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# ALGEBRAIC STRUCTURE OF GENERALIZED POSITIVE REAL MATRICES*

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**Abstract.** Square matrices $Z(\cdot)$ of real rational functions of a complex variable are considered with two properties: (1) $Z(\infty)$ has finite elements; (2) $Z(j\omega) + Z'(-j\omega)$ is nonnegative definite Hermitian for all real $\omega$, other than those for which $j\omega$ is a pole of an element of $Z(\cdot)$. Necessary and sufficient conditions for the nonnegativity property are derived which involve the existence of constant matrices satisfying several algebraic equations. The work thereby extends earlier results on the structure of rational positive real matrices.

**1. Introduction.** This paper investigates the structure of a class of matrices occurring in various systems theory problems. We shall work with $n \times n$ matrices $Z(\cdot)$ of real rational functions of a complex variable $s$; the matrices of particular interest are those for which $Z(\infty) < \infty$ (that is, no element of $Z(\cdot)$ has a pole at infinity) and for which $Z(j\omega) + Z'(-j\omega)$ is a nonnegative definite Hermitian matrix for all real $\omega$ with $j\omega$ not a pole of any element of $Z(\cdot)$.

The so-called positive real matrices [1] possess the aforementioned properties, but also possess additional properties restricting the nature of poles of the matrix elements. The structure of such matrices has been investigated from the systems theory point of view [2], [3], and applications of the structure properties have been discussed [4]. There are, however, system theoretic problems involving matrices $Z(\cdot)$ with the finite-at-infinity constraint and the $j\omega$-axis nonnegativity constraint, but without the additional constraints imposed by $Z(\cdot)$ being positive real.

We shall term such matrices *generalized positive real.* Examples of problems involving generalized positive real matrices as distinct from positive real matrices may be found in [5], which discusses system instability, and in [6] and [7], which discuss inverse optimal control problems.

In [5], a single-input, single-output, time-invariant, finite-dimensional system is considered, with a time-varying feedback gain coupling the output to the input. The Nyquist plot of the open-loop system is supposed not to intersect a certain disk in the complex plane; as a consequence of this, a certain scalar function $z(\cdot)$ of a complex variable $s$ is generalized positive real. The number of encirclements of the disk by the Nyquist plot determines whether $z(\cdot)$ is or is not positive real. When it is not positive real, an instability criterion is deduced.

To deal more effectively with systems theory problems involving rational

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generalized positive real matrices, we shall give a structural description parallel to that which is known for positive real matrices [3].

Section 2 presents the main result of the paper (see Theorem 1) while §3 discusses several implications of the results.

**2. Main results.** In this section we consider square matrices $Z(\cdot)$ of real rational functions of a complex variable $s$ subject to the constraint that $Z(\infty) < \infty$. A nonnegativity constraint will also be used extensively:

$$Z(j\omega) + Z'(-j\omega) \geqq 0 \quad \text{for almost all}^1 \text{ real } \omega \tag{1}$$

(the notation $A \geqq B$ [$A > B$] for Hermitian $A$ and $B$ means $A - B$ is nonnegative [positive] definite).

Before stating results for matrices $Z(\cdot)$ satisfying the above constraints, two results on rational positive real matrices will be reviewed.

LEMMA 1. *Suppose $Z(\cdot)$ is an $n \times n$ matrix of real rational functions of a complex variable s, with $Z(\infty) < \infty$. Suppose it is decomposed in the form* [3]

$$Z(s) = J + H'(sI - F)^{-1}G, \tag{2}$$

*where $F$, $G$, $H$, $J$ are real constant matrices. Then $Z(\cdot)$ is positive real if (1) holds and all eigenvalues of $F$ have negative real parts.*

*Proof.* This lemma is but a restatement of the definition of a positive real property.

LEMMA 2. *Let $Z(\cdot)$ be an $n \times n$ matrix of real rational functions of a complex variable s such that $Z(\infty) < \infty$. Let $\{F, G, H, J\}$ be a realization for $Z(\cdot)$, that is, a quadruple for which (2) holds. Suppose further that $[F, G]$ is completely controllable* [8] *and $[F, H]$ completely observable; then a necessary and sufficient condition for $Z(\cdot)$ to be positive real is that there exist real matrices $P = P' > 0$, $L$, $W_0$ such that*

$$PF + F'P = -LL', \tag{3a}$$

$$PG = H - LW_0, \tag{3b}$$

$$W_0'W_0 = J + J'. \tag{3c}$$

This lemma is proved in [3].

The following extended form of Lemma 2, relaxing simultaneously the complete observability requirement and the nonsingularity of $P$ requirement, will be required in the sequel.

LEMMA 3. *With the same hypothesis as in Lemma 2, save for requiring the complete observability of $[F, H]$, $Z(\cdot)$ is positive real if and only if there exist real matrices $P = P' \geqq 0$, $L$, $W_0$ such that* (3a, b, c) *hold.*

[1] We exclude those $\omega$ for which $j\omega$ is a pole of some element of $Z(\cdot)$.

*Proof*. Necessity follows as in [3]. Sufficiency only will be established here. Select a coordinate transformation $T$ (see [8]) such that

$$TFT^{-1} = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix}, \tag{4a}$$

$$H'T^{-1} = [H_1' \quad 0], \tag{4b}$$

$$TG = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \tag{4c}$$

with $[F_{11}, H_1]$ completely observable. Let $\hat{P}$, $\hat{L}$, $W_0$ be such that

$$\hat{P}F_{11} + F_{11}'\hat{P} = -\hat{L}\hat{L}', \tag{5a}$$

$$\hat{P}G_1 = H_1 - \hat{L}W_0, \tag{5b}$$

$$W_0'W_0 = J + J', \tag{5c}$$

with $\hat{P} = \hat{P}' > 0$. Existence of $\hat{P}$, $\hat{L}$, $W_0$ is guaranteed by Lemma 2 and the fact that $\{F_{11}, G_1, H_1, J\}$ is a completely controllable, completely observable realization of $Z(\cdot)$. Then it is straightforward to check that

$$P = T'\begin{bmatrix} \hat{P} & 0 \\ 0 & 0 \end{bmatrix} T, \tag{6a}$$

$$L = T'\begin{bmatrix} \hat{L} \\ 0 \end{bmatrix}, \tag{6b}$$

and $W_0$ satisfy (3a, b, c), where the zero blocks in (6a) augment $\hat{P}$ to be of the same dimension as $F$, and the zero block in (6b) augments the number of rows of $\hat{L}$ to equal the dimension of $F$.

The extension of Lemmas 2 and 3 to generalized positive real matrices, covered in Theorem 1 below, relies on associating with a generalized positive real $Z(s)$ a positive real $Y(s)$, applying Lemmas 2 and 3 to $Y(s)$ to conclude the existence of certain matrices, and then defining a set of matrices to be associated with $Z(s)$ using those associated with $Y(s)$.

THEOREM 1. *Let $Z(\cdot)$ be an $n \times n$ matrix of real rational functions of a complex variable $s$ such that $Z(\infty) < \infty$. Let $\{F, G, H, J\}$ be a realization of $Z(s)$ with $[F, G]$ completely controllable and $[F, H]$ completely observable. Then a necessary and sufficient condition for*

$$Z(j\omega) + Z'(-j\omega) \geqq 0 \tag{1}$$

*to hold for all real $\omega$ for which $j\omega$ is not a pole of any element of $Z(\cdot)$ is that there exist real matrices $P = P'$, $\det P \neq 0$, $L$ and $W_0$ such that*

$$PF + F'P = -LL', \tag{3a}$$

$$PG = H - LW_0, \tag{3b}$$

$$W_0'W_0 = J + J'. \tag{3c}$$

*Moreover, P is not positive definite if $Z(\cdot)$ is not positive real.*

The proof of sufficiency is relatively simple; the proof of necessity harder. We turn to the former first.

*Proof of sufficiency.* Explicit calculation yields

$$
\begin{aligned}
&Z(j\omega) + Z'(-j\omega)\\
&= J + J' + H'(j\omega I - F)^{-1}G + G'(-j\omega I - F')^{-1}H\\
&= J + J' + G'[P(j\omega I - F)^{-1} + (-j\omega I - F')^{-1}P]G\\
&\quad + W_0'L'(j\omega I - F)^{-1}G + G'(-j\omega I - F')^{-1}LW_0 \qquad \text{(using (3b))}\\
&= J + J' + G'(-j\omega I - F')^{-1}[-PF - F'P](j\omega I - F)^{-1}G\\
&\quad + W_0'L'(j\omega I - F)^{-1}G + G'(-j\omega I - F')^{-1}LW_0\\
&= W_0'W_0 + G'(-j\omega I - F')^{-1}LL'(j\omega I - F)^{-1}G\\
&\quad + W_0'L'(j\omega I - F)^{-1}G + G'(-j\omega I - F')^{-1}LW_0\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(using (3a), (3c))}\\
&= [W_0' + G'(-j\omega I - F')^{-1}L][W_0 + L'(j\omega I - F)^{-1}G].
\end{aligned} \tag{7}
$$

Because of the form of the right-hand side, (1) is established.

*Proof of necessity.* We begin by defining the matrix

$$S(s) = [Z(s) - I][Z(s) + I]^{-1}. \tag{8}$$

It is not hard to verify that if

$$S(s) = J_S + H_S'(sI - F_S)^{-1}G_S, \tag{9}$$

the matrices $F$, $G$, $H$, $J$ and $F_S$, $G_S$, $H_S$, $J_S$ are related through the invertible equations

$$F = F_S + G_S(I - J_S)^{-1}H_S', \tag{10a}$$

$$G = 2G_S(I - J_S)^{-1}, \tag{10b}$$

$$H' = (I - J_S)^{-1}H_S', \tag{10c}$$

$$J = -I + 2(I - J_S)^{-1}. \tag{10d}$$

*Note.* Because (1) holds and because $Z(\infty) < \infty$, the matrix $I + J$ or $I + Z(\infty)$ is nonsingular. This means that $J_S$ is well-defined as $I - 2[I + J]^{-1}$, and precisely because of the way $J_S$ is defined, $I - J_S = 2[I + J]^{-1}$ is nonsingular. These two facts guarantee that all quantities in (9) are well-defined or, equivalently, that (10a, b, c, d) are invertible.

The following sequence of implications should be noted:

$[F, G]$ is completely controllable implies $[F, \frac{1}{2}G(I - J_S)]$ or $[F, G_S]$ is completely controllable.

implies $[F - G_S K_S', G_S]$ is completely controllable for any $K_S$ of appropriate dimension.
implies $[F_S, G_S]$ is completely controllable, taking $K_S' = (I - J_S)^{-1} H_S'$.

It is also not hard to verify the following formula:

$$Z(s) + Z'(-s) = \tfrac{1}{2}[I + Z'(-s)][I - S'(-s)S(s)][I + Z(s)]. \tag{11}$$

Equations (1) and (11) together imply that

$$I - S'(-j\omega)S(j\omega) \geqq 0 \tag{12}$$

for all real $\omega$.

Now let a matrix $K$ be chosen so that the eigenvalues of the matrix $F_S - G_S K'$ all possess negative real part. Such a $K$ always exists when $[F_S, G_S]$ is completely controllable (see [9]). (Note that it may be possible to choose $K = 0$.)

Define

$$Q(s) = S(s)R(s), \tag{13}$$

where

$$R(s) = I - K'(sI - F_S + G_S K')^{-1} G_S. \tag{14}$$

Then simple manipulation yields

$$Q(s) = J_S + (H_S' - J_S K')(sI - F_S + G_S K')^{-1} G_S. \tag{15}$$

Equations (12) and (13) also imply

$$R'(-j\omega)R(j\omega) - Q'(-j\omega)Q(j\omega) \geqq 0 \quad \text{for all real} \quad \omega, \tag{16}$$

which, in full, is

$$\begin{aligned} &(I - J_S' J_S) - [K' + J_S'(H_S' - J_S K')](j\omega I - F_S + G_S K')^{-1} G_S \\ &- G_S'(-j\omega I - F_S' + K G_S')^{-1}[K + (H_S - K J_S')J_S] \\ &+ G_S'(-j\omega I - F_S' + K G_S')^{-1}[KK' - (H_S - K J_S')(H_S' - J_S K')] \\ &\quad \cdot (j\omega I - F_S + G_S K')^{-1} G_S \geqq 0 \quad \text{for all real} \quad \omega. \end{aligned} \tag{17}$$

Define now the matrix $P_Q$ as the unique symmetric solution of

$$\begin{aligned} &P_Q(F_S - G_S K') + (F_S' - K G_S')P_Q \\ &\qquad = KK' - (H_S - K J_S')(H_S' - J_S K'). \end{aligned} \tag{18}$$

The eigenvalue restriction on $F_S - G_S K'$ guarantees the existence of a unique and symmetric $P_Q$ satisfying (18) (see [10]). Then (17) becomes, using manipulations like those used in deducing (7),

$$Y(j\omega) + Y'(-j\omega) \geqq 0 \quad \text{for all real} \quad \omega, \tag{19}$$

where, if $Y(s) = J_Y + H_Y'(sI - F_Y)^{-1}G_Y$,

$$G_Y = G_S, \tag{20a}$$

$$F_Y = F_S - G_S K', \tag{20b}$$

$$H_Y = -K - H_S J_S + K J_S' J_S - P_Q G_S, \tag{20c}$$

$$J_Y = \tfrac{1}{2}(I - J_S' J_S). \tag{20d}$$

Because $F_S - G_S K' = F_Y$ has all eigenvalues with negative real parts, (19) implies by Lemma 1 that $Y(\cdot)$ is positive real. Lemma 3 may therefore be applied to yield the existence of matrices $P_Y = P_Y' \geqq 0$, $L_Y$ and $W_{0Y}$ for which

$$P_Y F_Y + F_Y' P_Y = -L_Y L_Y', \tag{21a}$$

$$P_Y G_Y = H_Y - L_Y W_{0Y}, \tag{21b}$$

$$W_{0Y}' W_{0Y} = J_Y + J_Y'. \tag{21c}$$

For convenience, these may be rewritten, using (20a, b, c, d), as

$$P_Y F_S + F_S' P_Y = -L_Y L_Y' + P_Y G_S K' + K G_S' P_Y, \tag{22a}$$

$$P_Y G_S = -K - H_S J_S + K J_S' J_S - P_Q G_S - L_Y W_{0Y}, \tag{22b}$$

$$W_{0Y}' W_{0Y} = I - J_S' J_S. \tag{22c}$$

Recapitulating, we have passed from $Z(s)$ to $S(s)$, thence to $R(s)$ and $Q(s)$, and finally to $Y(s)$. The quantities of further interest are, in order of their definition, $F$, $G$, $H$ and $J$, then $F_S$, $G_S$, $H_S$, $J_S$ (related to $F$, $G$, $H$ and $J$ via (10a, b, c, d), $K$ and $P_Q$ (here (18) is relevant), and finally $P_Y$, $L_Y$ and $W_{0Y}$ (see (22a, b, c)).

We now claim that matrices $P$, $L$ and $W_0$ satisfying (3a, b, c) are given by

$$P = \tfrac{1}{2}(P_Q + P_Y), \tag{23a}$$

$$L = (1/\sqrt{2})[L_Y + K W_{0Y}' + H_S(I - J_S')^{-1} W_{0Y}'], \tag{23b}$$

$$W_0 = \sqrt{2} W_{0Y}(I - J_S)^{-1}. \tag{23c}$$

Equation (3c) is easy to prove using (10d), (22c) and (23c); to prove (3a) and (3b) requires some manipulation, an outline of which will now be given.

Using (10a, b, c, d) and (23a, b, c), we have

$$PG - H + LW_0 = (P_Q + P_Y)G_S(I - J_S)^{-1} - H_S(I - J_S')^{-1} + L_Y W_{0Y}(I - J_S)^{-1} + KW_{0Y}'W_{0Y}(I - J_S)^{-1} + H_S(I - J_S)^{-1}W_{0Y}'W_{0Y}(I - J_S)^{-1}. \tag{24}$$

Now if (22b) and (22c) are used to substitute for $P_Y G_S$ and $W_{0Y}'W_{0Y}$, and all possible cancellations made, the right-hand side of (24) becomes zero. This proves (3b). At the same time, (10a, b, c, d) and (23a, b, c) give

$$\begin{aligned} PF + F'P + LL' &= \tfrac{1}{2}(P_Q + P_Y)[F_S + G_S(I - J_S)^{-1}H_S'] \\ &\quad + \tfrac{1}{2}[F_S + H_S(I - J_S')^{-1}G_S'](P_Q + P_Y) \\ &\quad + \tfrac{1}{2}[L_Y + KW_{0Y}' + H_S(I - J_S')^{-1}W_{0Y}][L_Y' + W_{0Y}K' \\ &\qquad + W_{0Y}'(I - J_S)^{-1}H_S']. \end{aligned} \tag{25}$$

The first and second terms on the right side of (25) may be manipulated to yield

$$\begin{aligned} &2(PF + F'P + LL') \\ &= P_Q(F_S - G_SK') + (F_S' - KG_S')P_Q + P_QG_SK' + KG_S'P_Q \\ &\quad + P_QG_S(I - J_S)^{-1}H_S' + H_S(I - J_S')^{-1}G_S'P_Q \\ &\quad + P_YF_S + F_S'P_Y + P_YG_S(I - J_S)^{-1}H_S' + H_S(I - J_S')^{-1}G_S'P_Y \\ &\quad + [L_Y + KW_{0Y}' + H_S(I - J_S')^{-1}W_{0Y}][L_Y' + W_{0Y}K' \\ &\qquad + W_{0Y}'(I - J_S)^{-1}H_S']. \end{aligned} \tag{26}$$

Equation (18) eliminates $P_Q$ from the first two terms. Equations (22a) and (22b) eliminate $P_Q$ and $P_Y$ from the next eight terms. What is left is then an expression involving $K$, $H_S$, $J_S$, $G_S$, $L_Y$ and $W_{0Y}$. Using (22c) causes the right-hand side then to equal zero.

Next, the nonsingularity of $P$ will be demonstrated. The symmetry of $P$ follows from (23a) and the symmetry of $P_Q$ and $P_Y$. Suppose $P$ is singular, so that there exists a nonsingular $T$ for which

$$\tilde{P} = T'PT = \begin{bmatrix} I_r & 0 & 0 \\ 0 & -I_s & 0 \\ 0 & 0 & 0_t \end{bmatrix}, \tag{27}$$

where $I_r$ is the $r \times r$ unit matrix and $0_t$ a zero $t \times t$ matrix ($t > 0$). Set $\tilde{F} = T^{-1}FT$, $\tilde{G} = T^{-1}G$, $\tilde{H} = H'T$, $\tilde{L}' = L'T$. Then (3a) and (3b) become

$$\tilde{P}\tilde{F} + \tilde{F}'\tilde{P} = -\tilde{L}\tilde{L}', \tag{28a}$$

$$\tilde{P}\tilde{G} = \tilde{H} - \tilde{L}W_0. \tag{28b}$$

Partition $\tilde{F}$, $\tilde{G}$, $\tilde{H}$ and $\tilde{L}$ as

$$\tilde{F} = \begin{bmatrix} F_{rr} & F_{rs} & F_{rt} \\ F_{sr} & F_{ss} & F_{st} \\ F_{tr} & F_{ts} & F_{tt} \end{bmatrix}, \tag{29a}$$

$$\tilde{G} = \begin{bmatrix} G_r \\ G_s \\ G_t \end{bmatrix}, \tag{29b}$$

$$H = \begin{bmatrix} H_r \\ H_s \\ H_t \end{bmatrix}, \tag{29c}$$

$$L = \begin{bmatrix} L_r \\ L_s \\ L_t \end{bmatrix}. \tag{29d}$$

Then it is easily checked that (27) and (28a) force $F_{rt}$, $F_{st}$, $F_{tr}$, $F_{ts}$ and $L_t$ to equal zero. Equation (28b) then forces $H_t$ to equal zero, which is incompatible with the complete observability of the pair

$$\begin{bmatrix} F_{rr} & F_{rs} & 0 \\ F_{st} & F_{ss} & 0 \\ 0 & 0 & F_{tt} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} H_r \\ H_s \\ H_t \end{bmatrix}.$$

Finally, note that if $P$ is positive definite, this fact, together with (3a, b, c), implies the positive real nature of $Z(\cdot)$ (see [3]). Thus if $Z(\cdot)$ is not positive real, $P$ is not positive definite.

Just as Lemma 3 extends the result of Lemma 2, so the following extension of Theorem 1 is possible.

COROLLARY. *With the same hypothesis as Theorem 1, save for the requirement that $[F, H]$ be completely observable, a necessary and sufficient condition for*

$$Z(j\omega) + Z'(-j\omega) \geqq 0 \tag{1}$$

*for all real $\omega$, where $j\omega$ is not a pole of any element of $Z(\cdot)$, is that there exist real matrices $P = P'$, $L$, $W_0$ such that* (3a, b, c) *hold. Moreover, $Z(\cdot)$ is positive real if and only if $P$ is nonnegative or positive definite.*

**3. Concluding remarks.** It is possible to give a simple frequency domain interpretation of the basic equations (3a, b, c). Defining

$$W(s) = W_0 + L'(sI - F)^{-1}G, \tag{30}$$

we have, using arguments appearing in the proof of sufficiency for Theorem

1, that

$$Z(j\omega) + Z'(-j\omega) = W'(-j\omega)W(j\omega). \tag{31}$$

The determination for a prescribed $Z(\cdot)$ of a $W(\cdot)$ satisfying (31) is termed spectral factorization. As is discussed in, for example, [11], for a prescribed $Z(\cdot)$, there are many possible $W(\cdot)$ satisfying (31); here we have elected to find a spectral factor $W(\cdot)$ which not only has the same poles as $Z(\cdot)$, but which can have two matrices of a realizing quadruple identical with those of $Z(\cdot)$.

It is of interest to observe how $P$, $L$ and $W_0$ in (3a, b, c) may be calculated, given $F$, $G$, $H$ and $J$. Section 2 shows how the determination of $P$ for a generalized positive real $Z(\cdot)$ can be made to depend on the determination of $P$ for a positive real $Z(\cdot)$, which is discussed in [12] and [13]; the former reference shows how to determine $P$ by solving a quadratic matrix equation, while the latter determines $P$ as the limiting solution of a matrix Riccati differential equation.

When $P$ in (3a, b, c) has been found, the determination of $L$ and $W_0$ proves straightforward.

For stability and instability studies, the positive definiteness or lack of positive definiteness of the $P$ matrix becomes important, since Lyapunov functions for systems with which a generalized positive real matrix is associated may well have a term $x'Px$ appearing in them. For examples, [5] and [14] can be consulted.

In inverse optimal control problems (see [6] and [7]) typically an equation such as (31) has to be solved, with the constraint that with $Z(s)$ of the form $J + H'(sI - F)^{-1}G$, then $W(s)$ should have the form $W_0 + L'(sI - F)^{-1}G$; usually $L$ has to be found, and the preceding two sections exhibit procedures for this.

## REFERENCES


[1] R. W. NEWCOMB, *Linear Multiport Synthesis*, McGraw-Hill, New York, 1966.

[2] R. E. KALMAN, *On a new characterization of linear passive systems*, Proc. First Allerton Conference on Circuit and System Theory, University of Illinois, Urbana, 1963, pp. 456–470.

[3] B. D. O. ANDERSON, *A system theory criterion for positive real matrices*, this Journal, 5 (1967), pp. 171–182.

[4] ———, *Development and applications of a system theory criterion for rational positive real matrices*, Proc. Fourth Allerton Conference on Circuit and System Theory, University of Illinois, Urbana, 1966, pp. 400–407.

[5] R. W. BROCKETT AND H. B. LEE, *Frequency domain instability criteria for time-varying and nonlinear systems*, Proc. IEEE, 55 (1967), pp. 604–619.

[6] R. E. KALMAN, *When is a linear control system optimal?* Trans. ASME Ser. D. J. Basic Engrg., 86 (1964), pp. 1–10.

[7] B. D. O. ANDERSON, *The inverse problem of optimal control*, Rep. SEL-66-038 (TR no. 6560-3), Stanford Electronics Laboratories, Stanford, California, 1966.

[8] R. E. Kalman, *Mathematical description of linear dynamical systems*, this Journal, 1 (1963), pp. 152–192.

[9] B. D. O. Anderson and D. G. Luenberger, *Design of multivariable control systems*, Proc. Inst. Elec. Engrs., 114 (1967), pp. 395–399.

[10] F. R. Gantmacher, *The Theory of Matrices*, Chelsea, New York, 1969.

[11] D. C. Youla, *On the factorization of rational matrices*, IRE Trans. Information Theory, IT-7 (1961), pp. 172–189.

[12] B. D. O. Anderson, *An algebraic solution to the spectral factorization problem*, Trans. IEEE Automatic Control, AC-12 (1967), pp. 410–414.

[13] ———, *Quadratic minimization, positive real matrices and spectral factorization*, submitted for publication.

[14] ———, *Stability of control systems with multiple nonlinearities*, J. Franklin Inst., 281 (1966), pp. 155–160.