Reprinted from *Electronics Letters,* Vol. 5, No. 22, 30th October 1969

STABILITY RESULTS FOR OPTIMAL SYSTEMS*

Linear optimal-control systems are known to have the property that they can tolerate large amounts of nonlinearity or time variation in the optimal-control law. This property is extended to other classes of optimal systems.

One of the popular ways of designing time-invariant linear feedback laws for constant linear systems of the form

$$\dot{x} = Fx + gu \quad (1)$$

is to find the control law which minimises a quadratic performance index

$$\int_{t_0}^{\infty} (u^2 + x'Qx) \quad (2)$$

Of course, Q is constant and nonnegative definite; normally $[F,g]$ is completely controllable, and $[F,D]$ is completely observable, for any D such that $DD' = Q$. These conditions ensure not only that the optimal closed-loop system is asymptotically stable, but that a substantial amount of nonlinearity or even time variation in the feedback loop will not disturb stability. More precisely, if the optimal feedback law is $u = -k'x$, the law $u = -\kappa(t)k'x$ with, for all t, $\frac{1}{2} + \epsilon \leqslant \kappa(t) \leqslant \bar{\kappa} < \infty$ with $\bar{\kappa}$ arbitrary will not disturb the asymptotic stability.[1]

Here we indicate briefly generalisations of this result, often alleged to be a strong justification for optimal control, to nonlinear systems. For the moment, we retain eqn. 1 but replace eqn. 2 by

$$V\{x(t_0),u(\cdot)\} = \int_{t_0}^{\infty} \{u^2 + m(x)\}dt \quad (3)$$

Here $m(x)$ is assumed nonnegative for all x, with the property that $m(e^{Ft}x_0) = 0$ for all t implies that $x_0 = 0$. We also assume that $m(x)$ is such that an optimal control exists and yields an asymptotically stable closed-loop system. Let $V^* = n(x)$ be the optimal performance index. Then the optimal control is

$$u^* = -\tfrac{1}{2}g'\nabla V^* = -\tfrac{1}{2}g'\nabla n(x) \quad (4)$$

and the Hamilton–Jacobi equation is

$$x'F'\nabla n(x) - \tfrac{1}{4}\nabla' n(x)gg'\nabla n(x) + m(x) = 0 \quad (5)$$

The closed-loop system resulting from implementation of the optimal control is

$$\dot{x} = Fx - \tfrac{1}{2}gg'\nabla n(x) \quad (6)$$

*Work supported by the Australian Research Grants Committee

However, we consider the introduction of a time-varying gain in the feedback loop, perhaps generated by a nonlinearity, and yielding a closed-loop system

$$\dot{x} = Fx - \tfrac{1}{2}gg'\nabla n(x) - \tfrac{1}{2}\kappa(t)gg'\nabla n(x) \quad (7)$$

where, for all t, $\frac{1}{2} + \epsilon \leqslant \kappa(t) \leqslant \kappa < \infty$ with $\bar{\kappa}$ arbitrary. We show now with a simple proof that asymptotic stability of eqn. 7 prevails, using the Lyapunov function $V_0 = n(x)$. Certainly V_0 is positive definite, as examination of the optimal-control problem associated with eqns. 1 and 3 shows. Further,

$$\dot{V}_0 = \nabla' n(x)\{Fx - \tfrac{1}{2}gg'\nabla n(x) - \tfrac{1}{2}\kappa(t)gg'\nabla n(x)\}$$

$$= x'F'\nabla n(x) - \tfrac{1}{2}\nabla' n(x)gg'\nabla n(x) - \frac{\kappa(t)}{2}\nabla' n(x)gg'\nabla n(x)$$

$$= -m(x) - \frac{\kappa}{2}\nabla' n(x)gg'\nabla n(x) \quad (8)$$

the last line following by use of the Hamilton–Jacobi equation (eqn. 5). Evidently $\dot{V}_0$ is always nonpositive. In fact it is nonzero along a trajectory, for suppose the contrary. Then $g'\nabla n(x) = 0$ identically, and thus eqn. 7 becomes $\dot{x} = Fx$. Consequently $\dot{V}_0 = -m(e^{Ft}x_0)$, which is known to not be identically zero. Asymptotic stability is thus established.

More can actually be proved. First, the performance index (eqn. 3) can be replaced by one with loss function $r(u) + m(x)$, where $r(u)$ must satisfy sufficient properties to ensure existence of V^* and asymptotic stability of the closed-loop system. If also the graph of $r(u)$ lies below that of any parabola ku^2, in the sense that $\frac{d}{du}(\ln u^2) \leqslant \frac{d}{du}\{\ln r(u)\}$ for all u, the result proved above for the special case $r(u) = u^2$ extends to the case of arbitrary $r(u)$. If $r(u)$ simply ensures existence of V^* and asymptotic stability of the closed-loop system, asymptotic stability of eqn. 7 holds with the same upper bound on $\kappa(t)$, but with a varied lower bound, always between $\frac{1}{2}$ and 1, but whose actual value depends on the particular $r(\cdot)$.

Secondly, the fact that, in eqn. 1, the term Fx is linear is irrelevant in drawing any of the conclusions. The results all go through for Fx replaced by $f(x)$, *mutatis mutandis*. In this case, some care is necessary in ensuring asymptotic stability of the closed-loop system and 'observability' of $m(x)$.

B. D. O. ANDERSON

Department of Electrical Engineering
University of Newcastle
NSW 2308, Australia

30th September 1969

Reference

1 SAGE, A. P.: 'Optimum systems control' (Prentice-Hall, 1968)