Passive reciprocal state-space synthesis using a minimum number of resistors

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Abstract

The paper presents a state-space synthesis of multiport passive reciprocal networks using a minimum number of resistors. The method relies on a Gauss-factorisation procedure similar to Bayard's classical synthesis procedure. After a brief review of state-space formulation of the synthesis problem, a construction of a state-space realisation for a prescribed symmetric impedance is developed with the aid of a Gauss factorisation. The realisation constructed is nonminimal in general, but it is passive and used a minimal number of resistors. The constraint on the Gauss factorisation imposed by symmetry is then examined and interpreted in state-space terms to give a reciprocal synthesis.

1 Introduction

Problems of network synthesis have historically been of interest to the filter designer, and, despite the advent of RC active filters, there still remain many situations where passive filters are preferred, or indeed must be employed. But, despite the enormous amount known about passive filters, much still remains to be discovered. For example, it is known that two ways of synthesising a lowpass filter are to use a ladder arrangement of LC elements or a cascade of lattices. One way offers sensitivity problems in the passband, the other in the stopband. The question arises as to whether the terminal behaviour are well understood. The reason the more differing equations describing networks with the same characteristic functions, (or more) networks possessing the same terminal behaviour. It also seems likely that the answer to questions such as this will depend on a number of components with an interconnection scheme) are few in number at present. Consequentially, a search for new state-space synthesis procedures appears justified on the grounds that, in practical applications may result.

The problem of synthesising a prescribed rational positive real impedance function or matrix using state-space ideas has been considered in References 1-3, while Reference 4 discusses the synthesis of a bounded real scattering function or matrix using state-space techniques. Both Reference 1, which is a generalisation of Reference 2, and Reference 3 present network realisations which use a minimal number of reactive elements and resistors, and both methods will handle non-reciprocal synthesis problems.

When a synthesis of a symmetric impedance matrix is required to be reciprocal, i.e. to use no gyrators, both methods (as also the method in Reference 4) raise difficulties. Indeed, the very nature of the methods—forcing the synthesising network to have a minimum number of resistors and reactive elements—rules out the possibility of always deriving a reciprocal network without modification of the method.

As Reference 5 points out, it is not in general the case that a symmetric positive real impedance matrix may be synthesised by a network that simultaneously is reciprocal, has the minimal number of resistors and has the minimal number of reactive elements. Hitherto available state-space syntheses have dropped the first requirement; here we shall drop the third.

Notwithstanding the above remarks, there are further reasons why the methods of References 1 and 3 tend to rule out reciprocal syntheses. In Reference 1, a lossless network is determined such that termination of some of the ports of this network in unit resistors leads to the prescribed impedance being observed at the remaining ports; the lossless network upon examination proves to have an impedance matrix $Z(s)$ with $Z_y(s)$ almost always skew. In Reference 3, a memoryless network is determined such that termination of some of the ports of this network in unit inductances leads to the prescribed impedance being observed at the remaining ports. Then there are never any capacitors present in the synthesising network, a constraint clearly often incompatible with reciprocity.

The difficulties with the methods of References 1 and 3 can be partly resolved by permitting the lossless and memoryless networks, respectively, to be described by hybrid matrices, rather than impedance matrices. But this does not resolve the basic difficulty, i.e. that it is not always possible to have reciprocity together with resistive and reactive-element minimality.

In the literature describing what may be termed classical synthesis procedures, reciprocal minimal-resistive syntheses may be found. For a good review, see Reference 5. One of the better known of these techniques is that due to Bayard, which can be made to depend on a Gauss-factorisation procedure discussed extensively in Reference 5. It is Bayard's procedure essentially that will be described here in state-space terms.

It may well be asked what the advantages are of carrying out in state-space terms a synthesis akin to a classical procedure. Computationally, there is not perhaps a great deal to choose. But the state-space technique does offer greater scope for extension to problems such as the equivalent network problem, as discussed earlier.

In passing from a state-space realisation of $Z(s)$ to a reciprocal synthesis, as we do here, we have in part been guided by the ideas of Layton, who, though unable to give a solution of the reciprocal-synthesis problem, gave a number of partial results.

The plan of the paper is as follows. In Section 2, we review the notions of the description of an impedance by state-space means, and the Gauss factorisation. Section 3 indicates straightforward preliminary simplifications for the synthesis problem, and Section 4 shows how, with the aid of a Gauss factorisation, to construct a state-space realisation for $Z(s)$ that eventually leads to a reciprocal synthesis. The realisation obtained is nonminimal, and results from using the Gauss factorisation. Section 5 is the core of the paper; here is examined the constraint in the Gauss factorisation imposed by reciprocity; an interpretation of the constraint is given in state-space terms, which yields a reciprocal synthesis. Sections 6 and 7 are devoted to examples and concluding remarks, respectively.

2 Review

In a state-space representation of an impedance $Z(s)$, the port current $i$ becomes the input $u$ and the port voltage $v$ the output $y$ of a set of state-space equations; thus

$$\dot{x} = Fx + Gu \quad \quad \quad \quad \quad (1a)$$
$y = Hx + Ju$ . . . . . . . . . (10)

with $u \equiv i$ and $v \equiv y$. Of course, $Z(s) = J + H'(si - F)^{-1}G$.

Any quadruple $[F; G, H, J]$ for which this formula holds is termed a realisation of $Z(s)$, and a minimal realisation if $F$ has minimal dimensions. We recall several key properties of realisations: (a) if $[F; G, H, J]$ is one realisation of $Z(s)$, others are provided by $\{TT^{-1}, T(G, T^{-1}y, H, J)\}$, where $T$ ranges over the set of nonsingular matrices; (b) if $[F; G, H, J]$ is minimal, all other minimal realisations are of the form $\{TT^{-1}, T(G, T^{-1}y, H, J)\}$ for some nonsingular $T$; (c) if $[F; G, H, J]$ is minimal, the realisation is completely observable and controllable, and rank $[H, F'H, ... , F^{-1}G] = \text{rank} [H, F', H, ... , F^{-(r-1)}] = p$, where $F$ is $p \times p$.

If $Z(s)$ is rational positive real, the matrix $Z(s) + Z'(-s)$ is nonnegative definite for almost all $s = j\omega$, with $\omega$ real. Further, there exists an infinity of rational matrices $W(s)$ so that

$Z(s) + Z'(-s) = W(s)W^*$ . . . . . . . . . (2)

There are many results covering the factorisation of eqn. 2; among these, we note the so-called Gauss factorisation:

**Lemma 1:** Let $Z(s)$ be a rational $n \times n$ positive real matrix. Suppose no element of $Z(s)$ possesses a purely imaginary pole, and that $Z(s) + Z'(-s)$ has rank $r$ almost everywhere. Then there exists an $r \times r$ diagonal matrix $N_1$ of real Hurwitz polynomials, and an $n \times r$ matrix $N_2$ of real polynomials, so that

$Z(s) + Z'(-s) = N_1(s)[N_2^*(s)]^{-1}N_1(s)^{-1}N_2(s)$ . . . . . . . . . (3)

Moreover, if $Z(s) = Z(s)'$, $N_1$ and $N_2$ exist as above, with, also, $N_2(s) = N_2(-s)$.

The proof of this lemma given in Reference 5 is essentially constructive; i.e. $N_1$ and $N_2$ are found which satisfy eqn. 3. The factor $W(s) = [N_1^*(s)]^{-1}N_2(s)$ so constructed will be used in Section 4 in the development of a state-space synthesis of $Z(s)$. Note that there is no claim that $N_1$ and $N_2$ are unique; indeed, they are not so, and neither is $[N_1^*(s)]^{-1}N_2(s)$.

Frequently in network-synthesis problems, solutions $W(s)$ of eqn. 2 are required with the property that the $W(s)$ have minimal degree and constant ranks in $\text{Re} \{s\} > 0$. Reference 5 shows that the construction of such $W(s)$ may be a good deal more difficult computationally than the construction of the Gauss factorisation $W(s)$.

## 3 Preliminary simplifications for synthesis problem

In this Section, we show how the problem of synthesising an arbitrary positive real $Z(s)$ with a minimal number of resistors may be reduced to the problem of synthesising, with a minimal number of resistors, a positive real $\tilde{Z}(s)$ so that (a) no element of $\tilde{Z}(s)$ possesses a purely imaginary pole (b) $\tilde{Z}(\infty)$ is nonsingular (c) normal rank $[\tilde{Z}(s) + \tilde{Z}'(-s)] = \text{normal rank} [\tilde{Z}(s) + \tilde{Z}'(-s)] = 0$ and (d) $\tilde{Z}(s)$ is symmetrical if $Z(\infty)$ is symmetrical.

We shall first reduce the problem of synthesising $Z(s)$ to the problem of synthesising $Z_3(s)$ with $Z_3(s)$ satisfying all but (a) above.

If $Z(\infty)$ is nonsingular, no reduction is required. But suppose $Z(\infty)$ is singular. Then $Z_3(s)$ may be singular or nonsingular (almost everywhere).

In the first case, there exists a constant matrix $T$ (not square) so that $Z(s) = T\tilde{Z}(s)T^*$, where $\tilde{Z}(s)$ is a nonsingular positive real matrix. Moreover, normal rank $[\tilde{Z}(s) + \tilde{Z}'(-s)] = \text{normal rank} [\tilde{Z}(s) + \tilde{Z}'(-s)]$, and $Z_3(s)$ is symmetrical if $Z(\infty)$ is symmetrical. Indeed, $Z_3(s)$ is the admittance matrix of a multipoit transformer connected to the input ports of a network synthesising $Z(s)$. Similar remarks apply to an admittance $\tilde{Z}(s)$ which is singular; i.e. the symmetry of $\tilde{Z}(s)$ is equivalent to the symmetry of an admittance which is nonsingular almost everywhere.

Moreover, the positive real property, normal rank $[\tilde{Y}(s) + \tilde{Y}'(-s)]$, and symmetry if present, are all preserved in replacing the singular admittance by a nonsingular one.

Turning to the second case, evidently $Y_1(s) = [Z(s)]^{-1}$ exists and is positive real; but, because $Z(\infty)$ is singular, $Y_1(s)$ must have some elements with a pole at infinity. Then, as is well known, we may write $Y_2(s) = Y_1(s) - sC$, where $Y_2(\infty)$ is finite, $Y_3(s)$ is positive real, $C$ is nonnegative definite and the degree of $Y_3(s)$ is less than the degree of $Y_3(s)$. Now the problem of synthesising $Y_3(s)$ is equivalent to the problem of synthesising $Y_2(s)$, and thus $Z(s)$; parallel connection of transformer-coupled capacitors to a synthesis of $Y_2(s)$ clearly yields a synthesis of $Y_3(s)$. Note that if $Y_2(s) + Y_2'(-s) = Y_3(s) + Y_2'(s)Z(s) + Z'(s)Y_2(-s)$, and thus normal rank $[Y_3(s) + Y_2'(-s)] = \text{normal rank} [Z(s) + Z'(-s)]$. Finally, note that $Y_3(s)$ and $Y_3(s)$ are symmetrical if $Z(s)$ is symmetrical.

Now if $Y_2(s)$ is singular, proceed by finding a nonsingular $Y_3(s)$ and a constant matrix $U$ so that $Y_3(s) = U'Y_2(s)U$. If $Y_2(s)$ is nonsingular, or if it is singular and $Y_2(s)$ has been found, examine $Y_2(s)[Y_2(s)]^{-1}$. If this matrix is singular, proceed by inverse to obtain $Z(s) = (Y_2(s))^{-1}$, and then $Z(s) = \tilde{Z}(s) - sL$ etc. Eventually, one must arrive at $Z(s)$ such that $Z(s)$ is nonsingular, or else the successive degree reductions lead to $Y_3(s)$ or $Z(s)$ being constant (and thus immediately synthesisable). Disregarding this second case, we observe that, by transformer coupling and pole extractions at infinity, $Z(s)$ has $Z(\infty)$ nonsingular, normal rank $[\tilde{Z}(s) + \tilde{Z}'(-s)] + \text{normal rank} [Z(s) + Z'(-s)]$, and $Z(s)$ is symmetrical if $Z(s)$ is symmetrical. A synthesis of $Z(s)$ follows from a synthesis of $Z(s)$ by addition of suitable transformers, capacitors and inductors.

To obtain $Z(s)$, a well-known device is used. Using a partial-fraction expansion or other technique, $Z(s)$ is written as the sum of two positive real impedances, one with elements possessing purely imaginary poles—call this $Z_3(s)$—and one with elements possessing purely imaginary poles together with $Z(\infty)$—this being $Z_2(s)$. The matrix $Z_2(s)$ is, of course, lossless positive real; synthesis procedures are well known; Reference 5 gives the classical approaches and Reference 9 gives the state-space approach.

Because $Z(\infty)$ is nonsingular, $Z(s)$ is nonsingular. A standard property of lossless positive real matrices yields $Z(s) + Z'(s) = 0$ for all $s$ and so $Z(s) + Z'(s) = Z(s) + Z'(s)$ and has normal rank equal to the normal rank of $Z(s) + Z'(s)$. Finally, symmetry of $Z(s)$ is readily found to guarantee symmetry of $Z(s)$ and $Z_2(s)$.

The fact that $Z(s) + Z'(s)$ and $Z(s) + A(s)$ have the same normal rank has a well known physical significance: the minimal number of resistors required for synthesising either impedance is the same, being precisely this rank. Since we know how to obtain a synthesis of $Z(s)$ from a synthesis of $Z(s)$, it follows that a synthesis of $Z(s)$ with a minimal number of resistors yields a synthesis of $Z(s)$ with a minimal number of resistors.

## 4 Construction of a state-space realisation for a prescribed $Z(s)$

In accordance with the results of Section 3, we shall consider, with no loss of generality, the problem of synthesising an $n \times r$ positive real matrix $Z(s)$ so that no element possesses a purely imaginary pole. The constraints, such as a requirement that $Z(\infty)$ be nonsingular, will not be imposed for the moment. Suppose that normal rank $[Z(s) + Z'(s)] = r$; then, by use of the Gauss-factorisation procedure of Reference 5, we may assume the availability of a diagonal $r \times r$ matrix of Hurwitz polynomials $N_1(s)$ and an $n \times r$ matrix of polynomials $N_2(s)$, even if $Z(s)$ is symmetric, so that one solution $W(s)$ of

$Z(s) + Z'(s) = W(s)W^*$ . . . . . . . . . (2)

is

$W(s) = [N_1(s)]^{-1}N_2(s)$ . . . . . . . . . (4)

The construction of a realisation of $Z(s)$ that eventually yields a synthesis will be preceded by the construction of a realisation for $W(s) = N_1(s)[N_1(s)]^{-1}$; we shall determine matrices $F, G, L$ and $W_0$ so that

$W(s) = W_0 + G'(s)(s - F)^{-1}L$ . . . . . . . . . (5)

[Subsequently, $F$ and $G$ will appear in a realisation of $Z(s)$].

$W_\infty$ is determined immediately from eqn. 4 on setting $s = \infty$.

It is a guaranteed part of the factorisation procedure inde-

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cated in lemma 1 that \( W_0 = W(\infty) < \infty. \) To determine \( F \) and \( L, \) write

\[
[N(s)]^{-1} = \text{diag} \left( \frac{1}{p_1(s)}, \frac{1}{p_2(s)}, \ldots, \frac{1}{p_r(s)} \right)
\]

where each \( p_i(s) \) is a Hurwitz polynomial. Define \( F_i' \) to be the companion matrix with \( p_i(s) \) as characteristic polynomial, i.e.

\[
F_i' = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
-1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-\prod_{j \neq i} p_j & -\prod_{j \neq i} p_j & -\prod_{j \neq i} p_j & \cdots & -\prod_{j \neq i} p_j
\end{bmatrix}
\]

in obvious notation, and define \( \bar{l}_i = [0 \ 0 \ \ldots \ 0 \ 1]^t \). It is readily verified that \([F_i', \bar{l}_i]\) is completely controllable for each \( i \), and that, for each \( i \),

\[
(sl - F_i')^{-1}\bar{l}_i = \frac{1}{p_i(s)}[1 \ s \ s^2 \ \cdots \ s^{n-1}]
\]

Now define \( F \) and \( L \) by

\[
F' = \begin{bmatrix}
F_1' & 0 & \cdots & 0 \\
0 & F_2' & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & F_r'
\end{bmatrix}
\quad \text{and} \quad
L = \begin{bmatrix}
\bar{l}_1 & 0 & \cdots & 0 \\
0 & \bar{l}_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \bar{l}_r
\end{bmatrix}
\]

The pair \([F', L]\) is completely controllable because each \([F_i', \bar{l}_i]\) is completely controllable.

We now seek a matrix \( G \) so that \( W'(s) = W_0' + G(sI - F'-1)L \). Subsequently we shall use \( F, G \) and \( L \) to define \( F, G \) and \( L \) satisfying eqn. 5. From eqn. 4, we observe that the \((i, j)\)th entry of \( V(s) = W'(s) \) is

\[
v_{ij}(s) = \frac{\nu_{ij}(s)}{p(s)} = v_{ij}(s) + \tilde{h}_i(s)
\]

where \( v_{ij}(s) \) is \( v_{ij}(s) \), and the degree of \( \tilde{h}_i(s) \) is less than the degree of \( p(s) \) with \( \tilde{h}_i(s) \) the residue of the polynomial \( \nu_{ij}(s) \), modulo \( p(s) \). Let \( \tilde{h}_i \) be the row vector obtained from the coefficients of \( \tilde{h}_i(s) \), arranged in ascending powers of \( s \). Then, since \( (sI - F'-1)L \) has the form

\[
\begin{bmatrix}
1 & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & s^{n-1}
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & \cdots & 0 \\
0 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
0 & \cdots & s^{n-1}
\end{bmatrix}
\]

it follows that

\[
G^* = \begin{bmatrix}
\tilde{h}_{i1} & \tilde{h}_{i2} & \cdots & \tilde{h}_{ir} \\
\tilde{h}_{i1} & \tilde{h}_{i2} & \cdots & \tilde{h}_{ir} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{h}_{i1} & \tilde{h}_{i2} & \cdots & \tilde{h}_{ir}
\end{bmatrix}
\]

In the construction procedure for \( F' \) and \( L \), if there exists a \( p_i(s) \) such that \( p_i(s) \) is a constant, the corresponding \( F_i' \) block must be of zero order; i.e. the rows and columns corresponding to the block \( F_i' \) in the \( F' \) matrix must be missing. Correspondingly, the rows in the \( L \) matrix occupied by the vector \( \bar{l}_i \) will evanesce, but the associated single column of zeros must remain to give the right dimension; since, if \( F' \) is \( p \times p \), \( L \) is \( p \times r \), where \( r \) is the normal rank of \( Z(s) + \tilde{Z}'(-s) \). Consequently, the number of columns in the matrix \( L \) is always \( r \). The matrix \( G^* \) will have \( \tilde{h}_{ki} \) zero for all \( m \), and the corresponding column will also evanesce because \( G^* \) has \( r \) columns.

Matrices \( F, G \) and \( L \) are determined from \( F, G \) and \( L \) as follows. For each \( i \), define \( P_i \) as the symmetric positive solution of \( P_i F_i' + F_i' P_i = -\prod_{j \neq i} P_j \). As shown in Reference 10, such a \( P_i \) exists because the characteristic polynomial \( (sI - F_i') \) is of the Hurwitz type and the pair \([F_i', \bar{l}_i]\) is completely controllable. Let \( T_i \) be any square matrix so that \( T_i^T F_i' = P_i \) and define \( F_i = T_i F_i' T_i^{-1} \) and \( l_i = (T_i^{-1})^t \bar{l}_i \). Also, define \( F \) to be the direct sum of the \( F_i \); \( T \) to be the direct sum of the \( T_i \) and

\[
F = T \prod_{i=1}^r (sI - F_i')^{-1} \quad \text{and, since} \quad F_i + F_i' = -l_i \bar{l}_i^t
\]

Finally, define \( G = T \tilde{G} \). Then \([F, G, L, W_0] \) is a realisation of \( W(s) \), with the additional properties that eqn. 6 holds and \([F, L] \) is completely observable (the latter property follows from the complete observability of \([F, L] \), or, equivalently, from the complete controllability of \([F', L'] \).

Define now

\[
H = G + LW_0
\]

Then, with \( F, G \) and \( H \) defined as above, and with \( J = Z(\infty) \), \([F, G, H, J]\) is a realisation of \( Z(s) \). To see this, observe that

\[
W'(s)W(s) = W_0'W_0 + W_0' LW(sI - F)^{-1} G + G'(-sI - F)^{-1} L W_0
\]

\[
+ G'(-sI - F)^{-1} L L' W_0 + G'(sI - F)^{-1} L Z(s)\]

From eqn. 6, the right-hand side becomes

\[
W_0'W_0 + W_0' L(sI - F)^{-1} G + G'(-sI - F)^{-1} L W_0
\]

\[
+ G'(-sI - F)^{-1} L L' W_0 + G'(sI - F)^{-1} L Z(s)
\]

From eqn. 7, this becomes

\[
W'(s)W(s) = W_0'W_0 + H'(sI - F)^{-1} G + G'(-sI - F)^{-1} H
\]

The above quantity is also \( Z(s) + Z'(s) \). The fact that \( F \) is a direct sum of matrices all of which have a Hurwitz characteristic polynomial then guarantees that \( Z(s) = Z(\infty) + H'(sI - F)^{-1} G \), since every element of \( Z(s) \) can only have left-half-plane poles.

The above procedure for the construction of a realisation of \( W(s) \) and \( Z(s) \) has nowhere made use of the symmetry of \( Z(s) \). The procedure up to this point is therefore valid for symmetric and nonsymmetric \( Z(s) \). The procedure yields a passive synthesis of an arbitrary positive real \( Z(s) \) which, in general, is nonreciprocal, in the following way.

The matrix

\[
Z_c = \begin{bmatrix}
J & -H' \\
G & -F
\end{bmatrix}
\]

may be regarded as the impedance matrix of a nondynamic network. When all but the first \( n \) ports are terminated in unit inductances, the impedance seen at the first \( n \) ports is readily found to be \( A = \text{diag}(a_0, a_1, \ldots, a_n) \) means that \( A \) is a diagonal matrix with diagonal entries in order \( a_0, a_1, \ldots, a_n \).
The synthesis is, in general, nonreciprocal, since $Z_0 = Z'_0$ will not, in general, hold.

5 Reciprocal synthesis of a symmetric positive real $Z(s)$

As indicated in Section 3, there is no loss of generality in assuming that $Z(\omega)$ is nonsingular, and that no element of $Z(\omega)$ has a purely imaginary pole. Because $Z(s)$ is positive real, $Z(wu) + Z'(\bar{w})(\bar{u})$ is nonsingular for all $w$, and thus $Z(s)$ is nonnegative definite for all $s$. If $\omega \to \infty$, it follows that $Z(\omega)$ is nonnegative definite, and, by its nonsingularity, positive definite.

Since, then, normal rank $[Z(s) + Z'(\omega)] = \text{rank} [Z(\omega) + Z'(\omega)] = \text{rank} [Z(\omega)] = n$. Then $Z(m)$, $N_i(s)$ and $N_0(s)$ as defined in Section 4 are $n \times n$ matrices. Further, $W_0 W_0 = 2I$, and thus $W_0$ is nonsingular.

We note the following result:

Lemma 2: With all quantities as defined in Section 4, $L(sI - F)^{-1} L'$ is symmetric, and

$$-H'(sI - F)^{-1} L = G'(sI - F)^{-1} L' \quad \ldots \quad (8)$$

Proof: Consideration of the definitions of $F$ and $L$ shows that $L(sI - F)^{-1} L'$ is a diagonal matrix, the $\delta$th diagonal entry being $\ell_\delta(sI - F)^{-1} L_\delta$. Hence the desired symmetry is present.

To prove eqn. 8 we shall, in fact, prove that

$$W_0' - H'(sI - F)^{-1} L = W_0 + G'(sI - F)^{-1} L' \quad . \quad (9)$$

Of course, the right-hand side of eqn. 9 is precisely $W(s)$.

The $\delta$th element of the right-hand side of eqn. 9 has already been written down as

$$n_{2\delta}(s) \quad \frac{d}{ds} \quad \frac{n_{2\delta}(s)}{p(s)} = v_{\delta j} + \hat{b}_j(s) = v_{\delta j} + n'_{\delta j}(sI - F)^{-1} L_\delta$$

for a certain $n'_{\delta j}$. Note that $n'_{\delta j}$ occupies that position in $G$ which corresponds to the position in $\tilde{G}$ occupied by $\tilde{n}''_{\delta j}$. Moreover, $n'_{\delta j} = \tilde{n}''_{\delta j} T_j$.

Evidently,*

$$n_{2\delta}(s) = v_{\delta j} \det \frac{(sI - F_1)}{p(s)} + n'_{\delta j} \det \frac{(sI - F_1)}{p(s)}$$

on using the even nature of $n_{2\delta}(s)$. Now the highest power of $s$ occurring in $n_{2\delta}(s)$ is $\rho$, where $F_1$ is $\rho \times \rho$, provided that $v_{\delta j}$ is nonzero. The nonsingularity of $W_0$ guarantees that, for any $\rho$, there is an $i$ (say $i = 1$) such that $v_{\delta j}$ is nonzero. The evenness of $s_{2\delta}(s)$ then implies that $\rho$ is even. The argument clearly holds for all $\rho$. From the evenness of $\rho$, we have

$$n_{2\delta}(s) = v_{\delta j} \det \frac{(sI - F_1)}{p(s)} - n'_{\delta j} \det \frac{(sI - F_1)}{p(s)}$$

$$\quad \frac{d}{ds} \quad \frac{n_{2\delta}(s)}{p(s)} = v_{\delta j} \det \frac{(sI - F_1)}{p(s)} + n'_{\delta j} \det \frac{(sI - F_1)}{p(s)}$$

on using the even nature of $n_{2\delta}(s)$. Now the highest power of $s$ occurring in $n_{2\delta}(s)$ is $\rho$, where $F_1$ is $\rho \times \rho$, provided that $v_{\delta j}$ is nonzero. The nonsingularity of $W_0$ guarantees that, for any $\rho$, there is an $i$ (say $i = 1$) such that $v_{\delta j}$ is nonzero. The evenness of $s_{2\delta}(s)$ then implies that $\rho$ is even. The argument clearly holds for all $\rho$. From the evenness of $\rho$, we have

$$n_{2\delta}(s) = v_{\delta j} \det \frac{(sI - F_1)}{p(s)} - n'_{\delta j} \det \frac{(sI - F_1)}{p(s)}$$

But the $(\rho - 1)$th element of the left-hand side of eqn. 9 can readily be shown (using $H = G + LW_0$) to be

$$v_{\delta j} = (v_{\delta j} + n'_{\delta j})(sI - F)^{-1} L_\delta$$

which proves lemma 2.

The idea behind the synthesis procedure is to replace the realisation $\{F, G, H, J\}$ of $Z(s)$ by a realisation $\{T F', T G, T H, J\}$, where $T$ is an orthogonal matrix, so that the new realisation possesses additional properties to those possessed by the realisation $\{F, G, H, J\}$. These additional properties make a reciprocal synthesis straightforward.

In order to generate $T$, we shall make use of lemma 2. First, we shall define a symmetric matrix $P$. Then we shall show that $P$ satisfies a number of constraints, including a constraint that all its eigenvalues be $+1$ or $-1$. Then we shall take $T$ as any orthogonal matrix so that $P = T^* \Sigma T$, where $\Sigma$ is a diagonal matrix with diagonal entries all $+1$ or $-1$.

Lemma 3: With $F, G, H, L$ and $W_0$ as defined in Section 4, the equations

$$FP = PF' \quad \ldots \quad . \quad . \quad (10a)$$

$$PL = L \quad \ldots \quad . \quad . \quad (10b)$$

define a unique matrix $P$. Moreover, $P$ is symmetric, and the following equations also hold:

$$PH = -G \quad . \quad \ldots \quad \ldots \quad . \quad (11a)$$

$$PG = -H \quad \ldots \quad \ldots \quad . \quad (11b)$$

$$PF = FP \quad \ldots \quad . \quad . \quad (11c)$$

$$P = P^{-1} \quad . \quad . \quad . \quad . \quad (11d)$$

Proof: We have, with $F$ a $p \times p$ matrix,

$$P[L \quad P' \quad (F')^2L \quad \ldots \quad (F')^{p-1}L] = \quad \ldots \quad . \quad (12)$$

from eqn. 10. The matrix $[L \quad P \quad (F')^2L \quad \ldots \quad (F')^{p-1}L]$ has rank $p$, since $[F', L]$ is completely controllable. Thus $P$ is uniquely defined.

We now observe that $[F', L]$ is completely controllable. The reasoning is as follows: because $[F', L]$ is completely controllable, $[F' - LK', L']$ is completely controllable for all $K$. Take $K = -L$; this yields the result that $[F' + LL', L]$ or $[-F, L]$ is completely controllable.

It then follows from eqn. 12 that $P$ must be nonsingular. Consequently,

$$L(sI - F)^{-1} L' = L'P(sI - P^{-1}FP)^{-1}P^{-1}L$$

from eqn. 10, while

$$L(sI - F)^{-1} L = L'P(sI - F)^{-1} L'$$

using lemma 2.

The complete controllability of $[F', L]$ implies that $L'P' = L'$ or $P' = L$. Eqn. 10a yields $FP' = FP'$ on transposition, and evidently $P$ and $P'$ both satisfy $FX = XF'$ and $XL = L$. Since the solution of these equations is unique, it follows that $P = P'$.

Now consider eqn. 8 of lemma 2. We have

$$G'(sI - F)^{-1} L = -H'(sI - F)^{-1} L$$

from which

$$PH = -G \quad . \quad \ldots \quad \ldots \quad . \quad (11a)$$

Since $H = G + LW_0$, we have, from eqn. 11a,

$$-G = PG + PLW_0 = PG + LW_0$$

or $PG = -H \quad . \quad \ldots \quad \ldots \quad . \quad (11b)$

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Next, from $F \neq F' = -LL'$, we have:

$$FP = -FP' + PLL'$$

$$= \frac{-FP - LL'}{F}$$

From eqns. 10a and $b$.

$$= \frac{-PP - LL'}{F}$$

$$= (F + LL')P$$

$$= F'P$$

From eqn. 6.

From eqns. 10b and 11c, we see that $P^{-1}L = L$ and $F^{-1} = F^{-1}$, or, in other words, $P^{-1}$ satisfies $FP = FX$ and $XL = L$. Hence eqn. 11f holds, proving lemma 3.

(b) Let $P$ be symmetric, there exists an orthogonal matrix $T$ such that $P = T^*\Sigma T$, with $\Sigma$ consisting of $T^*$ and $-1$ entries in the diagonal positions and zeros elsewhere.

Now we calculate $\{F_1, G_1, H_1, J\}$ of $Z(s)$ by $F_1 = FFT', G_1 = TG$ and $H_1 = TH$, and define the hybrid matrix

$$M = \begin{bmatrix} J & -H_1 \\
G_1 & -P_1 \end{bmatrix}$$

Lemma 4: With quantities as defined above, it is claimed that

(a) $M + M'$ is non-negative definite with rank $n$.

(b) With $+$ denoting a direct sum, $(I_e + \Sigma)M$ is symmetric.

Proof:

(a) $M + M' = (I_e + T)
\begin{bmatrix} J & -H_1 \\
G_1 & -P_1 \end{bmatrix} + (I_e + T)
\begin{bmatrix} J & -H_1 \\
G_1 & -P_1 \end{bmatrix}$

(b) The rank condition is obviously fulfilled, since $\{W_0 - L\}$ has rank no less than $W_0$, which has rank $n$.

\begin{align*}
Z(s) + Z'(-s) &= 2 \begin{bmatrix} 1 & 1 \\
4 & 4 - s^2 \\
6 - 3s^2 & 4 - s^2 \end{bmatrix} \\
&= 2 \begin{bmatrix} 4 - s^2 & 0 \\
4 - s^2 & 2(1 - s^2)(4 - s^2) \\
1 & \sqrt{2(1 - s^2)} \end{bmatrix} \begin{bmatrix} \frac{1}{4 - s^2} \\
0 \\
\frac{1}{\sqrt{2}(1 - s^2)} \end{bmatrix} \\
&= 2 \begin{bmatrix} 4 - s^2 & 0 \\
0 & 2(1 - s^2)(4 - s^2) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\
0 & \sqrt{2}(1 - s^2) \end{bmatrix}
\end{align*}

Thus

$$W(s) = \begin{bmatrix} 1 & 0 \\
0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\
0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & \sqrt{2} \\
0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\
0 & \sqrt{2} \end{bmatrix}$$

and

$$N_2(s) = \begin{bmatrix} 1 & 0 \\
0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\
0 & \sqrt{2} \end{bmatrix}$$

The common factor $(1 + s)$ of $[N_2(s)]^{-1}$ and $N_2(s)$ is left uncanceled, so that $N_2(s) = N_2(s)$; otherwise $Z(s)$ cannot be synthesized with reciprocal elements only.
Note that $p_1(s) = 1$ is a constant and no $F_1$ is needed. Thus

$$F' = F_2' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad G' = \begin{bmatrix} 6 & 6 \end{bmatrix}$$

Form $P_2$ from $P_2F_2 + F'_2P_2 = -i \omega P_2 = -LL'$ to give

$$P = P_2 = \begin{bmatrix} 1 \\ 12 \\ 0 \\ 1 \\ 6 \end{bmatrix}$$

From $T_2T_2 = P_2$, we obtain

$$T_2 = \begin{bmatrix} \frac{1}{2\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Hence

$$F = T_2F_2T_2^{-1} = \begin{bmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & -3 \end{bmatrix}$$

$$L = (T_2)^{-1}L = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \sqrt{6} \end{bmatrix}$$

and $G = T_2G_0 = \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix}$

Define

$$H = G + LW_0 = \begin{bmatrix} 0 & \sqrt{3} \\ 0 & -\sqrt{6} \end{bmatrix}$$

By direct computation, it is readily verified that

$$H(sI - F)^{-1}G = \begin{bmatrix} 0 & 0 \\ 0 & \frac{3(1 + s)}{s^2 + 3s + 2} \end{bmatrix}$$

and $J + H(sI - F)^{-1}G$ is the prescribed $Z(s)$, where

$$J = Z(\infty) = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

Thus $\{F, G, H, J\}$ is a (nonminimal) realisation of $Z(s)$, which readily yields a passive-network realisation since $\hat{M} = \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix}$ can be shown to have a nonnegative definite symmetric part.

To generate a reciprocal passive synthesis, define a unique symmetric matrix $P$ by means of the equations

$$FP = PF'$$

$$PL = L$$

from which $P$ is seen to be

$$P = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore, from $P = T'\Sigma T$, we are led to take

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

The final required realisation $\{F_1, G_1, H_1, J\}$ of $Z(s)$ is

$$F_1 = TFT' = \begin{bmatrix} -3 & \sqrt{2} \\ -\sqrt{2} & 0 \end{bmatrix}$$

$$G_1 = TG = \begin{bmatrix} 0 & \sqrt{6} \\ 0 & \sqrt{3} \end{bmatrix} \quad H_1 = TH = \begin{bmatrix} 0 & -\sqrt{6} \\ 0 & \sqrt{3} \end{bmatrix}$$

[Note that $\{F, G, H, J\}$ actually readily yields a reciprocal passive network, since one can take $\Sigma = P$ and $T = I_p$; then]

$$(I_p + \Sigma) \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix}$$

is symmetric, so that, in effect, no basis change is required for $\{F, G, H, J\}$. However, to synthesise the hybrid matrix

$$\begin{bmatrix} J & -H' \\ G & -F \end{bmatrix}$$

it may be desirable to group the open-circuited ports together and the short-circuited ports together.

Next form the hybrid matrix

$$M = \begin{bmatrix} J & -H' \\ G_1 & -F_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & \sqrt{6} & -\sqrt{3} \\ 0 & \sqrt{6} & 3 & -\sqrt{2} \\ 0 & \sqrt{3} & \sqrt{2} & 0 \end{bmatrix}$$

which is a positive real matrix with $(I_p + \Sigma) M$ symmetric.

To synthesise the network $N_1$ from $M$ (see Reference 9), we first realise

$$Z_1 = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \sqrt{6} \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

by the network of Fig. 1; $Y_1 = [0]$ merely represents an open circuit, and $T_1 = [0 \ \sqrt{3}]$ and $T_2 = [\sqrt{2}]$ are the two coupling-transformer ratios.

Therefore the network $N_1$ of hybrid matrix $M$ has the synthesis of Fig. 2 with ports 1, 2, and 3 as open-circuit ports, and port 4 as a short-circuit port. The network synthesising $Z(s)$ is found by terminating port 3 of $N_1$ in a unit inductance and port 4 in a unit capacitance, as shown in Fig. 3.
The synthesis uses two resistors, which is the minimum number possible.

Obtain $P_1$ from $P_1 \bar{F} + \bar{F}P_1 = -LL'$, to give

$$P_1 = \begin{bmatrix}
\frac{5}{28} & 0 & -\frac{1}{7} & 0 \\
0 & \frac{1}{7} & 0 & -\frac{3}{14} \\
-\frac{1}{7} & 0 & \frac{3}{14} & 0 \\
0 & -\frac{3}{14} & 0 & \frac{4}{7}
\end{bmatrix}$$

One matrix $T_1$ such that $T_1^T T_1 = P$ is

$$T_1 = \begin{bmatrix}
\frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & 0 \\
-\sqrt{\frac{2}{21}} & 0 & \sqrt{\frac{3}{14}} & 0 \\
0 & -\frac{3}{4\sqrt{7}} & 0 & \frac{2}{\sqrt{7}}
\end{bmatrix}$$

and $T_1^{-1} = \begin{bmatrix}
\frac{4}{\sqrt{3}} & 0 & \sqrt{\frac{14}{3}} & 0 \\
0 & \frac{3}{2} & 0 & \sqrt{\frac{7}{2}}
\end{bmatrix}$

Therefore

$$F = T_1 \bar{F} T_1^{-1} = \begin{bmatrix}
0 & -\sqrt{3} & 0 & -\sqrt{7} \\
\sqrt{3} & 0 & 0 & -\frac{3\sqrt{7}}{8} \\
0 & \frac{3}{2} & 0 & -2\sqrt{2} \\
\frac{\sqrt{7}}{2 \sqrt{3}} & -\frac{3 \sqrt{7}}{8} & 2 \sqrt{\frac{7}{3}} & -\frac{7}{8}
\end{bmatrix}$$

We write $a$ realisation of which is

$$G = T_1 \bar{G} = \begin{bmatrix}
\sqrt{\frac{2}{3}} \\
\frac{3}{2} \\
-\sqrt{\frac{7}{3}} \\
-\sqrt{\frac{7}{2}}
\end{bmatrix}$$

Define

$$H = G + LW_0 = \begin{bmatrix}
\sqrt{\frac{3}{2}} \\
-\frac{3}{2 \sqrt{2}} \\
-\sqrt{\frac{7}{3}} \\
-\frac{\sqrt{7}}{2 \sqrt{2}}
\end{bmatrix}$$

$$W'(s) = \frac{\sqrt{2}}{s^2 + s + 1}$$

Obtain $W(s)$ to be an even function, multiply $z(s) + z(-s)$ by $(s^2 + 3s^2 + 4)(s^2 - s + 1)$, so that

$$W(s) = \frac{\sqrt{2}(s^2 + s + 2)}{s^2 + s + 1}$$

The process is equivalent to modifying the impedance $z(s)$ by multiplication by $(s^2 + s + 2)(s^2 - s + 1)$, a standard technique used in the classical Darlington synthesis. Therefore, the Gauss factorisation which gives $N(z) = N(z(-s)$ for $Z(s) = Z(s)$ in multiports, often only by inserting common factors into $N(z)$ and $N(z)$, may be regarded as a generalisation of the classical 1-port Darlington procedure.

We write

$$W'(s) = \frac{\sqrt{2} + -2\sqrt{2}s^3 - 2\sqrt{2}s^2 - 3\sqrt{2}s + 2\sqrt{2}}{s^4 + 2s^3 + 4s^2 + 3s + 2}$$

a realisation of which is

$$F = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-2 & -3 & -4 & -2
\end{bmatrix}, \quad L = \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}$$

$$G = [2\sqrt{2} \quad -3\sqrt{2} \quad -\sqrt{2} \quad -2\sqrt{2}]$$
The solution of \( FP = PF' \) and \( PL = L \) is

\[
P = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]

Hence

\[
T = T' - T^{-1} = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

and \( \Sigma = \text{diag} \{1, 1, -1, -1\} \)

Also

\[
M = \begin{bmatrix}
J & -H' T \\
T G & -T F T
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -\sqrt{7} / 2\sqrt{2} & -3 / 2\sqrt{2} \\
-\sqrt{7} / 2\sqrt{2} & 7 / 8 & -3\sqrt{7} / 8 \\
-3 / 2\sqrt{2} & 3\sqrt{7} / 8 & 9 / 8 \\
\sqrt{7} / 2\sqrt{3} & \sqrt{7} / 2\sqrt{3} & 0 & 0
\end{bmatrix}
\]

Consider the impedance matrix

\[
Z_1 = \begin{bmatrix}
1 & -\sqrt{7} / 2\sqrt{2} & -3 / 2\sqrt{2} \\
-\sqrt{7} / 2\sqrt{2} & 7 / 8 & 3\sqrt{7} / 8 \\
-3 / 2\sqrt{2} & 3\sqrt{7} / 8 & 9 / 8
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & -\sqrt{7} / 2\sqrt{2} & -3 / 2\sqrt{2} \\
-\sqrt{7} / 2\sqrt{2} & 1 & -3 / 2\sqrt{2} \\
-3 / 2\sqrt{2} & -3 / 2\sqrt{2} & 1
\end{bmatrix}
\]

which has a synthesis of Fig. 4. The network \( N_1 \) of hybrid matrix \( M \) then has a synthesis shown in Fig. 5.

Fig. 4
Synthesis of \( Z_1 \) of example 2

The network synthetising \( z(s) \) is formed by terminating ports 2 and 3 of \( N_1 \) in unit inductances and ports 4 and 5 in unit capacitances, leading to Fig. 6.

The complete network uses one resistor.

A closer parallel to the Darlington synthesis would involve determination of a 2-port lossless network (in transfer-function matrix or state-space form), so that termination of one port in a unit resistance would yield \( z(s) \). Current work is exploring this approach, which can be expected to cut down on the number of transformers, yielding a far more acceptable network topology.

7 Conclusions

The preceding material suggests a number of problems. For example, it is not clear, other than because the reciprocal Bayard synthesis works, why the procedure presented here should work. The solution of the problem of obtaining all reciprocal syntheses via state-space procedures is apparently not in sight, and it is not even clear how reciprocal syntheses with a minimal number of resistors differ from the state-space point of view. Certainly, it is known that the Gauss-factorisation procedure does not yield a unique \( W(s) \), and it is clear that differing \( W(s) \) will yield differing state-space syntheses; but this is some distance away from identifying the difference between these syntheses.

It might be imagined that the theory of minimal state-space realisations, more highly developed than the theory of (arbitrary dimension) realisations, should offer promise in the obtaining of a reciprocal synthesis using a minimal number of reactive elements and a non-minimal number of
resistors. Such a synthesis would depend on factorising $Z(s) + Z'(-s)$ to obtain a $W(s)$ with a nonminimal number of columns. Such factorisations have recently been studied from the state-space viewpoint, but they have not been deeply studied in network theory.

When minimality of both the number of reactive elements and the number of resistive elements is not required, reactance extraction and Darlington-type synthesis are possible, based on the work of Koga on multivariable positive real synthesis.

8 References