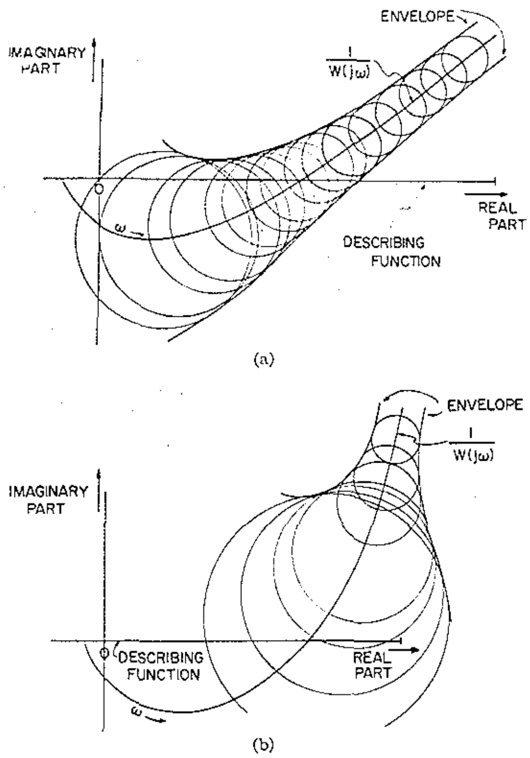

Fig. 3. Sketch illustrating the lack (a) of a left-hand intersection of $R(A, 0)$ and the envelope, and (b) of both intersections.

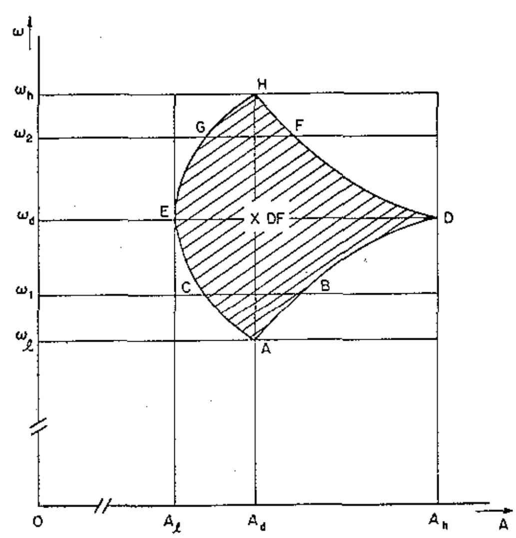

Fig. 4. The $A-\omega$ diagram corresponding to the system of Fig. 2.

Conclusion

The G-R method may be criticized on the following bases.

1) A strict interpretation must include the "doubling-back" phenomenon which apparently is of common occurrence.

2) One or more bounds may not exist because of the phenomena of Fig. 3.

3) The amplitude bounds when they do exist are typically 20 to 40 percent higher than the actual measured values; the frequency bounds are tighter being typically 3 to 7 percent high [4].

4) Attempts to improve significantly the efficacy of the method itself failed to do so, the cause for the failure being the dominance of A_h in the error bounds and the fact that the bound (7) includes, in itself, two other estimates, (8) and (9).

Further details on these assessments may be found in [4].

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Convergence Properties of Riccati Equation Solutions

Abstract—Existence results are developed for Riccati equations. In particular, it is shown that the existence of one solution to a Riccati equation implies the existence of a whole family of solutions whose initial condition lies in a cone determined by the initial condition associated with the known solution.

In a number of areas of system theory, Riccati equations of the following type appear:

$$\dot{P} = PAP + PB + B'P + C \quad \text{(1a)}$$

$$P(t_0) = P_0 \quad \text{(1b)}$$

where $A(t)$, $B(t)$, and $C(t)$ are known continuous matrices. Frequently, the matrices $A(t)$, $C(t)$, and P_0 are symmetric, and either positive definite, non-negative definite, nonpositive definite, or negative definite.

The Riccati equation may be solved backwards or forwards from t_0. Of special interest is the question of whether solutions exist globally, i.e., on the interval $[t_0, \infty)$ or $(-\infty, t_0]$, rather than in some neighborhood of t_0.

In this correspondence, we show how existence of a solution to (1a) with initial condition (1b) may often be used to conclude existence of solutions with the initial condition P_0 replaced by $\overline{P}_0$. More precisely, we have the following theorem.

Theorem 1: Consider equation (1a) with symmetric A, C, and P_0, and suppose a solution $P(\cdot)$ exists on the interval $[t_0, t_1)$ where $t_1 > t_0$ and t_1 may be $+\infty$. If $A(t)$ is non-negative (nonpositive) definite for all t, a solution $\overline{P}(\cdot)$ of (1a) exists on the interval $[t_0, t_1)$ with initial condition $\overline{P}(t_0) = \overline{P}_0$ for all $\overline{P}_0$ such that $P_0 - \overline{P}_0$ is non-negative (nonpositive) definite. Moreover, $P(t) - \overline{P}(t)$ is non-negative (nonpositive) definite for all t.

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The proof of this uses a result appearing elsewhere.[1,2] It is shown that

$$P(t) - \overline{P}(t) = RQ^{-1}R' \tag{2}$$

where

$$\dot{R} = (B' + PA)R \tag{3}$$

and

$$\dot{Q} = R'AR \tag{4}$$

and the initial conditions for (3) and (4) are chosen so that

$$R(t_0)Q^{-1}(t_0)R'(t_0) = P_0 - \overline{P}_0. \tag{5}$$

Consider first the case where $A(t)$ is nonnegative definite. Choose $Q(t_0) = I$ and $R(t_0)$ as any matrix satisfying $R(t_0)R'(t_0) = P_0 - \overline{P}_0$. Evidently $R(t)$ exists for all $t \epsilon [t_0, \infty)$, and thus $\dot{Q}$ is well defined for all t and is a non-negative definite matrix. Hence, $Q(t) - Q(t_0)$ is non-negative definite, and thus $Q^{-1}(t)$ exists for all t and is positive definite. Then $\overline{P}(t)$ in (2) is well defined, with $P(t) - \overline{P}(t)$ non-negative definite.

If $A(t)$ is nonpositive definite for all t, choose $Q(t_0) = -I$ and $R(t_0)$ as any matrix satisfying $R(t_0)R'(t_0) = \overline{P}_0 - P_0$. Then it is straightforward to show that $Q^{-1}(t)$ exists for all t and is negative definite. Then $\overline{P}(t)$ is well defined, and $P(t) - \overline{P}(t)$ is nonpositive definite. This proves the result.

Note that if $P(t)$ exists everywhere, the only way in which $\overline{P}(t)$ can fail to exist is through $Q(t)$ being singular. The constraints on $P_0 - \overline{P}_0$ and on A serve to prevent this possibility.

It is also interesting to observe from (2), (3), and (4) that the stability or otherwise of the difference between two solutions to (1a) is independent of C, and insofar as our sufficiency conditions are concerned, is also independent of B. Theorem 1 constrains only the initial conditions and A.

The following result may be established similarly to Theorem 1.

Theorem 2: Consider equation (1a) with symmetric A, C, and P_0, and suppose a solution $P(\cdot)$ exists on the interval $(t_{-1}, t_0]$ where $t_{-1} < t_0$ and t_{-1} may be $-\infty$. If $A(t)$ is nonnegative (nonpositive) definite, a solution $\overline{P}(\cdot)$ of (1a) exists on the interval $(t_{-1}, t_0]$ with initial condition $\overline{P}(t_0) = \overline{P}_0$ for all $\overline{P}_0$ such that $P_0 - \overline{P}_0$ is nonpositive (non-negative) definite. Moreover, $P(t) - \overline{P}(t)$ is nonpositive (non-negative) definite for all t.

Application of these results is being made to problems of simulating prescribed nonstationary covariances and of synthesizing passive time-variable networks.

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On Time Optimal Trajectories

Abstract—A time optimal steering policy can be computed for a given rocket vehicle that defines a trajectory from a given initial state to the desired final state by using the minimum principle and solving the associated two-point boundary value problem. In this correspondence, it is shown that application of the minimum principle can yield both a locally minimum solution and a locally maximum solution. Numerical evidence has been obtained to substantiate the findings.

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Consider the application of the minimum principle to determine the optimal steering policy of a point mass in a central force field.

The equations of motion are

$$\dot{r} = v \tag{1}$$

$$\dot{v} = -\frac{\mu r}{r^3} + a(t)\frac{u}{u} \tag{2}$$

where r and v are the position and velocity vectors in an inertial coordinate frame with the center at the origin of the attracting mass, μ is the gravitational constant, $a(t)$ is the time history of thrust acceleration magnitude, and u/u is the unit vector in the thrust direction.

The problem is to find the optimal value of the control variable $u(t)$ such that it transfers the point mass from the initial state (r_0, u_0, t_0) to the desired target set in minimum time.

The cost functional to be minimized is

$$J(u) = \int_{t_0}^{T} dt \tag{3}$$

where t_0 is the known initial time and the final time T is free. The Hamiltonian function H for the system is given by

$$H(r(t), v(t), q(t), s(t), u(t)) = 1 + \langle q(t), \dot{r}(t)\rangle + \langle s(t), \dot{v}(t)\rangle \tag{4}$$

where $s(t)$ and $q(t)$ are three-dimensional costate vectors.

In order that $u^*(t)$ be optimal, it is necessary that there exist a $q^*(t)$ and $s^*(t)$ such that

1) $s^*(t)$ and $q^*(t)$ correspond to $u^*(t)$, $r^*(t)$ and $v^*(t)$, so that $s^*(t)$, $q^*(t)$, $r^*(t)$, and $v^*(t)$ are a solution of the canonical system

$$\dot{r}^*(t) = \left[\frac{\partial H}{\partial s}(r^*(t), v^*(t), s^*(t), q^*(t), u^*(t))\right] \tag{5}$$

$$\dot{v}^*(t) = \left[\frac{\partial H}{\partial q}(r^*(t), v^*(t), s^*(t), q^*(t), u^*(t))\right] \tag{6}$$

$$\dot{q}^*(t) = -\left[\frac{\partial H}{\partial r}(r^*(t), v^*(t), s^*(t), q^*(t), u^*(t))\right] \tag{7}$$

$$\dot{s}^*(t) = -\left[\frac{\partial H}{\partial v}(r^*(t), v^*(t), s^*(t), q^*(t), u^*(t))\right]; \tag{8}$$

2) $H(r^*(t), v^*(t), s^*(t), q^*(t), u^*(t)) \leq H(r^*(t), v^*(t), s^*(t), q^*(t), u)$ for all u;

3) equations (5), (6), (7), and (8) satisfy all the boundary conditions, i.e., given initial state, final constraints, and the transversality conditions.

The Hamiltonian for the problem is

$$H(r, v, q, s, u) = \left[1 + q^T v + s^T\left(-\frac{\mu r}{r^3} + a(t)\frac{u}{u}\right)\right]. \tag{9}$$

The equations for the costates are

$$\dot{s} = -q \tag{10}$$

$$\dot{q} = r(-3\mu r^{-5} r^T s) + s(\mu r^{-3}). \tag{11}$$

Minimizing H with respect to $u(t)$ requires that

$$u(t) = -K(t)s(t) \tag{12}$$

where $K(t)$ is some positive-valued function. Since the magnitude of $u(t)$ is arbitrary, let $K(t) = 1$. Substituting (12) into (10) and (11) results in

$$\ddot{u} = -\frac{\mu u}{r^3} + \frac{3\mu(r^T u)r}{r^5}. \tag{13}$$

The problem, therefore, reduces simply to solving (1) and (2) together with (13), and satisfying appropriate boundary conditions. The preceding formulation has appeared in the literature before and most recently it was reported by Brown and Johnson.[1]

It is well known that the necessary conditions furnished by the minimum principle are local in nature, i.e., the set of controls which satisfy all the necessary conditions will consist of both locally time optimal controls and globally time optimal controls.

However, it has been observed that, if instead of minimizing the Hamiltonian, the Hamiltonian is maximized, the resulting expression of u will be the same as (13). That is, setting $u(t) = +s(t)$ and substituting $u(t)$ for $s(t)$ in (10) and (11) again yields (13).

The solution of the boundary value problem described by (1), (2), and (13), therefore, can be a locally minimum, a globally minimum, or a locally maximum time solution. The globally maximum time solution is meaningless.

A rocket flight to a polar orbit was considered. Thrust mass data and initial and terminal conditions were obtained from the

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