Note that the synthesis of $Z(s)$ uses a minimal number of resistors as required, because n = normal rank $[Z(s) + Z'(-s)]$.

As a summary of the synthesis procedure, we separate the following steps from the preceding ideas, eliminating the existence results:

(a) By transformer, inductance and capacitance extractions, reduce the general problem to one of synthetising an $n \times n$ $Z(s)$, with $Z(\infty)$ nonsingular, rank $[Z(s) + Z'(-s)] = n$, no element of $Z(\cdot)$ possessing a pole on the $j\omega$ axis and, of course, $Z(s) = Z'(s)$.

(b) Perform a Gauss factorisation, as applicable for symmetric $Z(s)$.

(c) Construct $\hat{F}$ using companion matrices, and $\hat{L}$ and $\hat{G}$ from $W(s)$. From $\hat{F}$, $\hat{L}$ and $\hat{G}$, determine F, L and G by solving a number of equations like $P_i\hat{F}_i + \hat{F}_i'P_i = - \hat{l}_i\hat{l}_i'$.

(d) Find $H = G + LW_0$, and P as the unique solution of $FP = PF'$ and $PL = L$.

(e) Find an orthogonal matrix T reducing P to a diagonal matrix Σ of $+1$ and -1 elements; i.e. $P = T'\Sigma T$.

(f) With $F_1 = TFT'$, $G_1 = TG$ and $H_1 = TH$, define the hybrid matrix M of eqn. 13 and give a reciprocal synthesis. Terminate the appropriate ports in unit inductances and capacitances to generate $Z(s)$.

For synthesis purposes, it is convenient to choose T so that all the $+1$ elements on the main diagonal of Σ precede all the -1 elements. This is, of course, always possible.

6 Synthesis examples

6.1 Example 1: synthesis of the symmetric positive real impedance

$$Z(s) = \begin{bmatrix} 1 & 1 \\ 1 & \dfrac{3s+3}{s+2} \end{bmatrix}$$

Since $Z(s)$ has no poles on the $j\omega$ axis, and $Z(\infty)$ is nonsingular, we factorise $Z(s) + Z'(-s)$ as follows:

$$Z(s) + Z'(-s) = 2\begin{bmatrix} 1 & 1 \\ 1 & \dfrac{6-3s^2}{4-s^2} \end{bmatrix}$$

$$= 2\begin{bmatrix} 4-s^2 & 0 \\ 4-s^2 & 2(1-s^2)(4-s^2) \end{bmatrix}\begin{bmatrix} \dfrac{1}{(4-s^2)^2} & 0 \\ 0 & \dfrac{1}{2(1-s^2)(4-s^2)^3} \end{bmatrix}\begin{bmatrix} 4-s^2 & 4-s^2 \\ 0 & 2(1-s^2)(4-s^2) \end{bmatrix}$$

$$= 2\begin{bmatrix} 1 & 0 \\ 1 & \sqrt{2}(1-s^2) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{(1-s^2)(4-s^2)} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & \sqrt{2}(1-s^2) \end{bmatrix}$$

Thus

$$W(s) = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{(1+s)(2+s)} \end{bmatrix}\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 2(1-s^2) \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{6s+6}{s^2+3s+2} \end{bmatrix}$$

i.e. $$[N_1(s)]^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{s^2+3s+2} \end{bmatrix}$$

and $$N_2'(s) = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 2(1-s^2) \end{bmatrix}$$

The common factor $(1 + s)$ of $[N_1(s)]^{-1}$ and $N_2'(s)$ is left uncancelled, so that $N_2(s) = N_2(-s)$; otherwise $Z(s)$ cannot be synthetised with reciprocal elements only.

Note that $p_1(s) = 1$ is a constant and no F_1' is needed. Thus

$$\hat{F}' = \hat{F}_2' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad \hat{L} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \hat{G}' = \begin{bmatrix} 0 & 0 \\ 6 & 6 \end{bmatrix}$$

Form P_2 from $P_2\hat{F}_2 + \hat{F}_2'P_2 = -\hat{l}_2\hat{l}_2' = -\hat{L}\hat{L}'$ to give

$$P = P_2 = \begin{bmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$$

From $T_2'T_2 = P_2$, we obtain

$$T_2 = \begin{bmatrix} \frac{1}{2\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Hence

$$F = T_2\hat{F}T_2^{-1} = \begin{bmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & -3 \end{bmatrix}$$

$$L = (T_2')^{-1}\hat{L} = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{6} \end{bmatrix}$$

and $\quad G = T_2\hat{G} = \begin{bmatrix} 0 & \sqrt{3} \\ 0 & \sqrt{6} \end{bmatrix}$

Define

$$H = G + LW_0 = \begin{bmatrix} 0 & \sqrt{3} \\ 0 & -\sqrt{6} \end{bmatrix}$$

By direct computation, it is readily verified that

$$H'(sI - F)^{-1}G = \begin{bmatrix} 0 & 0 \\ 0 & \frac{-3(1+s)}{s^2+3s+2} \end{bmatrix}$$

and $J + H'(sI - F)^{-1}G$ is the prescribed $Z(s)$, where

$$J = Z(\infty) = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

Thus $\{F, G, H, J\}$ is a (nonminimal) realisation of $Z(s)$, which readily yields a passive-network realisation since $\hat{M} = \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix}$ can be shown to have a nonnegative definite symmetric part.

To generate a reciprocal passive synthesis, define a unique symmetric matrix P by means of the equations

$$FP = PF'$$
$$PL = L$$

from which P is seen to be

$$P = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Therefore, from $P = T'\Sigma T$, we are led to take

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $\quad \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

The final required realisation $\{F_1, G_1, H_1, J\}$ of $Z(s)$ is

$$F_1 = TFT' = \begin{bmatrix} -3 & \sqrt{2} \\ -\sqrt{2} & 0 \end{bmatrix}$$

$$G_1 = TG = \begin{bmatrix} 0 & \sqrt{6} \\ 0 & \sqrt{3} \end{bmatrix} \qquad H_1 = TH = \begin{bmatrix} 0 & -\sqrt{6} \\ 0 & \sqrt{3} \end{bmatrix}$$

[Note that $\{F, G, H, J\}$ actually readily yields a reciprocal passive network, since one can take $\Sigma = P$ and $T = I_p$; then

$$(I_n + \Sigma)\begin{bmatrix} J & -H' \\ G & -F \end{bmatrix}$$

is symmetric, so that, in effect, no basis change is required for $\{F, G, H, J\}$. However, to synthetise the hybrid matrix

$$\begin{bmatrix} J & -H' \\ G & -F \end{bmatrix}$$

it may be desirable to group the open-circuited ports together and the short-circuited ports together.]

Next form the hybrid matrix

$$M = \begin{bmatrix} J & -H_1' \\ G_1 & -F_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & \sqrt{6} & -\sqrt{3} \\ 0 & \sqrt{6} & 3 & -\sqrt{2} \\ 0 & \sqrt{3} & \sqrt{2} & 0 \end{bmatrix}$$

which is a positive real matrix with $(I_n + \Sigma)\,M$ symmetric. To synthetise the network N_1 from M (see Reference 9), we first realise

$$Z_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & \sqrt{6} \\ 0 & \sqrt{6} & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & \sqrt{2} & \sqrt{3} \end{bmatrix}$$

by the network of Fig. 1; $Y_1 = [0]$ merely represents an open circuit, and $T_1 = [0 \;\; \sqrt{3}]$ and $T_2 = [\sqrt{2}]$ are the two coupling-transformer ratios.

Therefore the network N_1 of hybrid matrix M has the synthesis of Fig. 2 with ports 1, 2 and 3 as open-circuit ports, and port 4 as a short-circuit port. The network synthetising $Z(s)$ is found by terminating port 3 of N_1 in a unit inductance and port 4 in a unit capacitance, as shown in Fig. 3.

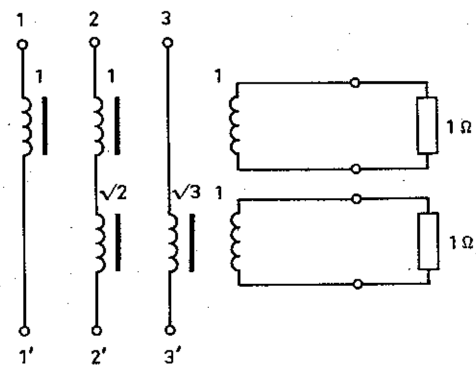

Fig. 1
Synthesis of Z_1 for example 1

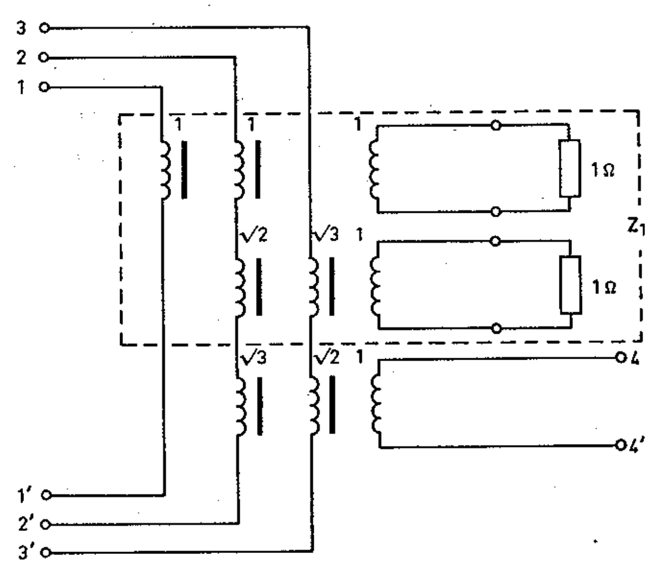

Fig. 2
Synthesis of M for example 1

The synthesis uses two resistors, which is the minimum number possible.

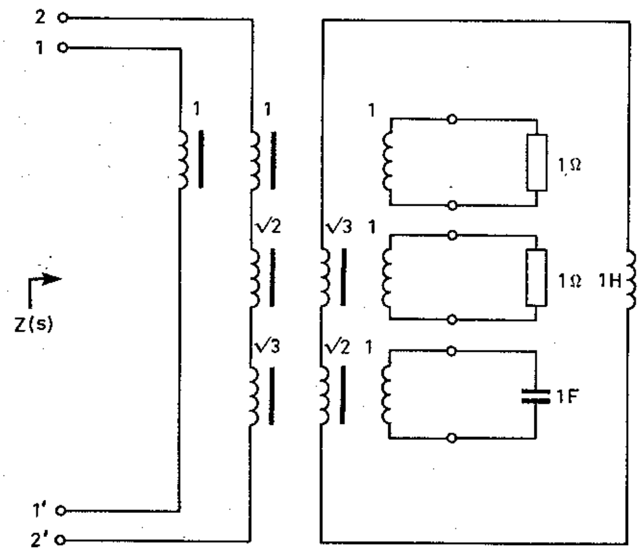

Fig. 3
Final circuit synthetising $Z(s)$ of example 1

6.2 Example 2: synthesis of the scalar positive real impedance

$$z(s) = \frac{s^2+2s+4}{s^2+s+1}$$

Since $z(s)$ has no poles on the $j\omega$ axis, and $z(\infty)$ is not zero,

$$z(s)+z(-s) = \frac{2(s^4+3s^2+4)}{(s^2+s+1)(s^2-s+1)}$$

$$= \frac{2(s^2+s+2)(s^2-s+2)}{(s^2+s+1)(s^2-s+1)}$$

It is readily seen that candidates for $W(s)$ are

$$\frac{\sqrt{2}\,(s^2+s+2)}{s^2+s+1} \text{ or } \frac{\sqrt{2}\,(s^2-s+2)}{s^2+s+1}$$

But the numerators are not even functions. To get the numerator of a $W(s)$ to be an even function, multiply $z(s)+z(-s)$ by $(s^4+3s^2+4)/(s^4+3s^2+4)$, so that

$$W(s) = \frac{\sqrt{2}\,(s^2-s+2)(s^2+s+2)}{(s^2+s+1)(s^2+s+2)}$$

$$= \frac{\sqrt{2}\,(s^4+3s^2+4)}{s^4+2s^3+4s^2+3s+2}$$

The process is equivalent to modifying the impedance $z(s)$ by multiplication by $(s^2+s+2)/(s^2+s+2)$, a standard technique used in the classical Darlington synthesis. [Therefore, the Gauss factorisation which gives $N_2(s) = N_2(-s)$ for $Z(s) = Z'(s)$ in multiports, often only by inserting common factors into $N_1(s)$ and $N_2(s)$, may be regarded as a generalisation of the classical 1-port Darlington procedure.]

We write

$$W'(s) = \sqrt{2} + \frac{-2\sqrt{2}s^3 - \sqrt{2}s^2 - 3\sqrt{2}s + 2\sqrt{2}}{s^4+2s^3+4s^2+3s+2}$$

a realisation of which is

$$\hat{F}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -4 & -2 \end{bmatrix} \qquad \hat{L} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\hat{G}' = [2\sqrt{2} \quad -3\sqrt{2} \quad -\sqrt{2} \quad -2\sqrt{2}]$$

Obtain P_1 from $P_1\hat{F} + \hat{F}'P_1 = -\hat{L}\hat{L}'$, to give

$$P_1 = \begin{bmatrix} \frac{5}{28} & 0 & -\frac{1}{7} & 0 \\ 0 & \frac{1}{7} & 0 & -\frac{3}{14} \\ -\frac{1}{7} & 0 & \frac{3}{14} & 0 \\ 0 & -\frac{3}{14} & 0 & \frac{4}{7} \end{bmatrix}$$

One matrix T_1 such that $T_1'T_1 = P$ is

$$T_1 = \begin{bmatrix} \frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ -\sqrt{\left(\frac{2}{21}\right)} & 0 & \sqrt{\left(\frac{3}{14}\right)} & 0 \\ 0 & -\frac{3}{4\sqrt{7}} & 0 & \frac{2}{\sqrt{7}} \end{bmatrix}$$

$$\text{and } T_1^{-1} = \begin{bmatrix} 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ \frac{4}{\sqrt{3}} & 0 & \sqrt{\left(\frac{14}{3}\right)} & 0 \\ 0 & \frac{3}{2} & 0 & \frac{\sqrt{7}}{2} \end{bmatrix}$$

Therefore

$$F = T_1\hat{F}T_1^{-1} = \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{7}}{2\sqrt{3}} \\ \frac{\sqrt{3}}{2} & -\frac{9}{8} & 0 & -\frac{3\sqrt{7}}{8} \\ 0 & 0 & 0 & -\frac{2\sqrt{2}}{\sqrt{3}} \\ \frac{\sqrt{7}}{2\sqrt{3}} & -\frac{3\sqrt{7}}{8} & 2\sqrt{\left(\frac{2}{3}\right)} & -\frac{7}{8} \end{bmatrix}$$

$$L = T_1^{-1}\hat{L} = \begin{bmatrix} 0 \\ \frac{3}{2} \\ 0 \\ \frac{\sqrt{7}}{2} \end{bmatrix} \qquad G = T_1\hat{G} = \begin{bmatrix} \sqrt{\frac{2}{3}} \\ -\frac{3}{2\sqrt{2}} \\ -\sqrt{\frac{7}{3}} \\ -\frac{\sqrt{7}}{2\sqrt{2}} \end{bmatrix}$$

Define

$$H = G + LW_0 = \begin{bmatrix} \sqrt{\left(\frac{2}{3}\right)} \\ -\frac{3}{2\sqrt{2}} \\ -\sqrt{\left(\frac{7}{3}\right)} \\ -\frac{\sqrt{7}}{2\sqrt{2}} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{3}{\sqrt{2}} \\ 0 \\ \sqrt{\left(\frac{7}{2}\right)} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{\left(\frac{2}{3}\right)} \\ \frac{3}{2\sqrt{2}} \\ -\sqrt{\left(\frac{7}{3}\right)} \\ \frac{\sqrt{7}}{2\sqrt{2}} \end{bmatrix}$$

The solution of $FP = PF'$ and $PL = L$ is

$$P = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Hence

$$T = T' = T^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and $\Sigma = \mathrm{diag}\{1, 1, -1, -1\}$

Also

$$M = \begin{bmatrix} J & -H'T' \\ TG & -TFT' \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{\sqrt{7}}{2\sqrt{2}} & -\frac{3}{2\sqrt{2}} & \sqrt{\left(\frac{7}{3}\right)} & -\sqrt{\left(\frac{2}{3}\right)} \\ -\frac{\sqrt{7}}{2\sqrt{2}} & \frac{7}{8} & \frac{3\sqrt{7}}{8} & -2\sqrt{\left(\frac{2}{3}\right)} & -\frac{\sqrt{7}}{2\sqrt{3}} \\ -\frac{3}{2\sqrt{2}} & \frac{3\sqrt{7}}{8} & \frac{9}{8} & 0 & -\frac{\sqrt{3}}{2} \\ -\sqrt{\left(\frac{7}{3}\right)} & 2\sqrt{\left(\frac{2}{3}\right)} & 0 & 0 & 0 \\ \sqrt{\left(\frac{2}{3}\right)} & \frac{\sqrt{7}}{2\sqrt{3}} & \frac{\sqrt{3}}{2} & 0 & 0 \end{bmatrix}$$

Consider the impedance matrix

$$Z_1 = \begin{bmatrix} 1 & -\frac{\sqrt{7}}{2\sqrt{2}} & -\frac{3}{2\sqrt{2}} \\ -\frac{\sqrt{7}}{2\sqrt{2}} & \frac{7}{8} & \frac{3\sqrt{7}}{8} \\ -\frac{3}{2\sqrt{2}} & \frac{3\sqrt{7}}{8} & \frac{9}{8} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -\frac{\sqrt{7}}{2\sqrt{2}} \\ -\frac{3}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & -\frac{\sqrt{7}}{2\sqrt{2}} & -\frac{3}{2\sqrt{2}} \end{bmatrix}$$

which has a synthesis of Fig. 4. The network N_1 of hybrid matrix M then has a synthesis shown in Fig. 5.

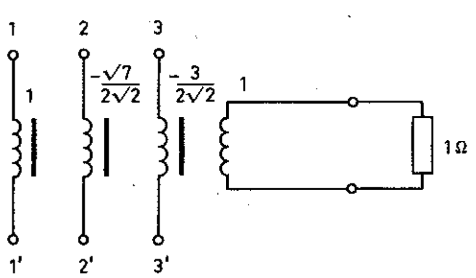

Fig. 4
Synthesis of Z_1 of example 2

The network synthetising $z(s)$ is formed by terminating ports 2 and 3 of N_1 in unit inductances and ports 4 and 5 in unit capacitances, leading to Fig. 6.

The complete network uses one resistor.

A closer parallel to the Darlington synthesis would involve determination of a 2-port lossless network (in transfer-function matrix or state-space form), so that termination of one port in a unit resistance would yield $z(s)$. Current work is exploring this approach, which can be expected to cut down on the number of transformers, yielding a far more acceptable network topology.

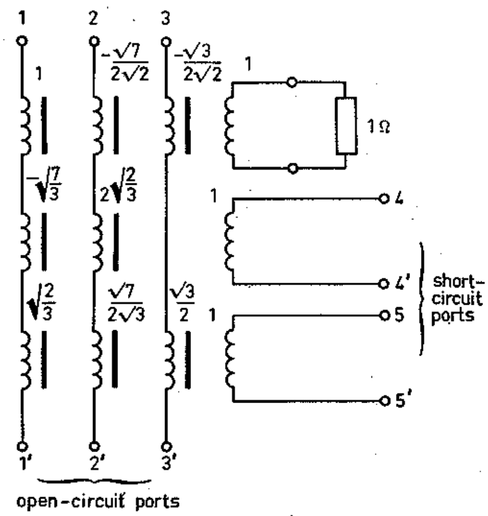

Fig. 5
Synthesis of M for example 2

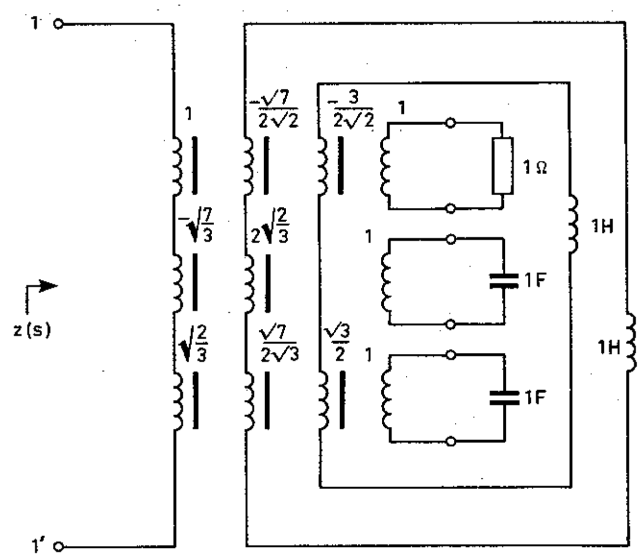

Fig. 6
Circuit yielding $z(s)$ of example 2

7 Conclusions

The preceding material suggests a number of problems. For example, it is not clear, other than because the reciprocal Bayard synthesis works, why the procedure presented here should work. The solution of the problem of obtaining all reciprocal syntheses via state-space procedures is apparently not in sight, and it is not even clear how reciprocal syntheses with a minimal number of resistors differ from the state-space point of view. Certainly, it is known that the Gauss-factorisation procedure does not yield[5] a unique $W(s)$, and it is clear that differing $W(s)$ will yield differing state-space syntheses; but this is some distance away from identifying the difference between these syntheses.

It might be imagined that the theory of minimal state-space realisations, more highly developed than the theory of (arbitrary dimension) realisations, should offer promise in the obtaining of a reciprocal synthesis using a minimal number of reactive elements and a nominimal number of resistors. Such a synthesis would depend on factorising $Z(s) + Z'(-s)$ to obtain a $W(s)$ with a nonminimal number of columns. Such factorisations have recently been studied from the state-space viewpoint,[11] but they have not been deeply studied in network theory.

When minimality of both the number of reactive elements and the number of resistive elements is not required, reactance extraction and Darlington-type synthesis are possible,[12] based on the work of Koga[13] on multivariable positive real synthesis.

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