On Reciprocity and Time-Variable Networks

Physically, the notion of reciprocity for networks means that an interchange of dual variable source and load leaves the load variable unchanged ([1], pp. 148–151). Viewed in this manner, the assumption of time-invariance on networks seems unnecessary, but—except for a definition ([2], p. 9)—we are unaware of any treatment of the properties of time-variable reciprocal networks. To investigate such situations, which is the purpose of this correspondence, we make this physical notion mathematically precise and valid for n-ports by following McMillan [3], p. 236).

Consider an n-port network N of allowed pairs $[v, i]$ of (time-domain) voltage v and current i n-vectors [2], [4]. Then N is called *reciprocal* if

$$\tilde{v}_1 * i_2 = \tilde{v}_2 * i_1 \tag{1a}$$

for all allowed pairs $[v_1, i_1]$ and $[v_2, i_2]$. The superscript tilde, ~, denotes matrix transposition, while the asterisk, $*$, denotes convolution. In (1a) one can think of the subscripts 1 and 2 as corresponding to a first and second set of measurements to correlate with the physical notion. Convolution is used because time-domain quantities are considered, whereas McMillan worked in the frequency-domain. McMillan also considered a restricted class of networks, while (1a) holds for any arbitrary network. By defining incident, v^i, and reflected, v^r, variables [5] through $v = v^i + v^r$ and $i = v^i - v^r$, (1a) is seen by direct substitution to be completely equivalent to

$$\tilde{v}_1^i * v_2^r = \tilde{v}_2^i * v_1^r. \tag{1b}$$

With (1b) in mind, and to obtain general but concrete results, we assume that N is linear and completely solvable, in which case a scattering matrix $s(t, \tau)$ exists [5], defined by

$$v^r(t) = \int_{-\infty}^{\infty} s(t, \tau) v^i(\tau) d\tau. \tag{2}$$

In (2) and the following, the integral is to be rigorously considered as a distributional mapping ([2], p. 10, and [6], p. 221). Substituting (2) into (1b) yields

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{v}_1^i(\lambda) s(t-\lambda, \tau) v_2^i(\tau) d\lambda d\tau &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{v}_2^i(x) s(t-x, y) v_1^i(y) dx dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{v}_1^i(\lambda) \tilde{s}(t-\tau, \lambda) v_2^i(\tau) d\lambda d\tau \end{aligned} \tag{3}$$

where the final term is obtained by transposing and letting $x = \tau$, $y = \lambda$ in the middle term. As solvability and reciprocity require consideration of all allowed v_1^i and v_2^i, (3) shows

$$s(t-\lambda, \tau) = \tilde{s}(t-\tau, \lambda) \tag{4}$$

for all t, λ, τ. Setting $\lambda = 0$ shows that a linear, completely solvable network N is reciprocal (under the given definition) if, and only if,

$$s(t, \tau) = \tilde{s}(t-\tau, 0). \tag{5}$$

This shows that N is time-invariant and that s is symmetric—and consequently so are the impedance z and admittance y, if they exist ([2], p. 10).

At this point one wonders if the proper definition of reciprocity has been used. However, one intuitively feels that a non-time-invariant network is nonreciprocal. Furthermore, one has the following two striking examples.

Example 1: The time-varying network of Fig. 1 has

$$s(t, \tau) = -\delta(t-\tau) I_2 + \phi(t) \cdot \frac{1}{\int_{\tau}^{\infty} \tilde{\phi}(\lambda)\phi(\lambda) d\lambda} \tilde{\phi}(\tau) u(t-\tau) \tag{6a}$$

with

$$\tilde{\phi}(t) = [\exp(-t), \exp(-2t)] \tag{6b}$$

for which $\tilde{s} \neq s$ (here $\delta = u'$, u = unit step function, ~ = transpose, and $I_2 = 2 \times 2$ identity). The transformer turns-ratios are

$$n_1(t) = e^{-t}(e^{-2t} + \tfrac{1}{2}e^{-4t})^{1/2} \tag{7a}$$

$$n_2(t) = e^{-2t}(e^{-2t} + \tfrac{1}{2}e^{-4t})^{1/2}. \tag{7b}$$

Although the transformer has a symmetric matrix [5], as does the capacitor, the interconnection does not.

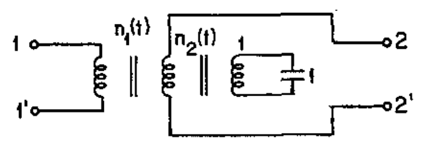


Fig. 1. Two-port with nonsymmetric s [turns-ratios as in (7)].

Example 2: The time-invariant nonreciprocal network of Fig. 2(a) has the equivalent of Fig. 2(b). Both structures have

$$Z(p) = \frac{p\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix} + \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}}{p^2+1} \tag{8a}$$

and

$$z(t, \tau) = \begin{bmatrix}\cos(t-\tau) & \cos(t-\tau) - \sin(t-\tau)\\ \cos(t-\tau) + \sin(t-\tau) & 2\cos(t-\tau)\end{bmatrix} u(t-\tau) \tag{8b}$$

$$= \begin{bmatrix}\cos t & -\sin t\\ \cos t + \sin t & \cos t - \sin t\end{bmatrix}\begin{bmatrix}u(t-\tau) & 0\\ 0 & u(t-\tau)\end{bmatrix}\begin{bmatrix}\cos\tau & \cos\tau + \sin\tau\\ -\sin\tau & \cos\tau - \sin\tau\end{bmatrix}. \tag{8c}$$

Here $Z(p)$ and $z(t, \tau)$ are the frequency- and time-domain impedance matrices; s can be found by taking the inverse Laplace transform of

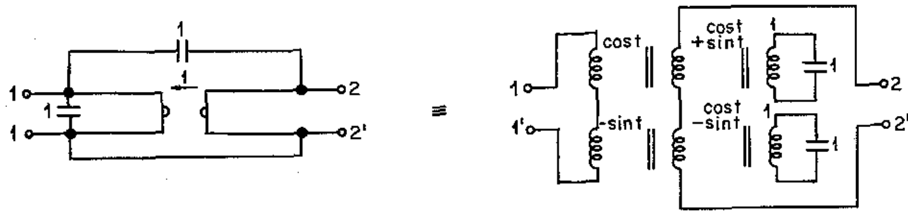
$$S(p) = [Z(p) + 1_2]^{-1}[Z(p) - 1_2]$$

and is seen to be nonsymmetric. Consequently, the interconnection of what may look like reciprocal (time-varying) elements can yield a network which would customarily be called nonreciprocal.

It seems, from these two examples, that the use of a definition in terms of symmetric describing matrices, as may be found in the time-invariant case (reference [7], p. 90), is meaningless. Furthermore, such definitions are to be deprecated on the grounds that they do not permit consideration of such networks as the nullator and norator (reference [2], p. 7). In actual fact, (1a) corresponds to the Lorentz reciprocity for sinusoidal electromagnetic fields in time-invariant media (reference [8], p. 454). It therefore seems that more detailed study in terms of electromagnetic and thermodynamic laws may prove worth while [9]–[11].

In summary, we have shown that what appears as the most reasonable definition of reciprocity, (1a), places symmetry and time-invariance constraints, (5), on the scattering matrix.

B. D. O. Anderson
R. W. Newcomb
Stanford Electronics Labs.
Stanford, Calif



(a) (b)

Fig. 2. Nonreciprocal equivalents.

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