Discrete positive-real functions and their application to system stability

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Abstract

A class of discrete-time transfer-function matrices termed discrete positive-real matrices is defined, and a system theoretic description of them is given, analogous to that for ordinary positive-real matrices. This description is applied to analysing the stability of a discrete-time system with linear forward part, and time-varying memoryless feedback.

Introduction

Since the classic study by Popov[1] in 1961, an extensive literature has arisen from the application of the concept of positive-real functions and matrices to the theory of linear dynamical systems. The concept originated in network theory as the frequency-domain formulation of the fact that the time integral of the energy input to a passive network must be positive. About 1962 Kalman[2] and Yacubovich[3] independently discovered an algebraic criterion for the transfer function of a continuous-time system to be positive-real. This criterion was the key to the solution of the problem of Lur'e: to state under what conditions a special class of Lyapunov functions would guarantee the stability of systems with a memoryless nonlinearity in the feedback loop. Closely analogous results were subsequently shown by Kalman and Szegö[4] to hold for single-input single-output discrete-time systems. These ideas were recently extended to multiple-input, multiple-output continuous-time systems by Anderson,[5] who obtained an algebraic criterion for a matrix of transfer functions to be positive real. This has proved useful in stability studies of systems with multiple nonlinearities,[6] in the theory of optimal linear control systems and in spectral factorisation problems.[7]

This paper presents a similar development for discrete-time systems. After a review in Section 2 of some results required in later proofs, the notion of a 'discrete positive-real' matrix will be defined in Section 3, followed by a discussion of the behaviour of such matrices on the unit circle, and the presentation of an algebraic criterion for a matrix of real rational functions to be discrete positive-real. Finally, in Section 4, a criterion analogous to the circle criterion for single-input single-output continuous systems[8] is derived for the stability of discrete-time systems with multiple time-varying nonlinearities as feedback elements. The criterion is similar to one of the results in the extensive discussion of the problem by Jury and Lee.[9]

2 Preliminaries

Before discussing discrete-time systems, we review some definitions and results from the theory of continuous-time systems. Any $m \times n$ matrix $G(s)$ of real rational functions of the complex variable s, with the property that $G(\infty)$ is finite, may be interpreted as the transfer-function matrix of a time-invariant lumped-parameter linear system having the state-space equations

$$\left.\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) \\ y(t) &= C'x(t) + Ju(t) \end{aligned}\right\} \quad \ldots \ldots \quad (1)$$

where $u(t)$ is an n vector, the input; x is a p vector, the state; $y(t)$ is an m vector, the output; B is $p \times n$, A is $p \times p$, C is $p \times m$ and J is $m \times n$. All vectors are real, and all matrices are real and constant. The prime denotes matrix transposition.

Paper 5713 C, first received 17th June and in revised form 11th September 1968

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The set (A, B, C, J) is called a 'realisation' of $G(s)$, and must be related to $G(s)$ via

$$G(s) = J + C'(sI - A)^{-1}B \quad \ldots \ldots \quad (2)$$

Every $G(s)$ has infinitely many realisations; those in which the state vector has the smallest possible number of components are termed 'minimal'.

A square matrix $G(s)$ of real rational functions of the complex variable s is called positive real (p.r.) if it satisfies the following conditions:

$$G(s) \text{ has elements analytic in } \mathrm{Re}\,(s) > 0 \quad \ldots \quad (3)$$

$$G(s) + G'^{*}(s) \geqslant 0 \text{ in } \mathrm{Re}\,(s) > 0 \quad \ldots \ldots \quad (4)$$

Here the superscript asterisk denotes the operation of complex conjugation and $D \geqslant 0$ (> 0) means that D is nonnegative definite (positive definite). Condition 4 can be shown to be equivalent to (Reference 10, chap. 5):

$$G(s) \text{ has simple poles on } \mathrm{Re}\,(s) = 0 \quad \ldots \ldots \quad (5)$$

For any pole of an element of $G(s)$ at which $\mathrm{Re}\,(s) = 0$, the residue matrix is nonnegative definite Hermitian . . . . (6)

$G(j\omega) + G'(-j\omega) \geqslant 0$ for all real ω for which $j\omega$ is not a pole of an element of $G(s)$ . . . (7)

It is possible to consider positive-real matrices having other than real-rational elements. However, as this paper is concerned only with lumped-parameter systems, the definition given above will suffice here. Note that, because the elements of $G(\cdot)$ are real rational, $G^{*}(s) = G(s^{*})$ for all s.

Lemma 1:[5] A square matrix $G(\cdot)$ of real rational functions, analytic in $\mathrm{Re}\,(s) > 0$, with simple poles only on $\mathrm{Re}\,(s) = 0$, and with finite $G(\infty)$, is positive real if, and only if, there exist a real symmetric positive definite matrix P and real matrices L and W such that

$$PA + A'P = -LL' \quad \ldots \ldots \quad (8a)$$

$$PB = C - LW \quad \ldots \ldots \quad (8b)$$

$$W'W = J + J' \quad \ldots \ldots \quad (8c)$$

where the set (A, B, C, L) is a minimal realisation of $G(\cdot)$.

3 Discrete positive real matrices

A matrix $G(z)$ of real rational functions of z may also be interpreted as the z transform of the impulse response matrix of a discrete-time system described by the difference equations

$$x(n+1) = Ax(n) + Bu(n) \quad \ldots \ldots \quad (9a)$$

$$y(n) = C'x(n) + Ju(n) \quad \ldots \ldots \quad (9b)$$

Here the input, state and output vectors are pulse sequences defined only at discrete instants of time, the symbol $x(n)$ being shorthand for $x(nT)$, where T is the sampling interval and

$n = 0, 1, 2, \ldots$ Again, we call the set (A, B, C, J) a 'realisation' of $G(\cdot)$, requiring

$$G(z) = J + C'(zI - A)^{-1}B \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

By analogy with the continuous-time case, we shall call a square matrix $G(z)$ of real-rational functions 'discrete positive real' (d.p.r.) if it has the following properties:

$$G(z) \text{ has elements analytic in } |z| > 1 \quad . \quad . \quad . \quad (11)$$

$$G^*(z) + G(z) \geq 0 \text{ in } |z| > 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Much as the positive real nature of a matrix $G(s)$ can be partly defined in terms of its behaviour on $\mathrm{Re}\, s = 0$, the discrete positive real nature of $G(\cdot)$ can be related to its properties on the unit circle.

Lemma 2: A square matrix $G(z)$ whose elements are real rational functions analytic in $|z| > 1$ is discrete positive real if, and only if, it satisfies all the following conditions:

Poles of elements of $G(z)$ on $|z| = 1$ are simple (13)

$G(e^{j\omega}) + G'(e^{-j\omega}) \geq 0$ for all real ω at which $G(e^{j\omega})$ exists . . . . (14)

If $z_0 = e^{j\omega_0}$, ω_0 real, is a pole of an element of $G(z)$, and if K is the residue matrix of $G(z)$ at $z = z_0$, the matrix $Q = e^{-j\omega_0}K$ is nonnegative definite Hermitian . . . . (15)

This result is the discrete-time counterpart of Reference 10, theorem 5.1: In the proof, one demonstrates corresponding assertions for the matrix $H(z) = G(1/z)$. This detour is necessary since $G(z)$ is not analytic in a closed bounded region, while $H(z)$ is. One then considers the function $f_x(z) = x'^*H(z)x$, with x some complex n vector, on a closed contour lying on the unit circle except for arbitrarily small semicircular indentations into $|z| < 1$ around each of the poles of $f_x(z)$ on $|z| = 1$. The remainder of the proof is a straightforward adaptation of the arguments employed in Reference 10; details will be omitted here.

Lemma 3: Let $G(z)$ be a square matrix of real rational functions of z with no poles in $|z| > 1$ and simple poles only on $|z| = 1$, and let (A, B, C, J) be a minimal realisation of $G(z)$. Then necessary and sufficient conditions for $G(z)$ to be discrete positive real are that there exist a real symmetric positive definite matrix P and real matrices L and W such that

$$A'PA - P = -LL' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16a)$$

$$A'PB = C - LW \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16b)$$

$$W'W = J + J' - B'PB \quad . \quad . \quad . \quad . \quad . \quad (16c)$$

Proof of necessity: We consider first the case where $G(z)$ is analytic at $z = -1$. By means of the bilinear transformation

$$s = \frac{z - 1}{z + 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

the matrix 10 is transformed into a matrix $G_c(s) = J_c + C_c'(sI - A_c)^{-1}B_c$, where

$$\left.\begin{aligned} A_c &= (A + I)^{-1}(A - I) \\ B_c &= 2(A + I)^{-2}B \\ C_c &= C \\ J_c &= J - C'(A + I)^{-1}B \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

It is not hard to show that eqns. 18 define a minimal realisation of $G_c(s)$, and that $G_c(s)$ is positive real if, and only if, $G(z)$ is discrete positive real. From lemma 1, therefore, there exist real matrices $P_c = P_c' > 0$, L_c and W_c such that

$$\left.\begin{aligned} &P_c(A + I)^{-1}(A - I) + (A' - I)(A' + I)^{-1}P_c = -L_cL_c' \\ &2P_c(A + I)^{-1}B = C - L_cW_c \\ &J + J' - C'(A + I)^{-1}B - B'(A' + I)^{-1}C = W_c'W_c \end{aligned}\right\} (19)$$

With the definitions

$$\begin{aligned} P &= 2(A' + I)^{-1}P_c(A + I)^{-1} \\ L &= L_c \\ W &= W_c - L_c'(A + I)^{-1}B \end{aligned}$$

eqns. 19 immediately reduce to eqns. 16.

The general case, where $G(z)$ has a simple pole at $z = -1$, can be treated by an expansion of $G(z)$ which separates out this pole. Thus

$$G(z) = G_1(z) + G_2(z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where

$$G_1(z) = \frac{z - 1}{z + 1}M = M - \frac{2M}{z + 1} \quad (M \text{ a constant matrix}) \quad (21)$$

and where $G_2(z)$ has no pole at $z = -1$. Application of lemma 2 then shows that M is real and nonnegative definite symmetric and that $G(z)$ is discrete positive real if, and only if, both $G_1(z)$ and $G_2(z)$ are. These two matrices will each have some minimal realisation, say (A_1, B_1, C_1, J_1) and (A_2, B_2, C_2, J_2), respectively, in terms of which a minimal realisation for $G(z)$ is given by

$$G(z) = J_1 + J_2 + [C_1'C_2']\left[zI - \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\right]^{-1}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad . \quad (22)$$

By hypothesis, $G_2(z)$ is discrete positive real, and hence, from the previous arguments, there exist matrices $P_2 = P_2' > 0$, L_2 and W_2 which satisfy eqns. 16 for the realisation (A_2, B_2, C_2, J_2). Since the matrix M in eqn. 21 is nonnegative definite symmetric, there exists a nonsingular T such that (Reference 11, p. 37)

$$M = T^{-1}\left(\sum_{i=1}^{r} x_ix_i'\right)(T')^{-1} = \sum_{i=1}^{r} y_iy_i'$$

where r is the rank of M and the x_i are linearly independent real vectors. Therefore $G_1(z)$ has a minimal realisation (I_r denotes the $r \times r$ identity matrix):

$$\left\{-I_r, (y_1, y_2, \ldots, y_r)', -2(y_1, y_2 \ldots, y_r)', \sum_{i=1}^{r} y_iy_i'\right\}$$

for which the matrices $P_1 = 2I_r$, $L_1 = 0$, $W_1 = 0$ are readily seen to satisfy eqn. 16.

It is now easily checked that the matrices

$$P = \begin{bmatrix} 2I_r & 0 \\ 0 & P_2 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 \\ L_2 \end{bmatrix}$$

$$W = W_2$$

satisfy eqn. 16 for the minimal realisation (eqn. 22) of $G(z)$. We have exhibited the matrices P, L and W for this particular realisation of $G(z)$. However, from the fact that minimal realisations are algebraically equivalent,[12] it follows immediately that such matrices exist for all minimal realisations.

Sufficiency: It suffices to show that eqns. 16 imply eqn. 12. Now, from eqn. 16a, it is readily verified that

$$(z^*I - A')P(zI - A) + (z^*I - A')PA + A'P(zI - A) = (|z|^2 - 1)P + LL'$$

After some manipulation, and use of eqns. 16b and 16c, there results

$$\begin{aligned} J + J' + C'(zI - A)^{-1}B + B'(z^*I - A')^{-1}C &= (|z|^2 - 1)B'(z^*I - A')^{-1}P(zI - A)^{-1}B \\ &\quad + \{W' + B'(z^*I - A)^{-1}L\}\{W + L'(zI - A)^{-1}B\} \end{aligned}$$

The right-hand side is clearly nonnegative definite in $|z| > 1$, while the left-hand side is precisely $G(z) + G'^*(z)$. This completes the proof of the lemma.

4 Stability of discrete-time systems with memoryless feedback

We examine the stability of the null solution (Lyapunov stability) of discrete-time feedback systems described by state-space difference equations of the form

$$\begin{aligned} x(n + 1) &= Ax(n) + B\phi \\ y(n) &= C'x(n) \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Here x is a p vector, the state; y is an m vector, the output; and Φ is an m vector, being a feedback signal produced by a group of memoryless nonlinearities in the feedback path. It will be assumed that (A, B) is completely controllable and (A, C') is completely observable.[12] Therefore (A, B, C) is a minimal realisation of the z transform $W(z)$ of the impulse-response matrix of the linear part of the system:

$$W(z) = C'(zI - A)^{-1}B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Note that $W(z)$ is a square matrix and $W(\infty) = 0$. It will be assumed that the ith element of the feedback vector is a function of the ith element of the output only, and even though it can be, in general, nonlinear and time-varying, it is restricted to lie always in a sector of the first and third quadrants:

$$\Phi(n; 0) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25a)$$

and for $y_i \neq 0$,

$$k_{ii}^{-1} y_i^2(n) > \phi_i\{n, y_i(n)\}y_i(n) > 0 \quad i = 1, 2, \ldots, m \quad (25b)$$

where each k_{ii} is a positive constant. Eqns. 25 can readily be seen to imply

$$\Phi\{n, y(n)\}y(n) > \Phi\{n, y(n)\}K\Phi\{n, y(n)\} \quad . \quad . \quad . \quad (26)$$

for $y \neq 0$, where K is a diagonal positive definite matrix formed from the k_{ii}.

Theorem: If the matrix $G(z) = K + W(z)$ is discrete positive real, the system (eqns. 23 and 25) is uniformly stable in the large.

Proof: By hypothesis, $G(z) = K + C'(zI - A)^{-1}B$ is d.p.r. Therefore, from lemma 3, there exist matrices $P = P' > 0$, L and U such that

$$\begin{aligned} A'PA - P &= -LL' \\ A'PB &= C - LU \\ U'U &= K + K' - B'PB \quad . \quad . \quad . \quad . \quad . \quad (27) \end{aligned}$$

Thus there exists a positive definite function

$$V(x) = x'Px$$

whose change along a solution of eqn. 23 is, from eqns. 23 and 27, given by

$$\begin{aligned} \Delta V(x) = &-[L'x(n) - U\Phi\{n, y(n)\}]'[L'x(n) - U\Phi\{n, y(n)\}] \\ &- 2[\Phi'\{n, y(n)\}y(n) - \Phi'\{n, y(n)\}K\Phi\{n, y(n)\}] \\ &\quad . \quad . \quad . \quad . \quad (28) \end{aligned}$$

The first term in eqn. 28 is clearly nonpositive, while the second term is nonpositive from eqn. 25. Thus V is a positive definite function for all x, with a nonnegative change along any trajectory, establishing the stability of the system.[13]

Corollary 1: If the matrix $G(z) = K + W(z)$ is d.p.r., and if the nonlinearities are time-invariant but otherwise restricted as in eqn. 25, the system (eqn. 23) is asymptotically stable in the large.

Proof: To establish asymptotic stability in the autonomous case, it is sufficient[13] to show that $\Delta V(x)$ cannot vanish identically along any solution of eqn. 23. From eqns. 26 and 28, suitably modified to remove the explicit dependence of Φ on n, it is clear that $\Delta V(x)$ vanishes identically only if the output y is identically zero. But this is impossible while the state is nonzero, since the system is assumed to be completely observable.

In the time-varying case, detailed information on the structure of the matrices B and C is, in general, required before asymptotic stability can be shown. However, if it is possible to find a ρ, $0 < \rho < 1$, such that $G(\rho z)$ is d.p.r., asymptotic stability holds. We have

Corollary 2: If there exists a number ρ, $0 < \rho < 1$, such that the matrix $G(\rho z) = K + \rho^{-1}C'(zI - \rho^{-1}A)^{-1}B$ is d.p.r., the system (eqns. 23 and 25) is uniformly asymptotically stable in the large.

This follows by applying lemma 3 to the realisation of $G(\rho z)$; a simple calculation shows that the change of the positive definite form $V(x) = x'Px$ is negative definite along any trajectory of eqn. 23.

5 Conclusions

One application, to stability theory, has been given of an algebraic description of discrete positive real *functions*.[4] It would not be unreasonable to hope for further applications along the lines of Reference 7, and to hope that these applications should make use of the description of discrete positive-real matrices, as distinct from functions, presented in this paper.

Just as for continuous-time systems,[14] it should be possible to show that, in a nominally linear asymptotically stable system, small amounts of nonlinearity can be tolerated without affecting stability. The proof of this result in the continuous-time case relies on the fact that, given an $n \times n$ matrix $W(s)$ with elements analytic in $\mathrm{Re}(s) \geq 0$, there exists a diagonal matrix $K \geq 0$ and a real scalar $\sigma > 0$ such that $K + W(s + \sigma)$ is positive real. The equivalent discrete-time theorem would be: given an $n \times n$ matrix $G(z)$ with elements analytic in $|z| \geq 1$, there exists a diagonal matrix $K \geq 0$ and a real scalar ρ, with $0 < \rho < 1$ such that $K + G(\rho z)$ is positive real.

6 References

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