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CONTROLLABILITY, OBSERVABILITY AND STABILITY OF LINEAR SYSTEMS*

L. M. SILVERMAN† AND B. D. O. ANDERSON‡

1. Introduction. Of the many types of stability which may be defined for dynamical systems, at least two are of special importance when the systems are linear. These are bounded-input bounded-output (BIBO) stability [1] and exponential stability [2]. The aim of this paper is to establish an equivalence between these two types of stability for a large class of linear time-variable systems. The basic system description we shall consider is an *impulse response matrix* H which maps the system inputs u into the system outputs y via the formula

$$y(t) = \int_{t_0}^{t} H(t, \tau)u(\tau)\,d\tau, \tag{1}$$

when the system is in the zero state at time t_0. An alternate description is provided by a set of state equations of the form

$$\dot{x} = Ax + Bu, \tag{2a}$$

$$y = Cx, \tag{2b}$$

where A, B and C are time-variable matrices, and x is the state vector associated with the coordinate basis used in setting up (2). The dimensions of the vectors x, u and y will be taken to be n, r and m, respectively. The well-known [3] relationship between the two representations is that $H(t, \tau) = C(t)\Phi(t, \tau)B(\tau)$ for $t \geqq \tau$, where Φ is the transition matrix of the homogeneous part of (2a).

Recall (see [1], [4]–[6]) that a system of the above type is (zero-state) BIBO stable if and only if there exists a positive constant c_1 such that

$$\int_{-\infty}^{t} \| H(t, \tau) \| \, d\tau \leqq c_1 \quad \text{for all} \quad t, \tag{3}$$

where $\| \cdot \|$ denotes the Euclidean norm. It should be noted that this type of stability is independent of the particular *realization* (2) of H. In con-

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† Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, California 94720.

‡ Department of Electrical Engineering, University of Newcastle, Newcastle, New South Wales, Australia.

trast, exponential stability is a characteristic of the internal structure of the system. As is well known [1], the realization (2) is exponentially stable if and only if there exist positive constants c_2 and c_3 such that

$$(4) \qquad \| \Phi(t, \tau) \| \leqq c_2 e^{-c_3(t-\tau)} \quad \text{for all } \tau \text{ and for all } t \geqq \tau.$$

In the time-invariant case (A, B and C constant matrices), relations between the two types of stability are well known. Exponential stability implies BIBO stability, while BIBO stability, together with complete controllability [3] and complete observability [3], implies exponential stability.[1] Hence, in synthesizing a time-invariant impulse response matrix one is assured that all minimal realizations will have appropriate stability properties. Unfortunately, no such simple and analogous statements can be made in the time-variable case. Indeed, as was observed by Kalman [7], who first investigated this problem, it is impossible to conclude the existence of any sort of relation between the two types of stability without further constraints on the realizations (2). The reason for this is that one can construct a realization for H with an essentially arbitrary A matrix. If (2) is to represent a practical physical system (e.g., an analogue computer), then a natural restriction is that the elements of the coefficient matrices be bounded functions. Consequently, we shall assume *that a constant K* exists such that for all t,

$$(5) \qquad \| A(t) \| \leqq K, \quad \| B(t) \| \leqq K, \quad \| C(t) \| \leqq K.$$

A system representation satisfying (5) will be termed a *bounded realization*.

Even with the restriction to bounded realizations, complete controllability and observability do not suffice to insure the equivalence of BIBO and exponential stability. It is shown below, however, that a somewhat more stringent, but physically reasonable, set of constraints does provide a connection between the two types of stability. Several important classes of systems which satisfy these constraints are also derived.

2. Uniform controllability and observability. As defined by Kalman [8], the system representation (2) is *uniformly completely controllable* if, for some $\delta_c > 0$, any two of the following three conditions hold for all s (any two imply the third):[2]

$$(6) \qquad 0 < \alpha_1(\delta_c)I \leqq M(s - \delta_c, s) \leqq \alpha_2(\delta_c)I,$$

$$(7) \quad 0 < \alpha_3(\delta_c)I \leqq \Phi(s - \delta_c, s)M(s - \delta_c, s)\Phi'(s - \delta_c, s) \leqq \alpha_4(\delta_c)I,$$

[1] As pointed out by a reviewer, a proof of this widely used result does not seem to exist in the literature. Such a proof is provided here by Theorem 3 and Theorem 4 specialized to time-invariant systems.

[2] If A and B are symmetric matrices, $A > B$ ($A \geqq B$) means $A - B$ is positive (nonnegative) definite.

$$\|\Phi(t,\tau)\| \leqq \alpha_5(|t-\tau|) \cdot \text{ for all } t, \tau, \tag{8}$$

where

$$M(s-\delta_c, s) = \int_{s-\delta_c}^{s} \Phi(s,t)B(t)B'(t)\Phi'(s,t)\,dt. \tag{9}$$

The above criteria greatly simplify for bounded realizations. Condition (8) is immediately implied by the bound on A, and it is a routine matter to show that the upper bound in (6) is always satisfied. Hence, we have the following lemma.

LEMMA 1. *A bounded system of the form* (2) *is uniformly completely controllable if and only if there exists* $\delta_c > 0$ *such that for all* s,

$$M(s-\delta_c, s) \geqq \alpha_1(\delta_c)I > 0. \tag{10}$$

Uniform complete observability is defined [8] in a dual [3] manner to the above in terms of the matrix

$$W(s-\delta_0, s) = \int_{s-\delta_0}^{s} \Phi'(t, s-\delta_0)C'(t)C(t)\Phi(t, s-\delta_0)\,dt, \tag{11}$$

so that we need not state the definition explicitly here.

3. Equivalence of BIBO and exponential stability. If the system (2a) with x considered as the output is BIBO stable, it will be said that the system is bounded-input bounded-state (BIBS) stable. We shall first establish an equivalence between BIBS and exponential stability. As a preliminary, we prove the following lemma which gives a useful alternate characterization of uniform complete controllability.

LEMMA 2. *A bounded realization* (2a) *is uniformly completely controllable if and only if there exists* $\delta_c > 0$ *such that for every state* $\xi \in R^n$ *and for any time* s, *there exists an input* u *defined on* $(s-\delta_c, s)$ *such that if* $x(s-\delta_c) = 0$, *then* $x(s) = \xi$ *and* $\|u(t)\| \leqq \gamma(\delta_c, \|\xi\|)$ *for all* $t \in (s-\delta_c, s)$.

Proof. If the system (2) is uniformly completely controllable, then the input $u(t) = B'(t)\Phi'(s,t)M^{-1}(s-\delta_c, s)\xi$ will transfer the system from the zero state at time $s-\delta_c$ to the state ξ at time s. From (5), (6) and (8) it is clear that a constant γ independent of s and t exists such that $\|u(t)\| < \gamma$ for all $t \in (s-\delta_c, s)$.

The converse will be established by contradiction. If the system is not uniformly completely controllable, then Lemma 1 implies that, for each $\delta > 0$ and for any $\alpha > 0$, there is a vector $\lambda \in R^n$, with $\|\lambda\| = 1$, such that for some s, $\lambda' M(s-\delta, s)\lambda < \alpha$, or equivalently, for some s,

$$\int_{s-\delta}^{s} \|\lambda'\Phi(s,\tau)B(\tau)\|^2\,d\tau < \alpha. \tag{12}$$

Suppose that a bounded control u exists which transfers the zero state at time $s - \delta$ to the state λ at time s. Then,

$$\lambda = \int_{s-\delta}^{s} \Phi(s, \tau)B(\tau)u(\tau)\, d\tau,$$

which together with the Schwarz inequality implies

$$(13)\qquad \|\lambda\|^2 \leqq \left[\int_{s-\delta}^{s} \|\lambda'\Phi(s, \tau)B(\tau)\|^2\, d\tau\right]^{1/2} \left[\int_{s-\delta}^{s} \|u(\tau)\|^2\, d\tau\right]^{1/2}.$$

If $\| u(t) \| < \gamma(\delta, 1)$ for all $t \in (s - \delta, s)$ and for all s, then (12) and (13) imply that for some s, $\gamma\sqrt{\alpha\delta} \geqq 1$, a contradiction since α can be made arbitrarily small. This completes the proof.

THEOREM 1. *If* (2a) *is bounded and uniformly completely controllable, then it is BIBS stable if and only if it is exponentially stable.*

Proof. It is well known and straightforward to show [1] that if B is bounded, then exponential stability implies BIBS stability.

To prove the converse, let λ be any unit norm vector in R^n. It follows from Lemma 2 that if (2a) is uniformly completely controllable and bounded, then there exists a $\delta_c > 0$ such that, for all s, an input u exists which satisfies

$$(14)\qquad \lambda = \int_{s-\delta_c}^{s} \Phi(s, \tau)B(\tau)u(\tau)\, d\tau,$$

and $\| u(\tau) \| \leqq \gamma_1(\delta_c)$ for $t \in (s - \delta_c, s)$. Multiplying both sides of (14) by $\Phi(t, s)$ and integrating the norm of the result yields the inequality

$$(15)\qquad \int_{t_0}^{t} \|\Phi(t, s)\lambda\|\, ds \leqq \gamma_1 \int_{t_0}^{t} \left\{\int_{s-\delta_c}^{s} \|\Phi(t, \tau)B(\tau)\|\, d\tau\right\} ds.$$

Letting $r = \tau - s + \delta_c$, and interchanging the order of integration on the right-hand side of (15), it is then seen that

$$(16)\qquad \int_{t_0}^{t} \|\Phi(t, s)\lambda\|\, ds \leqq \gamma_1 \int_{0}^{\delta_c} \left\{\int_{t_0+r-\delta_c}^{t+r-\delta_c} \|\Phi(t, \tau)B(\tau)\|\, d\tau\right\} dr,$$

and for $0 \leqq r \leqq \delta_c$ it is clear that

$$(17)\qquad \int_{t_0+r-\delta_c}^{t+r-\delta_c} \|\Phi(t, \tau)B(\tau)\|\, d\tau \leqq \int_{-\infty}^{t} \|\Phi(t, \tau)B(\tau)\|\, d\tau.$$

Since (2a) is assumed BIBS, it follows from (3) that the right-hand side of (17) is bounded by a constant γ_2 so that (15)–(17) imply

$$(18)\qquad \int_{-\infty}^{t} \|\Phi(t, s)\lambda\|\, ds \leqq \gamma_1\gamma_2\delta_c \quad \text{for all} \quad t.$$

Hence, if the supremum of (18) over all $\| \lambda \| = 1$ is taken, the bound

$$\int_{-\infty}^{t} \| \Phi(t, s) \| \, ds \leq \gamma_1 \gamma_2 \delta_0 \quad \text{for all} \quad t \tag{19}$$

is obtained. But (19) together with the bound (5) on A suffices to imply exponential stability [4], [9]. This completes the proof.

To complement the above theorem, we now relate BIBO and BIBS stability.

THEOREM 2. *If (2) is bounded and uniformly completely observable, then it is BIBO stable if and only if it is BIBS stable.*

Proof. Suppose that BIBO stability does not imply BIBS stability, i.e., there exists a bounded input u which produces both a bounded output and an unbounded state. Then, corresponding to an arbitrary positive number N, there is a value of time $s - \delta_0$ for which $\| x(s - \delta_0) \| > N$. Set u equal to zero over the interval $(s - \delta_0, s)$. Then the output y over this interval is given by $y(t) = C(t)\Phi(t, s - \delta_0)x(s - \delta_0)$. Consequently, using the dual of (10),

$$\int_{s-\delta_0}^{s} y'(t)y(t) \, dt = x'(s - \delta_0)W(s - \delta_0, s)x(s - \delta_0) \geq \beta_1(\delta_0)N^2.$$

Hence, at some point t in $(s - \delta_0, s)$, $\| y(t) \| > N\sqrt{\beta_1/\delta_0}$. But since N is arbitrary, while u is bounded this contradicts the assumption of BIBO stability. This completes the proof of the theorem, since it is obvious that BIBS implies BIBO stability if C is bounded.

Following immediately from Theorems 1 and 2 is the main result, as given in the following theorem.

THEOREM 3. *If (2) is bounded, uniformly completely controllable and uniformly completely observable, then it is BIBO stable if and only if it is exponentially stable.*

A valid question at this point is whether the boundedness constraint of Theorem 3 is essential to the conclusion. It is clear that the constraint on the matrix A can be relaxed since (8) holds under somewhat weaker conditions [8] than (5). However, as shown by the following example, the constraints on B and C are essential.

Example. Consider the system realization $\dot{x} = -x + u$, $y = gx$, where $g(t) = k$ for $t \in (k, k + (1/k))$, $k = 1, 2, \cdots$, and is zero elsewhere. It is easily verified that this system is uniformly completely controllable and observable, yet it is simultaneously exponentially stable and BIBO unstable.

4. Periodic systems. Periodic systems (A, B and C periodic with the same period) are an important subclass of linear systems. It is shown below that minimal (completely controllable and observable) periodic systems are uniformly completely controllable and observable. This together with Theorem 3 establishes the apparently known [10] but previously unproven fact that BIBO and exponential stability are equivalent in periodic systems.

THEOREM 4. *If (2) is periodic, then it is uniformly completely controllable (observable) if and only if it is completely controllable (observable).*

Proof. If (2) is completely controllable, there must exist a finite $\sigma > 0$ such that $M(0, \sigma) \geqq \epsilon I > 0$. Let k be a positive integer such that $kT > \sigma$, where T is the period of the matrices A, B and C. Clearly, for $s \in (kT, 2kT)$, $M(s - 2kT) \geqq \epsilon I$. It is easily verified, however, that $M(s - 2kT, s)$ is periodic in s with period T. Hence, $M(s - 2kT, s) \geqq \epsilon I$ for all s. By Theorem 3, therefore, (2) is uniformly completely controllable. Since the converse is obviously true, this completes the proof.

5. Classes of uniformly completely controllable systems. In order to apply the results of the previous sections in stability analysis or system synthesis, it is useful to have criteria for uniform complete controllability which do not require calculation of the transition matrix.[3] Such criteria are derived below, and it is shown that several broad classes of systems have the uniform complete controllability property. A basic tool in this development is the following lemma establishing the invariance of uniform complete controllability under bounded state-variable feedback of the form $u = Gx + Fv$, where v is the input to the closed loop system.

LEMMA 3.[4] *A bounded realization (2) is uniformly completely controllable if and only if the system $(A + BG, BF, C)$ is uniformly completely controllable, where G is any $r \times n$ bounded matrix and F is any $r \times r$ bounded matrix whose inverse is also bounded.*

Proof. Let (2) be uniformly completely controllable. Then by Lemma 2 there are a $\delta > 0$ and an input u_1 which takes $x(s - \delta) = 0$ to $x(s) = \xi$, such that $\| u_1(t) \| \leqq \gamma(\delta, \|\xi\|)$ for all $t \in (s - \delta, s)$ and for all s. It is readily verified that if $v_1(t) = F^{-1}u_1(t) - Gx_1(t)$ is the input to $(A + BG, BF, C)$, where x_1 is the trajectory in (A, B, C) due to u_1, then $z_1(s - \delta) = 0$ and $z_1(s) = \xi$, where z_1 is the trajectory of $(A + BG, BF, C)$ due to v_1 (in fact, $z_1(t) = x_1(t)$ for all $t \in (s - \delta, s)$). Since for all $t \in (s - \delta, s)$,

$$\| v_1(t) \| \leqq \| F^{-1}(t) \| \| u_1(t) \| + \| G(t) \| \int_{s-\delta}^{s} \| \Phi(t, \tau) B(\tau) u_1(\tau) \| \, d\tau,$$

it is easily shown that $\| v_1(t) \| \leqq \gamma_1(\delta, \|\xi\|)$. Hence, by Lemma 2, $(A + BG, BF, C)$ is uniformly completely controllable.

[3] Such criteria are also applicable in other problems [8] which involve uniform complete controllability.

[4] The proof of this lemma is based on an argument of Brockett [11] used in proving the invariance of complete controllability in time-invariant systems under time-invariant state-variable feedback.

The converse follows by a similar argument.

Remark. Lemma 3 can be applied directly to a class of problems studied extensively in recent years—stability analysis of a constant linear system with bounded time-variable feedback from output to input. If G is a bounded $r \times m$ matrix, then such a system has the form $(A + BGC, B, C)$, where (A, B, C) is a time-invariant completely controllable and observable system. It follows immediately from Lemma 3 and its dual that the closed loop system is uniformly completely controllable and observable so that BIBO and exponential stability are equivalent in this class of systems. Consequently, only one of the two types of stability need be examined and several existing results can be strengthened. For example, a recent criterion for Lyapunov instability given by Brockett and Lee [12, Theorem 1] extends to a criterion for BIBO instability.

THEOREM 5. *If A and B are bounded and B contains an $n \times n$ submatrix $\tilde{B}$ whose inverse is also bounded, then* (2a) *is uniformly completely controllable.*

Proof. Without loss of generality, we may take $\tilde{B} = B$. Letting $G = -AB^{-1}$ and $F = B^{-1}$, we obtain the time-invariant closed loop system (O, I, C). The result then follows from Lemma 3.

A corollary to the above is the well-known result of Perron [4], [9] that BIBS and exponential stability are equivalent in systems satisfying the hypothesis of the theorem.

The constraint on B in Theorem 5 is quite restrictive. A much weaker condition under which the result holds will now be presented for single-input systems $(B = b$ in (2)). Let $Q_c = [p_0\, p_1 \cdots p_{n-1}]$, where $p_0 = b$ and $p_{k+1} = -Ap_k + \dot{p}_k$, $k = 1, 2, \cdots$. In terms of this controllability matrix [13] we have the following theorem.

THEOREM 6. *If* (2a) *is a bounded, single-input realization and Q_c is a Lyapunov transformation*[5] [5], *then the system is uniformly completely controllable.*

Proof. Let λ be an arbitrary constant vector, and let $g(s, \tau) = \lambda'\Phi(s, \tau)b(\tau)$. Also, let

$$\tilde{M}(s - \delta, s) = \int_{s-\delta}^{s} \Phi(s, \tau)Q_c(\tau)Q_c'(\tau)\Phi'(s, \tau)\, d\tau.$$

It is easily shown that

$$\frac{\partial^i}{\partial\tau^i} g(s, \tau) = \lambda'\Phi(s, \tau)p_i(\tau),$$

so that

$$\lambda'\tilde{M}(s - \delta, s)\lambda = \sum_{i=0}^{n-1} \int_{s-\delta}^{s} \left[\frac{\partial^i}{\partial\tau^i} g(s, \tau)\right]^2 d\tau. \tag{20}$$

[5] For time-invariant systems, this condition on Q_c is equivalent to complete controllability.

It can also be shown [14] that for all s, each element of $\Phi(s, t)b(t)$, and hence $g(s, t)$, is a solution of the differential equation

$$z^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t) z^{(i)}(t) = 0, \tag{21}$$

where $[a_0\, a_1 \cdots a_{n-1}]' = -Q_c^{-1} p_n$. By virtue of the assumptions on A and Q_c, the coefficients $a_i(t)$ are bounded for all t, so that the following inequality holds [14]:

$$\int_{s-\delta}^{s} \left[\frac{\partial^i}{\partial \tau^i} g(s, \tau)\right]^2 d\tau \leqq K_1 \int_{s-\delta}^{s} g^2(s, \tau)\, d\tau \quad \text{for} \quad 1 \leqq i \leqq n, \tag{22}$$

where K_1 is a constant which depends only on δ. From (20) and (22), therefore, it follows that

$$\lambda' M(s - \delta, s)\lambda \geqq \frac{1}{nK_1} \lambda' \tilde{M}(s - \delta, s)\lambda. \tag{23}$$

Since the system (A, Q_c, C) satisfies the hypothesis of Theorem 5, (A, B, C) must be uniformly completely controllable.

A second class of uniformly completely controllable systems is delineated by the following theorem, the proof of which is an immediate consequence of Lemma 3.

THEOREM 7. *The (phase-variable) canonical form*

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{24}$$

where the coefficients $a_i(t)$ are bounded for all t, is uniformly completely controllable.

Theorem 7 implies that BIBS and exponential stability are equivalent in systems represented in phase-variable canonical form (this result was established previously in [15]). It should be noted that any representation which can be transformed to this form via a Lyapunov transformation also has this property. A general method for calculating a transformation to phase-variable form was given in [16], and it is clear from the form of this transformation that with some additional constraints on the derivatives of the matrices A and b, the classes of systems considered in Theorems 6 and 7 are equivalent. Without such constraints, however, they are distinct.

An interesting corollary to Theorems 6 and 7 is the following for systems represented in the form

(25) $$y^{(n)} + \sum_{i=0}^{n-1} a_i y^{(i)} = u,$$

where the $a_i(t)$ are bounded for all t, $0 \leqq i \leqq n-1$.

COROLLARY. *If the system represented by* (25) *is BIBO stable, then there exist positive constants c_1 and c_2 such that for any solution y of the homogeneous part of* (25),

(26) $$\| \bar{y}(t) \| \leqq c_1 \| \bar{y}(t_0) \| e^{-c_2(t-t_0)}$$

for all $t \geqq t_0$, where $\bar{y} = [y\, y^{(1)} \cdots y^{(n-1)}]$.

Proof. If we let $x_i = y^{(i)}$, $0 \leqq i \leqq n-1$, then (25) has the state representation (24), with $y = [1\, 0 \cdots 0]x$. From Theorem 7, this representation is uniformly completely controllable, and from the dual version of Theorem 6 it is uniformly completely observable. Hence, by Theorem 3, the result (26) follows.

A weaker version of the above corollary was established by Kaplan [17, Chap. 8, Theorem 25]. He showed that (26) holds under the more restrictive condition that the impulse response matrix of (25) is exponentially bounded.

In conclusion, we note that Theorems 4–7 are applicable to the synthesis of impulse response matrices. Under appropriate conditions [18], H can be realized as a member of one of the classes discussed above. Thus the internal stability of the corresponding physical realizations is guaranteed, if H represents a BIBO stable system.

REFERENCES

[1] L. A. ZADEH AND C. A. DESOER, *Linear System Theory*, McGraw-Hill, New York, 1963.

[2] N. P. BHATIA, *On exponential stability of linear differential systems*, this Journal, 2 (1965), pp. 181–191.

[3] R. E. KALMAN, *Mathematical description of linear dynamical systems*, this Journal, 1 (1963), pp. 152–192.

[4] R. E. KALMAN AND J. E. BERTRAM, *Control system analysis and design via the 'second method' of Lyapunov. I. Continuous-time systems*, Trans. ASME Ser. D. J. Basic Engrg., 82 (1960), pp. 371–393.

[5] D. C. YOULA, *On the stability of linear systems*, IEEE Trans. Circuit Theory, CT-10 (1963), pp. 276–279.

[6] C. A. DESOER AND A. J. THOMASIAN, *A note on zero-state stability of linear systems*, Proc. 1st Allerton Conference on Circuit and System Theory, University of Illinois, Urbana, 1963, pp. 50–52.

[7] R. E. KALMAN, *On the stability of time-varying linear systems*, IRE Trans. Circuit Theory, CT-9 (1962), pp. 420–422.

[8] ———, *Contributions to the theory of optimal control*, Bol. Soc. Mat. Mexicana (2), 5 (1960), pp. 102–119.

[9] O. Perron, *Die Stabilitätsfrage der Differentialgleichungen*, Math. Z., 32 (1930), pp. 703–728.

[10] R. W. Brockett, *The status of stability theory for deterministic systems*, IEEE Trans. Automatic Control, AC-11 (1966), pp. 596–606.

[11] ———, *Poles, zeros and feedback; state space interpretation*, Ibid., AC-10 (1965), pp. 129–134.

[12] R. W. Brockett and H. B. Lee, *Frequency-domain instability criteria for time-varying and nonlinear systems*, Proc. IEEE, 55 (1967), pp. 604–618.

[13] L. M. Silverman and H. E. Meadows, *Controllability and observability in time-variable linear systems*, this Journal, 5 (1967), pp. 64–73.

[14] L. H. Haines and L. M. Silverman, *Internal and external stability of linear systems*, J. Math. Anal. Appl., to appear.

[15] B. D. O. Anderson, *Stability properties of linear systems in phase-variable form*, in preparation.

[16] L. M. Silverman, *Transformation of time-variable systems to canonical (phase-variable) form*, IEEE Trans. Automatic Control, AC-11 (1966), pp. 300–303.

[17] W. Kaplan. *Operational Methods for Linear Systems*, Addison-Wesley, Reading, Massachusetts, 1962.

[18] L. M. Silverman, *Stable realization of impulse response matrices*, IEEE International Convention Record, 15 (1967), pp. 32–36.