Discrete positive-real functions and their application to system stability

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Abstract
A class of discrete-time transfer-function matrices termed discrete positive-real matrices is defined, and a system theoretic description of them is given, analogous to that for ordinary positive-real matrices. This description is applied to analyzing the stability of a discrete-time system with linear forward part and time-varying memoryless feedback.

Introduction
Since the classic study by Popov in 1961, an extensive literature has arisen from the application of the concept of positive-real functions and matrices to the theory of linear dynamical systems. The concept originated in network theory as the frequency-domain formulation of the fact that the time integral of the energy input to a passive network must be positive. About 1962 Kalman and Yakubovich independently discovered an algebraic criterion for the transfer function of a continuous-time system to be positive-real. This criterion was the key to the solution of the problem of Lyapunov, to state under what conditions a special class of Lyapunov functions would guarantee the stability of systems with a memoryless nonlinearity in the feedback loop. Closely analogous results were subsequently shown by Kalman and Szegö to hold for single-input single-output discrete-time systems. These ideas were recently extended to multiple-input, multiple-output continuous-time systems by Anderson, who obtained an algebraic criterion for a matrix of transfer functions to be positive real. This has proved useful in stability studies of systems with multiple nonlinearities, in the theory of optimal linear control systems and in spectral factorization problems.

This paper presents a similar development for discrete-time systems. After a review in Section 2 of some results required in later proofs, the notion of a 'discrete positive-real' matrix will be defined in Section 3, followed by a discussion of the behaviour of such matrices on the unit circle, and the presentation of an algebraic criterion for a matrix of real rational functions to be discrete positive-real. Finally, in Section 4, a criterion analogous to the circle criterion for single-input single-output continuous systems is derived for the stability of discrete-time systems with multiple time-varying nonlinearities as feedback elements. The criterion is similar to one of the results in the extensive discussion of the problem by Jury and Lee.

2 Preliminaries
Before discussing discrete-time systems, we review some definitions and results from the theory of continuous-time systems. Any \( m \times n \) matrix \( G(s) \) of real rational functions of the complex variable \( s \), with the property that \( G(0) \) is finite, may be interpreted as the transfer-function matrix of a time-invariant lumped-parameter linear system having the state-space equations

\[
\begin{align*}
\frac{dx(t)}{dt} &= Ax(t) + Bu(t) \\
y(t) &= C^T x(t) - Ju(t)
\end{align*}
\]

where \( x(t) \) is an \( n \) vector, the input: \( x \) is a \( p \) vector, the state; \( y(t) \) is an \( m \) vector, the output: \( B \) is \( p \times n \), \( A \) is \( p \times p \), \( C \) is \( p \times m \), and \( J \) is \( m \times n \). All vectors are real, and all matrices are real and constant. The prime denotes matrix transposition.

\[PA = A^P = -LL'\]  
\[PB = C - LW\]  
\[W'W = J - J'\]

where the set \((A, B, C, L)\) is a minimal realisation of \( G(\cdot) \).

3 Discrete positive real matrices
A matrix \( G(z) \) of real rational functions of \( z \) may also be interpreted as the \( z \) transform of the impulse response matrix of a discrete-time system described by the difference equations

\[
x(n - 1) = Ax(n) + Bu(n) \\
y(n) = C^T x(n) + Ju(n)
\]

Here the input, state and output vectors are pulse sequences defined only at discrete instants of time, the symbol \( x(n) \) being shorthand for \( x(nT) \), where \( T \) is the sampling interval and
\( n = 0, 1, 2, \ldots \) again, we call the set \( (A, B, C, J) \) a 'realisation' of \( G(z) \), requiring

\[
G(z) = J + C(zI - A)^{-1}B.
\]

By analogy with the continuous-time case, we shall call a square matrix \( G(z) \) of real-rational functions 'discrete positive real' (d.pr.) if it has the following properties:

\[
G(z) \text{ has elements analytic in } |z| > 1
\]

\[
G(z) = 0 \text{ in } |z| > 1
\]

Much as the positive real nature of a matrix \( G(z) \) can be partly defined in terms of its behaviour on \( \Re z > 0 \), the discrete positive real nature of \( G(z) \) can be related to its properties on the unit circle.

**Lemma 2:** A square matrix \( G(z) \) whose elements are real rational functions analytic in \( |z| > 1 \) is discrete positive real if, and only if, it satisfies all the following conditions:

- Poles of elements of \( G(z) \) on \( |z| = 1 \) are simple

\[
G(z) + G(1/z) > 0 \text{ for all real } \omega \text{ at which } G(\omega) \text{ exists}
\]

\[
\text{if } z_0 = e^{j\omega} \text{ is a pole of an element of } G(z), \text{ and if } K \text{ is the residue matrix of } G(z) \text{ at } z = z_0, \text{ the matrix } Q = e^{-j\omega}K \text{ is nonnegative definite.}
\]

Hermitian

The result is the discrete-time counterpart of Reference 10-theorem 3.1. In the proof, one demonstrates corresponding assertions for the matrix \( H(z) = G(1/z) \). This decor is necessary since \( G(z) \) is not analytic in a closed bounded region, while \( H(z) \) is. One then considers the function \( f_z(z) = e^{-j\omega}H(z)x \), with \( x \) some complex vector, on a closed contour lying on the unit circle except for arbitrarily small semi-circular indentations into \( |z| < 1 \) around each of the poles of \( f_z(z) \) on \( |z| = 1 \). The remainder of the proof is a straightforward adaptation of the arguments employed in Reference 10; details will be omitted here.

**Lemma 3:** Let \( G(z) \) be a square matrix of real rational functions of \( z \) with no poles in \( |z| > 1 \) and simple poles only on \( |z| = 1 \), and let \( (A, B, C, J) \) be a minimal realisation of \( G(z) \). Then necessary and sufficient conditions for \( G(z) \) to be discrete positive real are that there exist a real symmetric positive definite matrix \( P \) and real matrices \( L \) and \( W \) such that

\[
A'PA = P = -LL',
\]

\[
A'PB = C - LW,
\]

\[
W'W = J' - B'PB.
\]

**Proof of necessity:** We consider first the case where \( C(z) \) is analytic at \( z = -1 \). By means of the bilinear transformation

\[
z = \frac{1}{1 - z}.
\]

The matrix 10 is transformed into a matrix \( G_z(s) = J_z + C_z(zI - A_z)^{-1}B_z \), where

\[
A_z = (A + 1)^{-1}(A - I),
\]

\[
B_z = 2(A + 1)^{-2}B,
\]

\[
C_z = C,
\]

\[
J_z = J - C'(A + 1)^{-1}B.
\]

It is not hard to show that eqns. 18 define a minimal realisation of \( G_z(s) \), and that \( G_z(s) \) is positive real if, and only if, \( G(z) \) is discrete positive real. From lemma 1, therefore, there exist real matrices \( P_z = P_z > 0, L_z \) and \( W_z \) such that

\[
P_z(A - I)^{-1}(A - I) + (A' - M'A + 1)^{-1}P_z = -L'_z
\]

\[
2P_z(A - I) = B = C - L_zW_z
\]

\[
J_z - C'(A + 1)^{-1}B - B'(A + 1)^{-1}C = W'_zW_z.
\]

With the definitions

\[
P = 2(A - I)^{-1}P_z(A - I)^{1}
\]

\[
L = L_z
\]

\[
W = W_z - L'_z(A - I)^{-1}B
\]

eqns. 19 immediately reduce to eqns. 16.

The general case, where \( G(z) \) has a simple pole at \( z = -1 \), can be treated by an expansion of \( G(z) \) which separates out this pole. Thus

\[
G(z) = G(z) + G_z(z).
\]

where

\[
G_z(z) = \frac{z - 1}{z + 1}M = \frac{2M}{z + 1} (M \text{ a constant matrix})
\]

and where \( G_z(z) \) has no pole at \( z = -1 \). Application of lemma 2 then shows that \( M \) is real and nonnegative definite symmetric and that \( G(z) \) is discrete positive real if, and only if, both \( G(z) \) and \( G_z(z) \) are. These two matrices will each have some minimal realisation, say \( (A_1, B_1, C_1, J_1) \) and \( (A_2, B_2, C_2, J_2) \), respectively, in terms of which a minimal realisation for \( G(z) \) is given by

\[
G(z) = J_1 + J_2 + [C_1C_2\left[ zI - (A_1 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right]^{-1}B_2 \right] .
\]

By hypothesis, \( G_z(z) \) is discrete positive real, and hence, from the previous arguments, there exist matrices \( P_z = P_z > 0, L_z \) and \( W_z \) which satisfy eqns. 16 for the realisation \( (A_1, B_1, C_2, J_2) \). Since the matrix \( M \) in eqn. 21 is nonnegative definite symmetric, there exists a nonsingular \( T \) such that (Reference 11, p. 37)

\[
M = T^\top \left[ \sum_{i=1}^{m} 2x_iz_i(T^\top)^{-1} \right] \sum_{i=1}^{m} y_iy_i^\top
\]

where \( r \) is the rank of \( M \) and the \( x_i \) are linearly independent real vectors. Therefore \( G_z(z) \) has a minimal realisation \( (J_2, P_z, L_z) \), denotes the \( r \times r \) identity matrix:

\[
\left\{ -I, x_1y_1, \ldots, x_my_m, -2(y_1y_2, \ldots, y_{m-1}y_m) \right\}
\]

for which the matrices \( P_z = 2L_z, L_z = 0, W_z = 0 \) are readily seen to satisfy eqn. 16.

It is now easily checked that the matrices

\[
P = \begin{bmatrix} 2L_z & 0 \\ 0 & P_z \end{bmatrix}
\]

\[
L = \begin{bmatrix} 0 \\ L_z \end{bmatrix}
\]

\[
W = W_z
\]

satisfy eqn. 16 for the minimal realisation (eqn. 22) of \( G(z) \). We have exhibited the matrices \( P, L \) and \( W \) for this particular realisation of \( G(z) \). However, from the fact that minimal realisations are algebraically equivalent,\(^{12}\) it follows immediately that such matrices exist for all minimal realisations.

**Sufficiency:** It suffices to show that eqns. 16 imply eqn. 12. Now, from eqn. 16a, it is readily verified that

\[
(x^2I - A')P(zI - A) = (x^2I - A')P + A'P(zI - A)
\]

\[
= ([x^2I - A']^2 - 1)P = LL'.
\]

After some manipulation, and use of eqns. 16b and 16c, there results

\[
J = J' + C'(zI - A)^{-1}B + B'(z^2I - A')^{-1}C
\]

\[
= \left( [Iz^2 - 1]B(z^2I - A')^{-1}C
\]

\[
+ \left( W' - B'(z^2I - A')^{-1}C \right) W + L'(zI - A)\right.
\]

The right-hand side is clearly nonnegative definite in \( |z| > 1 \), while the left-hand side is precisely \( G(z) = C^*zC \). This completes the proof of the lemma.

### 4 Stability of discrete-time systems with memoryless feedback

We examine the stability of the null solution (Lyapunov stability) of discrete-time feedback systems described by state-space difference equations of the form

\[
\begin{align*}
    x(n+1) &= Ax(n) + Bu(n) \\
    y(n) &= Cx(n)
\end{align*}
\]

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Here $x$ is a $p$ vector, the state; $y$ is an $m$ vector, the output; and $\Phi$ is an $m \times n$ vector, being a feedback signal produced by a group of memoryless nonlinearities in the feedback path. It will be assumed that $(A, B)$ is completely controllable and $(A, C)$ is completely observable. Therefore, $(A, B, C)$ is a minimal realization of the $z$ transform $W(z)$ of the impulse-response matrix of the linear part of the system:

$$W(z) = C(zI - A)^{-1}B$$

(Note that $W(z)$ is a square matrix and $H(z) = 0$. It will be assumed that the $i$th element of the feedback vector is a function of the $i$th element of the output only, and even though it can be, in general, nonlinear and time-varying, it is restricted to lies always in a sector of the first and three quadrants:

$$\Phi(n, 0) = 0$$

and for $y_i \neq 0$,

$$k_i^{-1}y_i(n) > \phi_i(n), y_i(n), y_i(n) > 0 \quad i = 1, 2, \ldots, m$$

(25b)

where each $k_i$ is a positive constant. Eqs. 25 can readily be implied by

$$\Phi(n, y(n)) > \Phi(n, 0)K\Phi(n, y(n))$$

(26)

for $y \neq 0$, where $K$ is a diagonal positive definite matrix formed from the $k_i$. Therefore, from Lemma 3, there exist matrices $P = P^T > 0$, $L$ and $U$ such that

$$AP - P = -LL'$$

$$APB - C = LU$$

$$U'U = K + K - B'PB$$

(27)

Thus there exists a positive definite function

$$V(x) = x'Px$$

whose change along a solution of eqn. 23 is, from eqns. 23 and 27, given by

$$\Delta V(x) = -[L'x(n) - U\Phi(n, y(n))] [L'x(n) - U\Phi(n, y(n))] + 2[\Phi(n, y(n))x(n) - \Phi(n, y(n))K\Phi(n, y(n))]$$

(28)

The first term in eqn. 28 is clearly nonpositive, while the second term is nonpositive from eqn. 25. Thus $V$ is a positive definite function for all $x$, with a nonnegative change along any trajectory, establishing the stability of the system.13

Corollary 1: If the matrix $G(z) = K + W(z)$ is d.p.r., and if the nonlinearities are time-invariant but otherwise restricted as in eqn. 25, the system (eqn. 23) is asymptotically stable in the large.

Proof: To establish asymptotic stability in the autonomous case, it is sufficient13 to show that $\Delta V(x)$ cannot vanish identically along any solution of eqn. 23. From eqns. 26 and 28, suitably modified to remove the explicit dependence of $\Phi$ on $n$, it is clear that $\Delta V(x)$ vanishes identically only if the output $y$ is identically zero. But this is impossible while the state is nonzero, since the system is assumed to be completely observable.

In the time-varying case, detailed information on the structure of the matrices $B$ and $C$ is, in general, required before asymptotic stability can be shown. However, if it is possible to find a $p$, $0 < p < 1$, such that $G(pz)$ is d.p.r., asymptotic stability holds. We have

Corollary 2: If there exists a number $p$, $0 < p < 1$, such that the matrix $G(pz) = K + p^{-1}C(zI - A)^{-1}B$ is d.p.r., the system (eqns. 23 and 25) is uniformly asymptotically stable in the large.

This follows by applying Lemma 3 to the realization of $G(pz)$; a simple calculation shows that the change of the positive definite form $V(x) = x'Px$ is negative definite along any trajectory of eqn. 23.

5 Conclusions

One application, to stability theory, has been given of an algebraic description of discrete positive real functions.14 It would not be unreasonable to hope for further applications along the lines of Reference 7, and to hope that these applications should make use of the description of discrete positive-real matrices, as distinct from functions, presented in this paper. Just as for continuous-time systems,14 it should be possible to show that, in a nominally linear asymptotically stable system, small amounts of nonlinearity can be tolerated without affecting stability. The proof of this result in the continuous-time case relies on the fact that, given an $n \times n$ matrix $W(z)$ with elements analytic in Re $z > 0$, there exists a diagonal matrix $K > 0$ and a real scalar $\sigma > 0$ such that $K + W(z + \sigma)$ is positive real. The equivalent discrete-time theorem would be: given an $n \times n$ matrix $G(z)$ with elements analytic in $|z| > 1$, there exists a diagonal matrix $K > 0$ and a real scalar $p$, $0 < p < 1$ such that $K + G(pz)$ is positive real.

8 References