Linear system optimisation with prescribed degree of stability

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Abstract

The paper presents a scheme for obtaining a linear-feedback law for a linear system as a result of minimising a quadratic-performance index; the resulting closed-loop system has the property that all its poles lie in a halfplane $\mathrm{Re}(s) < -\alpha$, where $\alpha > 0$ may be chosen by the designer. The advantages of this arrangement over conventional optimal design are considered. In particular, it is shown that the reduction of trajectory sensitivity to plant-parameter variations as a result of any closed-loop control is greater for $\alpha > 0$ than for $\alpha = 0$, that there is inherently a greater margin for tolerance of time delay in the closed loop when $\alpha > 0$, that there is greater tolerance of nonlinearity when $\alpha > 0$, and that asymptotically stable bang-bang control may be achieved simply by inserting a relay in the closed loop when $\alpha > 0$. The disadvantage of the scheme appears to be that, with $\alpha > 0$, more severe requirements are put on the power level at which input transducers should operate than for $\alpha = 0$.

1 Introduction

In this paper, we consider the time-invariant linear system

$$\dot{x} = Fx + Gu \qquad x(t_0) = x_0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

with linear-control laws of the form

$$u = -K'x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Here, F is constant and $n \times n$, G and K are constant and $n \times p$, x is an $n$vector, the state, and u is a $p$vector, the control.

At least two distinct approaches to the selection of control laws K have been used, which are motivated more or less by engineering requirements. The first is to choose K in order to minimise a quadratic performance index (defined in more detail subsequently).[1-3] The second is to choose K so that the poles of the closed-loop system

$$\dot{x} = (F - GK')x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

achieve certain desired values.[4-9]

In general, it is not possible to minimise a quadratic performance index and, at the same time, achieve arbitrary closed-loop poles. What we show here, however, is that it is possible to minimise a quadratic performance index and, at the same time, ensure that the closed-loop poles lie to the left of a line $\mathrm{Re}(s) = -\alpha$, for a prescribed $\alpha > 0$. There are certain other attendant advantages of the optimisation procedure, too, which relate to the sensitivity of the closed-loop system to variation of parameters in the forward part of the loop, the tolerance of time delay in the closed loop, and the tolerance of nonlinearities in the closed loop, and the introduction of a bang-bang control.

In Section 2, we summarise the standard results on the quadratic-regulator problem. In Section 3, we present the procedure for control-law selection which simultaneously minimises a quadratic performance index and constrains the closed-loop poles as described above. In Section 4 we obtain the additional properties of the closed loop already alluded to, and we conclude with some brief remarks on extending the ideas in Section 5.

2 Summary of quadratic-regulator results

2.1 Control law

In the quadratic-regulator problem, one seeks to minimise

$$V = \int_{t_0}^{\infty} (u'Ru + x'Qx)dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

(by appropriate choice of u), subject to eqns. 1. The matrix R is positive definite symmetric and constant while Q is nonnegative definite symmetric and constant.

It is intuitively clear that one can restrict attention to completely controllable systems (eqn. 1),[1-3] and this conclusion is borne out by the mathematics. (If eqns. 1 are not completely controllable, the uncontrollable states, and their contribution to the performance index (eqn. 4), can be separated out; the contribution to the performance index is independent of the control u used. This contribution may be infinite, which makes the optimisation problem meaningless, or, if the uncontrollable states are asymptotically stable, their contribution to eqn. 4 will be finite. In any case, any quadratic-regulator problem with uncontrollable states is at once reducible to one with controllable states; accordingly we shall assume that (F, G) is completely controllable.

The control u, which performs the minimisation, is a linear law of the form of eqn. 2, given by[1-3]

$$K = PGR^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and P is a nonnegative definite symmetric matrix defined either by

$$\dot{\Pi} = \Pi F + F'\Pi - \Pi GR^{-1}G'\Pi + Q \qquad \Pi(t_1) = 0 \quad . \quad . \quad . \quad . \quad (6a)$$

$$P = \lim_{t \to -\infty} \Pi(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6b)$$

or as the unique nonnegative definite solution of

$$XF + F'X - XGR^{-1}G'X + Q = 0 \quad . \quad . \quad . \quad . \quad (7)$$

Though it is known that $x_0'Px_0$ is the value of V when the optimal control is used, this is perhaps one of the less useful facts about the optimisation.

2.2 Stability of the closed-loop system

Almost always, closed-loop linear systems are required to be stable. Accordingly, one is led to ask whether the implementation of the optimal control law leads to an asymptotically stable system. The answer is as follows.

Let H be any matrix so that $Q = HH'$, and let (F, H) be completely observable (i.e. $H'e^{FT}a = 0$ for all t implies $a = 0$). Then the closed-loop system is asymptotically stable. (Also, P is positive definite.)

Clearly, this is an important result. But it fails to yield any measure of stability. The closed-loop poles are in the left half of the s plane, but how far they are from the imaginary axis is not known. In Section 3, of course, we seek to put a minimum distance between the closed-loop poles and the imaginary axis.

The requirement of complete observability is vital in the following sense; if eqns. 1 have unstable states which are not observable, then precisely because these states do not affect the performance index (eqn. 4), there will be no control action

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trying to stabilise these states. Accordingly, the closed loop will be unstable. In the event that unobservable states are all asymptotically stable, one can, however, rely on the closed-loop system being asymptotically stable too.

2.3 Trajectory sensitivity

The question of whether it is better to use open-loop or closed-loop control of a plant may depend on whether if a plant parameter varies, the resulting variation in the trajectories of the plant will be greater for open-loop or for closed-loop control. (The point is that the open-loop control is unaltered by plant-parameter variation, while the closed-loop control, being derived from measurements on the plant trajectory, will be changed; it may then compensate for or add to the variation in the plant trajectory. One hopes that it will compensate.)

The definition of sensitivity and the characterisation of its improvement is a complex question.[10–13] For scalar-input systems, classical theories suggest the advantages of having as large a return difference as possible, and the theories state that sensitivity improvement is obtained (i.e. closed-loop control is superior to open-loop control) if the return difference has magnitude greater than unity. Now, the return difference is quite simply $1 + \boldsymbol{K}'(j\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}$, and then the calculations detailed, e.g. as in Reference 3, yield, for the optimally designed system,

$$|1 + \boldsymbol{K}'(j\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}|^2 = 1 + \boldsymbol{G}'(-j\omega\boldsymbol{I} - \boldsymbol{F}')^{-1}\boldsymbol{Q}(j\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G} \quad . \quad . \quad . \quad . \quad (8)$$

$$\geqslant 1 \quad \text{for all real } \omega \quad . \quad . \quad . \quad . \quad (9)$$

Thus, there is always sensitivity improvement. In References 10–13, the significance is discussed of a generalisation to multiple-input systems of the return-difference concept in respect of sensitivity improvement. It is also shown in References 10–11 that

$$\{\boldsymbol{I} + \boldsymbol{R}^{1/2}\boldsymbol{K}'(-j\omega\boldsymbol{I} - \boldsymbol{F}')^{-1}\boldsymbol{G}\boldsymbol{R}^{-1/2}\}\{\boldsymbol{I} + \boldsymbol{R}^{1/2}\boldsymbol{K}'(j\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}\boldsymbol{R}^{1/2}\} = \boldsymbol{I} + \boldsymbol{R}^{-1/2}\boldsymbol{G}'(-j\omega\boldsymbol{I} - \boldsymbol{F}')^{-1}\boldsymbol{Q}(j\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}\boldsymbol{R}^{-1/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The quantity on the left of eqn. 10 turns out to be a measure of the magnitude of the return difference, and the fact that the second matrix on the right-hand side of eqn. 10 is non-negative definite leads again to the conclusion that there is sensitivity improvement.

For convenience, we shall confine discussion on the sensitivity question henceforth to single-input systems, with the understanding that appropriate technical definitions lead to results for multiple-input systems.

2.4 Gain margin, phase margin and time-delay tolerance

For convenience, we again restrict attention to single-input systems. A Nyquist diagram of the open-loop transfer

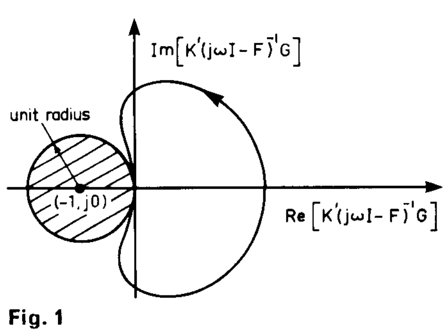

Fig. 1
Nyquist diagram corresponding to an asymptotically stable F matrix, with no encirclement of (−1, j0)

function $\boldsymbol{K}'(j\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}$ can be plotted; eqn. 9 guarantees that the plot does not enter a circle with centre $(-1, j0)$ and unity radius. Two sample Nyquist plots are shown in Figs. 1 and 2, corresponding to stable $\boldsymbol{F}$ and unstable $\boldsymbol{F}$, respectively.

With the aid of these diagrams, a little thought will quickly establish the validity of the following claims. Though we are

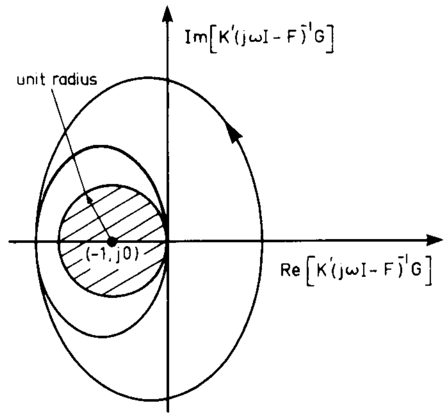

Fig. 2
Nyquist diagram corresponding to an unstable F matrix
The transfer function $K(sI - F)^{-1}G$ has two poles in $\text{Re}(s) > 0$ and encircles the point $(-1, j0)$ twice

unaware of a reference setting out these points, there is no doubt they are well known to many people.*

(*a*) Increase of the loop gain by a scalar, i.e. replacement of $\boldsymbol{K}'(j\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}$ by $\beta\boldsymbol{K}'(j\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}$ for any scalar $\beta > 1$, will not alter the encirclements of the $(-1, j0)$ point, and therefore the system stability. Thus, the gain margin is infinite. (This is borne out by Section 2.5.)

(*b*) When, for some ω, the condition $|\boldsymbol{K}'(j\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}| = 1$ holds, the angle through which a vector from the origin to the point $\boldsymbol{K}'(j\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}$ must be swung in a clockwise direction to reach the $(-1, j0)$ point is greater than $60°$. This is because the closest the point $\boldsymbol{K}'(j\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}$ can be to $(-1, j0)$ is the point A in Fig. 3; here a $60°$

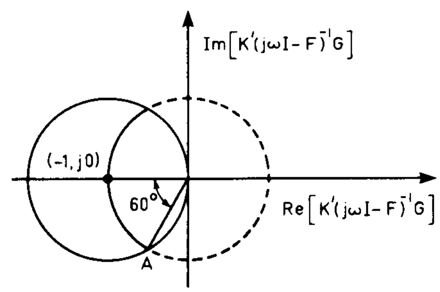

Fig. 3
Diagram illustrating lower bound on phase margin
Broken line denotes points at which $|K'(j\omega I - F)^{-1}G| = 1$ is permissible

rotation will do the trick. Away from A, more than $60°$ is required. Thus, even if there is more than one frequency for which $|\boldsymbol{K}'(j\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}| = 1$, the phase margin is at least $60°$.

(*c*) Let $\omega = \omega_1$ be the greatest frequency. There may only be one, or even none, for which $|\boldsymbol{K}'(j\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}| = 1$. Then, since a time delay T introduces a rotation of ωT radians at frequency ω in the Nyquist diagram, it follows that a time delay less than $\pi/3\omega_1$ can be tolerated without destroying stability. Of course, insertion of a time delay can destroy the property that the gain margin is infinite.

2.5 Tolerance of nonlinearities

Frequently, systems which are nonlinear may be treated as linear for the purposes of analysis. If the nonlinearity is small (where we shall here leave undefined the term small), one expects the linear analysis to be a reasonable prediction of performance in the nonlinear regime. For this reason, it is interesting to consider the effect of introducing nonlinearities into the closed-loop system.

Such nonlinearities may be anywhere in the loop, and it is possible to come to some quantitative conclusions,[14] which

* We acknowledge discussions with G. Franklin on these points

follow merely from the asymptotic stability of the closed-loop system. We shall however, note here the behaviour of the closed-loop system with gross nonlinearities in the input transducers. The conclusions follow from the optimality, as well as the asymptotic stability, of the closed-loop system.

If the system is behaving in the described linear fashion, let us say that the input transducers are linear and have a (normalised) gain of unity, meaning that, in fact, each entry u_i of $\boldsymbol{u}$ is precisely $-(\boldsymbol{K}'\boldsymbol{x})_i$, as required normally.

Now, by taking $\boldsymbol{x}'\boldsymbol{P}\boldsymbol{x}$ as a Lyapunov function for the closed-loop system, it is straightforward to show[14, 15] that a control law

$$u_i = -k_i(t)(K'x)_i$$

(where $\frac{1}{2} + \epsilon_1 \leqslant k_i(t) \leqslant 1/\epsilon_2$ for arbitrary $\epsilon_1, \epsilon_2 > 0$, all i and all t) does not disturb the asymptotic stability of the closed-loop system.* The point is that the unity gain of each transducer is replaced by a variable gain between a half and infinity. Further, though this gain has been indicated to be time variable, it can equally well be nonlinear—since any nonlinear gain can be replaced by a time-variable gain (though not vice versa).

By considering the case of constant k_i, we also obtain the conclusion of the previous Section, i.e. the system-gain margin is infinite.

2.6 Insertion of a bang-bang control law

We consider now the arrangement of Fig. 4, where a relay has been inserted to turn the otherwise optimal-control system into a bang-bang system.

We first observe that $s\boldsymbol{K}'(s\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G} = \boldsymbol{K}'(s\boldsymbol{I} - \boldsymbol{F})(s\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G} + \boldsymbol{K}\boldsymbol{F}(s\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}$ and so $\lim_{S\to\infty} s\boldsymbol{K}'(s\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G} = \boldsymbol{K}'\boldsymbol{G} = \boldsymbol{G}'\boldsymbol{P}\boldsymbol{G} > 0$. This implies that if $\boldsymbol{K}'(s\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}$ is

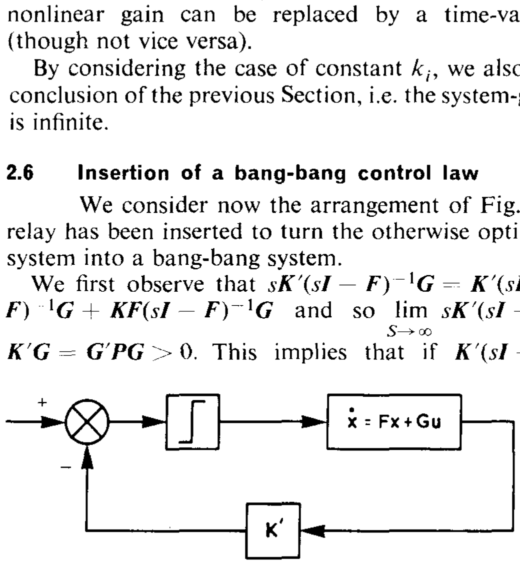

Fig. 4
Construction of bang-bang system from optimal system

expressed as the ratio of two polynomials, with the leading coefficient of the denominator polynomial unity, then the numerator polynomial has degree one less than the denominator polynomial and has a positive leading coefficient. Now, a necessary condition for (local) stability, given this just stated property of the transfer function, is that all zeros of $\boldsymbol{K}'(s\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}$ must have nonpositive real parts.[16, 17]

That this second condition is satisfied we see by using the following reasoning. We have just shown that system asymptotic stability is not impaired by the introduction of any gain greater than $\frac{1}{2}$. Hence, points on the root locus of $\boldsymbol{K}'(s\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}$ corresponding to gains greater than $\frac{1}{2}$ all lie in $\mathrm{Re}\,(s) < 0$. Consequently, points corresponding to infinite gain must lie in $\mathrm{Re}\,(s) \leqslant 0$; but these points are precisely the zeros of $\boldsymbol{K}'(s\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}$.

It is also shown in References 16 and 17 that a sufficient condition for (local) asymptotic stability for the case above is that all zeros of $\boldsymbol{K}'(j\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}$ have negative real parts. This condition is satisfied if $\boldsymbol{Q}$ is positive definite as the following reasoning shows. Reference to eqn. 8 shows that, if $\boldsymbol{K}'(j\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}$ is zero for some real ω, it must be true that $\boldsymbol{G}'(-j\omega\boldsymbol{I} - \boldsymbol{F}')^{-1}\boldsymbol{Q}(j\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}$ is also zero (and thus $\boldsymbol{Q}$ is not positive definite, since $(j\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}$ is readily proved never to be zero).

Note that the case $\boldsymbol{F} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, $\boldsymbol{G} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\boldsymbol{Q} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ leads to $\boldsymbol{K} = \begin{bmatrix} 0 \\ \sqrt{2} - 1 \end{bmatrix}$ and to a transfer function $\boldsymbol{K}'(s\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}$, which is zero for $s = 0$; consequently it is possible for the transfer-function zeros not to lie in $\mathrm{Re}\,(s) < 0$.

* R. Kalman pointed out this result in the scalar-input case in a private communication. Subsequently to this communication, the result has appeared in various locations (see Reference 15 for the most general form)

The above results are now summarised: necessary conditions for stability of the scheme of Fig. 4 are always satisfied; sufficient conditions for asymptotic stability are satisfied if the zero of $\boldsymbol{K}'(s\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G}$ lie in $\mathrm{Re}\,(s) < 0$, with the positive definiteness of $\boldsymbol{Q}$ guaranteeing this constraint.

3 Derivation of the optimal pole-positioning control law

In this Section, we define a quadratic-loss function for the system defined by eqns. 1 (still assumed completely controllable) which leads to a linear-control law of the type represented by eqn. 2, with the additional property that the closed-loop-system poles lie to the left of $\mathrm{Re}\,(s) = -\alpha$.

3.1 Construction of the control law

In place of the performance index (eqn. 4), we take the performance index

$$V = \int_{t_0}^{\infty} (\boldsymbol{u}'\boldsymbol{R}\boldsymbol{u} + \boldsymbol{x}'\boldsymbol{Q}\boldsymbol{x})e^{2\alpha t}dt \quad \ldots\ldots \quad (11)$$

As before, $\boldsymbol{R}$ is positive definite symmetric and constant, while $\boldsymbol{Q}$ is nonnegative definite symmetric and constant. Assuming for the moment that a *linear constant* control law will minimise V, we can give plausible reason for the closed-loop-system pole constraint. The complete controllability of eqns. 1 assures that there is some control for which V is finite (e.g. a control taking x_0 to the zero state at time $t = t_0 + 1$, with the control zero for $t > t_0 + 1$). Consequently the minimum value of V is finite. The assumption that the control law is linear and constant implies that $\boldsymbol{x}$ and $\boldsymbol{u}$ behave exponentially; to ensure finiteness of V, $\boldsymbol{x}$ and $\boldsymbol{u}$ must decay faster than $e^{-\alpha t}$, which, in turn, implies the closed-loop pole constraint.

To minimise eqn. 11 subject to the conditions of eqns. 1, set

$$\hat{\boldsymbol{x}} = \boldsymbol{x}e^{\alpha t} \qquad \hat{\boldsymbol{u}} = \boldsymbol{u}e^{\alpha t} \quad \ldots\ldots \quad (12)$$

Then eqns. 1 are equivalent to

$$\dot{\hat{\boldsymbol{x}}} = (\boldsymbol{F} + \alpha\boldsymbol{I})\hat{\boldsymbol{x}} + \boldsymbol{G}\hat{\boldsymbol{u}} \qquad \boldsymbol{x}(t_0) = e^{\alpha t_0}x_0 \quad \ldots \quad (13)$$

while $(\boldsymbol{u}'\boldsymbol{R}\boldsymbol{u} + \boldsymbol{x}'\boldsymbol{Q}\boldsymbol{x})e^{2\alpha t} = \hat{\boldsymbol{u}}'\boldsymbol{R}\hat{\boldsymbol{u}} + \hat{\boldsymbol{x}}'\boldsymbol{Q}\hat{\boldsymbol{x}}$, and thus minimisation with respect to eqns. 1 of eqn. 11 is equivalent to minimisation with respect to eqns. 13 of

$$V = \int_{t_0}^{\infty} (\hat{\boldsymbol{u}}'\boldsymbol{R}\hat{\boldsymbol{u}} + \hat{\boldsymbol{x}}'\boldsymbol{Q}\hat{\boldsymbol{x}})dt \quad \ldots\ldots \quad (14)$$

in the following senses:

(*a*) The minimum value of eqn. 11 (expressed in terms of x_0) is the same as the minimum value of eqn. 14 [expressed in terms of $\hat{\boldsymbol{x}}(t_0)$, taking account of $\hat{\boldsymbol{x}}(t_0) = e^{\alpha t_0}x_0$].

(*b*) If $\hat{\boldsymbol{u}} = f(x)$ is the optimal control for eqns. 13 and 14, $\boldsymbol{u} = e^{-\alpha t}f(\boldsymbol{x}e^{\alpha t})$ is the optimal control for eqns. 1 and 11, and conversely.

The first point is not as significant as the second. We know that, for eqns. 13 and 14, the optimal control is

$$\hat{\boldsymbol{u}} = -\boldsymbol{K}_\alpha'\hat{\boldsymbol{x}} \quad \ldots\ldots \quad (15)$$

where $\boldsymbol{K}_\alpha = \boldsymbol{P}_\alpha\boldsymbol{G}\boldsymbol{R}^{-1}$ $\quad \ldots\ldots \quad (16)$

and $\boldsymbol{P}_\alpha$ is the unique nonnegative definite solution of

$$\boldsymbol{X}(\boldsymbol{F} + \alpha\boldsymbol{I}) + (\boldsymbol{F}' + \alpha\boldsymbol{I})\boldsymbol{X} - \boldsymbol{X}\boldsymbol{G}\boldsymbol{R}^{-1}\boldsymbol{G}'\boldsymbol{X} + \boldsymbol{Q} = 0 \quad . \quad (17)$$

(Equivalently, $\boldsymbol{P}_\alpha$ could be found as the limiting solution of a Riccati equation.) The second remark above then yields that the optimal control for eqns. 1 and 11 is

$$\boldsymbol{u} = -\boldsymbol{K}_\alpha'\boldsymbol{x} \quad \ldots\ldots \quad (18)$$

Thus, the construction of the desired control law is essentially no more difficult than for the case when $\alpha = 0$.

3.2 Verification of the pole-positioning property

Let us now consider the closed-loop poles of $\dot{\boldsymbol{x}} = (\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}_\alpha')\boldsymbol{x}$.

We shall assume, as before, that with $\boldsymbol{H}$ any matrix such that $\boldsymbol{H}\boldsymbol{H}' = \boldsymbol{Q}$, the pair $(\boldsymbol{F}, \boldsymbol{H})$ is completely observable. Then the system defined by eqns. 13 and 15 is surely asymptotically stable, as indicated in Section 2. This system is $\dot{\hat{x}} = (F - GK'_\alpha + \alpha I)\hat{x}$, and since the poles of this system, being given by the eigenvalues of $F - GK'_\alpha + \alpha I$, have negative real parts, it follows that the poles of $\dot{x} = (F - GK'_\alpha)x$, being given by the eigenvalues of $F - GK'_\alpha$ (which are less by α than the eigenvalues of $F - GK'_\alpha + \alpha I$), all possess real parts less than $-\alpha$.

Note that with the observability constraint, it also follows that P_α is positive definite.

4 Additional properties of the new closed-loop law

In this Section, we reconsider the questions of trajectory sensitivity, gain and phase margins and time-delay tolerance, tolerance of nonlinearities and the introduction of a bang-bang control. In general, it is found that taking $\alpha > 0$ improves performance in each of these categories over the case $\alpha = 0$.

4.1 Trajectory sensitivity

As before, we shall restrict attention to the single-input case, and consider the magnitude of the return difference. Evidently, because the system is single-input, the weighting matrix R in eqns. 13 can be assumed to be unity (by adjusting Q if necessary). Then, from the fact that P_α satisfies eqn. 17, we obtain

$$P_\alpha(j\omega I - F) + (-j\omega I - F')P_\alpha + P_\alpha GG'P_\alpha = Q + 2\alpha P_\alpha$$

or

$$G'(-j\omega I - F)^{-1}P_\alpha G + G'P_\alpha(j\omega I - F)^{-1}G + G'(-j\omega I - F')^{-1}P_\alpha GG'P_\alpha(j\omega I - F)^{-1}G = G'(-j\omega I - F')^{-1}(Q + 2\alpha P_\alpha)(j\omega I - F)^{-1}G$$

Now we use the definition of K_α to yield

$$|1 + K'_\alpha(j\omega I - F)^{-1}G|^2 = 1 + G'(-j\omega I - F')^{-1}Q(j\omega I - F)^{-1}G + G'(-j\omega I - F')^{-1}2\alpha P_\alpha(j\omega I - F)^{-1}G \quad (19)$$

With K_0 denoting the control law derived in Section 2, corresponding to taking $\alpha = 0$ in Section 3, comparison of eqns. 8 and 9 yield

$$|1 + K'_\alpha(j\omega I - F)^{-1}G|^2 \geqslant |1 + K'_0(j\omega I - F)^{-1}G|^2 \quad . \quad (20)$$

the difference between these two quantities being $G'(-j\omega I - F')^{-1}2\alpha P_\alpha(j\omega I - F)^{-1}G$ which is positive for almost all ω.

The point to note is that replacement of $\alpha = 0$ by a positive α leads to a higher return difference (and thus greater sensitivity improvement).

It is also relevant to ask whether increasing α from one positive value other than zero to a larger positive value causes the return difference to increase. Inspection of eqn. 19 shows that this increase will occur if $P_{\alpha 1} - P_{\alpha 2}$ is positive definite with $\alpha_1 > \alpha_2$, or if $\dfrac{\partial P_\alpha}{\partial \alpha}$ is positive definite. We now show that this latter case holds. With P_α replacing X in eqn. 17, differentiation and rearrangement yield

$$\frac{\partial P_\alpha}{\partial \alpha}(F - GK'_\alpha + \alpha I) + (F - GK'_\alpha + \alpha I)'\frac{\partial P_\alpha}{\partial \alpha} = -Q - 2P_\alpha \quad . \quad . \quad . \quad . \quad (21)$$

The right-hand side is negative definite, and the eigenvalues of $F - GK'_\alpha + \alpha I$ all have negative real parts. Hence $\dfrac{\partial P_\alpha}{\partial \alpha}$ is positive definite, by the lemma of Lyapunov.[18]

4.2 Gain margin, phase margin and time-delay tolerance

Eqn. 2 and the material of the earlier Section lead to the same conclusion as Section 2, namely that the gain margin is infinite, and the phase margin at least 60°. Eqn. 20, in fact, implies that the Nyquist plot of $K'_\alpha(j\omega I - F)^{-1}G$ for $\alpha > 0$ is further from the unit-radius circle of centre $(-1, j0)$ than that of $K'_0(j\omega I - F)^{-1}G$ in the sense that the distances between $(-1, j0)$ and corresponding points on the two plots (i.e. points determined by the same ω) are ordered by the value of α. All points are, of course, outside the circle. Indeed, the remarks at the end of Section 4.1 confirm that the larger is α, the greater will be the distance of corresponding points from $(-1, j0)$.

Larger α do not immediately imply a large phase margin [because, in general, a frequency ω_1 for which $|K'_0(j\omega_1 I - F)^{-1}G| = 1$ is not such that $|K'_\alpha(j\omega_1 I - F)^{-1}G| = 1$]. But the above remarks do indicate that the larger is α, the more phase shift can, in general, be tolerated, and thus the more time delay, at a fixed frequency.

A further viewpoint of these properties is provided by an equation similar to eqn. 19. Using the fact that P_α satisfies the same equation as P_0, except with $F + \alpha I$ replacing F, we can deduce the following analogue of eqn. 8:

$$|1 + K'_\alpha(j\omega I - F - \alpha I)^{-1}G|^2 = 1 + G'(-j\omega I - F' - \alpha I)^{-1}Q(j\omega I - F - \alpha I)^{-1}G \quad . \quad . \quad . \quad . \quad . \quad (22)$$

implying that

$$|1 + K'_\alpha(j\omega I - F - \alpha I)^{-1}G|^2 \geqslant 1$$

Those remarks made concerning the Nyquist plot of $K'(j\omega I - F)^{-1}G$ in the $\alpha = 0$ case can now be made about the Nyquist plot of $W_\alpha(j\omega - \alpha)$, where $W_\alpha(s) = K'(sI - F)^{-1}G$. Thus the plot of $W_\alpha(j\omega - \alpha)$, rather than just $W_\alpha(j\omega)$, avoids the circle of centre $(-1, j0)$ and radius 1, and so on.

4.3 Tolerance of nonlinearities

Let us now consider the tolerance of nonlinearities in the input transducers. Instead of writing

$$u_i = -(K'_\alpha x)_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

we shall permit

$$u_i = -k_i(t)(K'_\alpha x)_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and subsequently present constraints on $k_i(t)$ which preserve stability. In effect, we shall be trading the extra stability of the linear closed-loop system, implied by having poles to the left of $\mathrm{Re}(s) = -\alpha$ instead of $\mathrm{Re}(s) = 0$, for looser constraints on $k_i(t)$ than were permitted in Section 3.

Adopt, as a Lyapunov function, $V = x'P_\alpha x$. For simplicity, we shall consider a single-input system, but the results may readily be extended. Then

$$\dot{V} = x'(P_\alpha F + F'P_\alpha) - x'P_\alpha GK'_\alpha xk(t) - x'K_\alpha G'P_\alpha xk(t) \quad (25)$$

Now use the definition of K_α and eqn. 17 with X replaced by P_α to yield

$$\dot{V} = -x'Mx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where

$$M = 2\alpha P_\alpha + Q - 2K_\alpha K'_\alpha\{k(t) - \tfrac{1}{2}\} \quad . \quad . \quad . \quad (27)$$

Asymptotic stability is retained if M is positive definite, and if $M - 2\beta P_\alpha$ is nonnegative definite for some positive β, i.e. $\dot{V} < -2\beta V$, then the states decay at least as fast as $e^{-\beta t}$.

The upper bound on $k(t)$ can remain as before; the lower bound was previously $\frac{1}{2}$, but in this instance we see that this will yield decay as fast as $e^{-\alpha t}$. But $k(t) < \frac{1}{2}$ will still guarantee $M - 2\beta P_\alpha$ nonnegative definite for some positive β; in fact, provided that

$$k(t) - \epsilon > \frac{1}{2} - \frac{\lambda_{min}(\alpha P_\alpha + \frac{1}{2}Q)}{\lambda_{max}(K_\alpha K'_\alpha)} \quad . \quad . \quad . \quad . \quad (28)$$

for fixed positive ϵ, asymptotic stability prevails. But the smaller the minimum $k(t)$ is permitted to become, i.e. the broader the range of nonlinearity of time variation permitted, the smaller β, or the greater the reduction in degree of stability of the closed-loop system.

It is not clear how the bound on the right of eqn. 28 varies with α. Certainly, as α increases, $\lambda_{min}(\alpha P_\alpha + Q)$ increases, but so also does $\lambda_{max}(K_\alpha K'_\alpha) = \lambda_{max}(P_\alpha GG'P_\alpha)$, both increases coming about because $\dfrac{\partial P_\alpha}{\partial \alpha}$ is positive definite. If Q is singular, as may well be the case, then certainly changing α from 0 to a positive value will increase the range of stability. This is because $\lambda_{min}(Q = 0)$, while $\lambda_{min}(\alpha P_\alpha + Q) > 0$ for all positive α. But one cannot apparently conclude that indefinite increase of α will continually increase the range of stability.

The conclusion of this Section could be useful in handling a situation where an input transducer was known to be nonlinear, with its (nonlinear) gain confined to a certain sector. Experimentation with different Q and α may lead to a control law K_α such that eqn. 28 holds. Then the linear design could be applied to the nonlinear system with the assurance that asymptotic stability of the closed-loop nonlinear system would follow.

4.4 Insertion of a bang-bang control law

We consider the situation of Fig. 4, with K_α replacing K. It is again true that the numerator polynomial of $K'_\alpha(sI - F)^{-1}G$ is of one less degree than the denominator polynomial, with the leading coefficients of both polynomials possessing the same sign. There is, however, a strengthened result concerning the zeros. From eqns. 26 and 27, it follows that, with $V = x'P_\alpha x$, a Lyapunov function $\dot{V} \leqslant -2\alpha V$ for all constant gains $k > \frac{1}{2}$. This means that points on the root locus of $K'_\alpha(sI - F)^{-1}G$ corresponding to finite gains greater than $\frac{1}{2}$ all lie to the left of $\mathrm{Re}(s) = -\alpha$; consequently the zeros of $K'_\alpha(sI - F)^{-1}G$ cannot lie to the right of this line, though they may lie on it.

We conclude that the relay system is asymptotically stable,[17] and that in the sliding regime[16, 17] it has degree of stability α.

5 Conclusions

We begin by summarising the previous material.

First, we have exhibited a procedure for obtaining a control law which simultaneously minimises a quadratic-loss function, and achieves closed-loop poles lying in a restricted region of the s plane. The procedure is very little different from that used in solving normal quadratic-loss problems.

Secondly, we have compared the properties of systems designed according to the new scheme and the old. We observe that the sensitivity to plant-parameter variation is better for the new scheme than the old, because the return difference is larger. Gain margin is unaltered, while more phase shift (or time delay) can be tolerated, without disturbing stability, for the new scheme than for the old. It appears that a wider range of nonlinearity in the input transducer can be tolerated with the new scheme, and we have obtained results suggesting application in the area of relay-control systems.

Though many of the results have only been established for single-input systems, with the appropriate modifications they apply for multiple-input systems. That they apply for time-varying systems, again with appropriate modifications, may also be checked.

If states are not available for feedback, an estimator can always be designed which will obtain the states. As is well known, the closed-loop plant dynamics and estimator dynamics are essentially independent, and so the introduction of state estimators would not be expected to alter the conclusions.

It would appear, on the face of it, that much is to be gained and little lost by choosing $\alpha > 0$ in place of $\alpha = 0$ in any optimisation problem. We can, however, note one potential disadvantage. In heuristic terms, the faster a state is controlled to the origin, the greater the control power (instantaneous value of $u'u$) required. Evidently, choice of a large α is tantamount to fast control of a state (because states decay at least as fast as $e^{-\alpha t}$) and thus high power u. Thus, a practical limitation is imposed by the power-handling capacity of the input transducers in dealing with large α.

There are two possible directions in which this work might be extended. The first would be to seek a technique for ensuring that all poles of a closed-loop system, designed using a quadratic performance-index minimisation, had a damping ratio with a prescribed lower bound. The second would be to seek functions $f(t)$, other than $e^{\alpha t}$, such that replacement of $e^{\alpha t}$ by $f(t)$ in the performance index (eqn. 11) would still yield a constant linear optimal control, but recent work of the authors does suggest that no such functions exist.

6 References

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