Design of Feedback Laws for Linear Control Systems

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Summary

This paper surveys techniques for choosing linear feedback laws for linear, time-invariant, continuous systems. The techniques discussed are for the purpose of achieving one or more of the following aims : reduction in the system sensitivity to internal parameter variations ; optimal design of the system ; control of the transient response of the system and minimisation of noise effects. Feedback laws are defined as linear transformations on the system state variables but may be implemented as a transformation on the system state estimate, derivable from the input and output of the system.

1. Introduction†

The classical approach to control system design has revolved around a consideration of linear single-input, single-output, time-invariant systems and the design of such systems has, in general, proceeded via graphical or ad hoc analytical techniques. Because of the nature of these design techniques, it has been almost impossible to formulate general design methods for systems that are non-linear or multi-loop or time-varying. Also, it has usually been impossible to formulate general design methods that achieve more than one design objective ; thus systems which are designed to perform well in the presence of external noise may perform badly in the presence of plant parameter variations.

This paper is intended to survey some of the most recent developments of modern control theory applicable to linear, time-invariant, continuous systems. Since the paper is only a survey, the reader must refer to the full treatments available in the literature, some of which are referenced in the paper, to obtain a true understanding of the use and power of these modern methods.

The plan of the paper is as follows. In section 2 we review some of the design objectives of the classical theory ; these objectives are the subject of later remarks in the paper. Sections 3 and 4 formulate, in mathematical terms, the systems considered, while sections 5 to 8 discuss in turn each of the design objectives listed in section 2. For completeness, state-variable estimators are briefly mentioned in section 9, while in section 10 future research directions are proposed.

2. Aims of System Control

It appears that two objectives of control can be specified. The first is to use the inherent properties of a system in such a way that we, as the users, like the end results. Unfortunately, in order to do this successfully, we are often faced with having to achieve the second objective, namely, the elimination of the effects of some essentially extraneous influences which cannot properly be associated with inherent system properties. An example of such an effect is externally introduced noise. A second example, and probably the one recognised for the longest time, is that of the slow variation of internal plant parameters, due to ageing or some other process.

Translating the somewhat general remarks above into more precise statements, we list the following four aims of control system design (without any claim that the list is exhaustive, of course).

(a) *Sensitivity Problem* : the reduction in the sensitivity of the input/output performance of the system to internal parameter variations.

(b) *Optimal Control Problem* : the optimisation of the system, that is, the minimisation of some performance index that measures the performance of the system with respect to some criterion chosen by the user. Typically classical treatments are restricted to minimising such quantities as an integral squared error ; modern control theory allows a far wider selection, for instance, fuel consumption, energy consumption or simply time.

(c) *Pole Positioning Problem* : the achieving of arbitrary dynamics of the system, that is, the control of the transient response of the system. Since the transient response is in essence determined by the pole positions of the system transfer function matrix, the above term is used.

(d) *Stochastic Control Problem* : the minimisation of the problems associated with noise, assumed to be introduced additively in two places : at the system input and at the system output.

An additional aim of the classical theory is to achieve stability (that is, non-oscillation) of the system or,

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†Since the date of original submission of this paper, a considerable time has elapsed. The indulgence of the readers is requested in presenting ideas which may now seem naive or outdated, and in omitting discussion of many pertinent references which have appeared more recently.

equivalently, to ensure that the poles lie in the left half plane (numerous plants are unstable without feedback; other plants may be such that the application of feedback to achieve some desired property, other than stability, renders the plant unstable). The aim of stability has been disguised in a variety of ways through such descriptive terms as gain margin, phase margin and so on.

One of the pleasing features of modern control theory is that in general the stability problem can be made to *take care of itself and the designer can turn his attention* to other design aims. Because of this fact, stability is not included in the list of design objectives but is to be understood as a basic requirement for any design.

While the four problems listed have been considered from a classical viewpoint, in some cases satisfactory solutions have been obtained only with modern control theory methods. The following sections briefly discuss these methods which have to do with the selection of appropriate feedback laws.

3. Brief Review of the State-Variable Approach

Until about five years ago, it was customary to relate the input and output of a linear system by a "transfer function" or "transfer function matrix". Thus if U(s) denotes the Laplace transform of the system input p-vector, u(t), and Y(s) the Laplace transform of the system output m-vector, y(t), then:

$$Y(s) = W(s)U(s) \tag{1}$$

where W(s) is the $m \times p$ transfer function matrix of the system.

An alternative description is obtained by introducing the notion of the state vector[1, 2] which at any one instant of time is a collection of numbers that sums up the entire past history of the system in so far as the future behaviour of the system is concerned. State variable (time-domain) equations may be written for a linear, time-invariant, continuous system as:

$$\frac{dx}{dt} = Fx + Gu \tag{2a}$$

$$y = Hx \tag{2b}$$

If x is an n-vector function of time, the constant matrices F, G and H are of dimension $n \times n$, $n \times p$ and $m \times n$ respectively; as before, u(t) and y(t) are the system input and output vectors. Fig. 1 illustrates the system arrangement; note that the loop in the diagram is not a feedback loop in the ordinary sense.

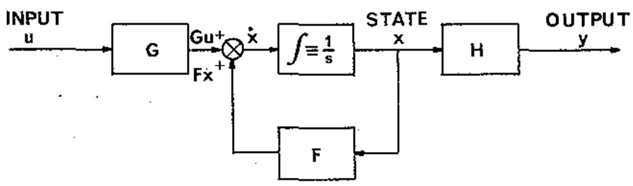

Figure 1.—System with state-variable description.

1. Zadeh, L. A. and Desoer, C. A., "Linear System Theory", McGraw-Hill, New York, (1963).
2. Kalman, R. E., "Mathematical Description of Linear Dynamical Systems", *J. Soc. Ind. and Appl. Math. Control*, Series A, Vol. 1, No. 2, 1963, p. 152.

It is evident from equation (2) that we may regard the input, u, as directly changing the state vector, x, rather than the output and the output, y, as being derived via a linear transformation on the state variables.

The matrices F, G and H are not independent of W(s). By taking the Laplace transform of equation 2 and eliminating X(s), explicit calculation yields

$$Y(s) = H(s\mathbf{1} - F)^{-1}GU(s) \tag{3}$$

which shows that

$$W(s) = H(s\mathbf{1} - F)^{-1}G \tag{4}$$

where $\mathbf{1}$ is the unit matrix. The determination of W from the matrices F, G and H is straightforward; the converse problem, that is, the determination of F, G and H from W, is harder (see references 2 and 3 for details). Some subtle problems are involved, especially with regard to determining the dimension of the F matrix.

An example of a second order system with one input and one output is provided by:

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{5a}$$

$$y = [b_1 \quad b_2]x \tag{5b}$$

for which the transfer function is

$$W(s) = \frac{b_2 s + b_1}{s^2 + a_2 s + a_1} \tag{5c}$$

Equations 5(a) and 5(b) constitute the "state description" and 5(c) is the transfer function.

4. Prototype Linear Feedback System

Feedback round a system has traditionally been applied by *operating on the output and adding (or subtracting)* the result of this operation at the input (see fig. 2). We shall however assume in this and later sections that the

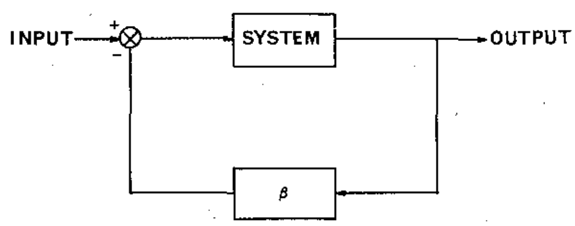

Figure 2.—Traditional view of feedback.

state vector can be measured and thus some function of it fed back (the legitimacy of this assumption will be discussed at the end of this section). It is also convenient to conceive of a linear transformation being applied to the state vector, the result of the linear transformation being then applied at the input of the system. With K a constant matrix, we shall say that the feedback law is K if the system input, u, is composed of an external input, v, summed with $-Kx$. Fig. 3 illustrates application of the feedback law K. The situation is in contrast to the classical case where the feedback, β, of fig. 2 is in

3. Anderson, B. D. O., Newcomb, R. W., Kalman, R. E. and Youla, D. C., "On the Equivalence of Linear, Time-Invariant, Dynamical System", *J. Franklin Institute*, Vol. 281, No. 5, May 1966, p. 371.

general a function of s multiplying (at least for a single-output system) a scalar quantity ; here, the feedback K is constant but multiplies a vector quantity.

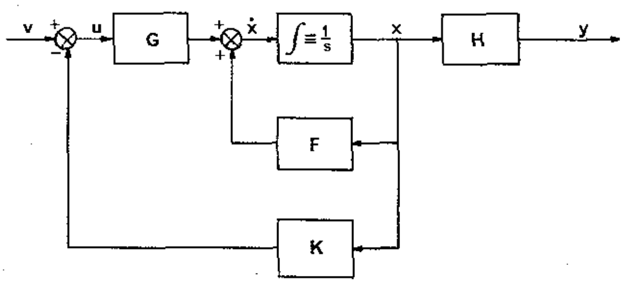

Figure 3.—System with state-variable feedback.

State-variable equations for the feedback system can then readily be determined as :

$$\frac{dx}{dt} = (F - GK)x + Gv \quad (6a)$$

$$y = Hx \quad (6b)$$

For the system as described by equations 5(a) to 5(c), if the feedback law is

$$K = [k_1 \quad k_2] \quad (7)$$

the closed loop equations become

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -a_1-k_1 & -a_2-k_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \quad (8a)$$

$$y = [b_1 \quad b_2]x \quad (8b)$$

with the closed loop transfer function, by analogy with equation 5(c),

$$W_c(s) = \frac{b_2 s + b_1}{s^2 + (a_2 + k_2)s + (a_1 + k_1)} \quad (8c)$$

It is important to note that feeding back an arbitrary linear function of the output, say K_1y, is merely a special case of state-variable feedback because identical feedback results from taking a state-variable feedback law, K, equal to K_1H. Even if derivatives of the output are fed back, it is possible to show that there is an equivalent state-variable feedback law, K, which is constant, since these derivatives are linear transformations of the states.[1, 2] Thus, state-variable feedback can do everything that output feedback can do and possibly more ; this is one basic reason for its utility.

As an example of a feedback system in which β contains a derivative, consider the system described by equations 5(a) to 5(c) and suppose it is desired to feed back $(c_2\dot{y} + c_1y)$. This is identical to feeding back

$$c_2[b_1 \quad b_2]\dot{x} + c_1[b_1 \quad b_2]x$$

$$= c_2[b_1 \quad b_2]\begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} x + c_1[b_1 \quad b_2]x + c_2b_2u \quad (9)$$

Apart from the term involving the input, u, which can easily be eliminated by scaling (that is, inserting a constant multiplier, $(1 - c_2b_2)$, at the system input), we observe that the feedback is of the form Kx.

In a given physical system the state-variable, x, may be directly measurable ; there are however many situations where it is not but in this latter instance it is possible to construct the state vector by operating on the input, u, and the output, y, of the system with a device called a "state-variable estimator" (these devices are discussed later in section 9). Thus there is justification for the statement made at the beginning of this section, that the state can be measured and some function of it fed back.

5. The Sensitivity Problem

The aim is to choose a feedback law, K, such that the resulting closed loop system is less sensitive to internal parameter variations than the equivalent open loop system (for a detailed statement of the problem see references 4 and 5). The first requirement is to characterise, mathematically, the sensitivity reduction. For some time it has been known that for a single loop system, sensitivity reduction occurs when, in broad terms, the amplitude of the loop gain is much larger than unity. Exact conditions for multi-loop systems are given in references 4 or 5 in terms of a "return difference matrix". For the system of fig. 3 to be less sensitive to parameter variations in the forward path than an equivalent open loop system, the condition is that for all frequencies, ω, of interest :

$$[1 + G'(-j\omega 1 - F')K'][1 + K(j\omega 1 - F)^{-1}G] - 1 \geq 0 \quad (10)$$

where the notation implies that the matrix on the left side is non-negative definite and the prime denotes matrix transposition. The matrix $[1 + K(j\omega 1 - F)^{-1}G]$ is the return difference matrix. For a single-input, single-output function, equation 10 reduces to $|T(j\omega)| \geq 1$, where $T(j\omega)$ is the return difference of Bode.

The choice of a matrix, K, to satisfy equation 10 is then the design objective. No systematic way of directly making such a choice has been developed. However, optimal control theory, as discussed below, provides a convenient route to equation 10 and constitutes a reasonable solution to the problem of choosing K.

Even if it were possible to choose K to satisfy equation 10, the problem still remains of selecting that particular K out of the set of all possible ones which leads to satisfactory dynamics or achieves some other stated design objective.

6. Optimal Linear Systems

An optimal control problem often begins by asking, with reference to the system of fig. 1, a question such as the following : given a scalar function, L(x,u,t), of the state variable, the input and time, what is the input which is such that the performance index, $V(x_0)$, is minimised where $V(x_0)$ is defined by :

$$V(x_0) = \int_{t_0}^{\infty} L(x,u,t)dt \quad (11)$$

4. Cruz, J. B. and Perkins, W. R., "A New Approach to the Sensitivity Problem in Multi-variable Feedback System Designing", *Trans. I.E.E.E.*, Vol. AC-9, No. 3, July 1964, p. 216.
5. Anderson, B. D. O. and Newcomb, R. W., "An Approach to the Time-Varying Sensitivity Problem", Report No. SEL-66-06, (TR No. 6560-1), Stanford Electronics Laboratories, Stanford, California, June 1966.

the system being in an initial state x_0 at initial time t_0.

While the appropriate selection of L and the existence of a corresponding problem solution are indeed deep and often unresolved questions (see for example references 6 and 7), some results applicable to linear control systems have been well studied.

In the linear regulator problem, the object is to return the state to zero from some non-zero value, x_0, at initial time, t_0, in such a way as to minimise $V(x_0)$ (by appropriate choice of the system input, u) where

$$V(x_0) = \int_{t_0}^{\infty} [x'(t)Qx(t) + u'(t)Ru(t)]dt \qquad (12)$$

Q being a non-negative definite constant symmetric matrix and R a positive definite constant symmetric matrix. The non-zero value, x_0, corresponds to an initial error in the system; the error might be a non-zero value of a system output or, if the latter is zero, a non-zero value of the derivative of the system output. The exact nature of the error and the way in which x_0 corresponds to it will not concern us here.

Note that we are not yet talking of a "feedback" regulator, that is a regulator which derives its input from feeding back some function of the output.

Both non-zero states and non-zero controls contribute towards producing a positive integrand in equation 12 and thus a positive V (the fact that Q is non-negative and R is positive means that $x'Qx$ and $u'Ru$ are non-negative and positive respectively, unless $x = 0$ or $u = 0$). Evidently, small values of u will cause $u'Ru$ to be smaller but at the same time $x'Qx$ will not tend to zero as fast as if a large value of u (corresponding to large $u'Ru$) were used to drive the integrator outputs to zero. Optimal control seeks that value which is "just right", representing the compromise between values of u which are too large or too small.

Very similar heuristic considerations apply to the servomechanism problem. In this problem, the object is to make the state follow some desired path, $x_d(t)$, in such a way as to minimise $V(x_0)$, where

$$V(x_0) = \int_{t_0}^{\infty} \{[x'(t) - x'_d(t)]Q[x(t) - x_d(t)] + u'(t)Ru(t)\}dt \quad (13)$$

with the same conditions on Q and R as before.

It is possible to formulate both problems with the state variable, x, replaced by the output, y, and possibly thereby reflect the system designer's aims more accurately. Thus in a position controller, if position but not velocity is important, the formulation of equation 13 with x replaced by y is sufficient. However if both position and velocity need to be controlled, either velocity must be regarded as a second output or else the state variable formulation must be used (the mathematics is the same for both cases).

It turns out[6, 7] that the minimisation in equations 12 and 13 is achieved by choosing the input, u(t), at time, t, to be a certain linear transformation of the state, x(t), at time, t, with, in the servomechanism problem, an additional additive external input being required which is independent of x_0 and computable in advance. In other words, optimality is achieved by implementing a certain linear control law, K_{opt} (in combination with an external input, v, in the servomechanism problem). Methods for computing K_{opt} in terms of the system matrices and the matrices Q and R are given in references 6 and 7.

The implementation of the control law has other very important advantages:

(a) For a very broad range of conditions (see reference 8 for the single loop case and reference 9 for the multi-loop case) the closed loop system is automatically stable, even if the open loop system is unstable.*

(b) For a very broad range of conditions, an optimal design is identical to a design where reduction in sensitivity to plant parameter variation is achieved (see references 8, 9 and 10).

(c) Again for a very broad range of conditions, an optimal design is very tolerant of system non-linearities, in the sense that the closed loop system remains stable if certain nominally linear elements become highly non-linear (see reference 11).

Thus in one stroke, optimal design achieves the two design objectives of optimality itself and improvement of the system's sensitivity, while, at the same time, it guarantees stability and leads to systems which can tolerate non-linearities without severe performance degradation.

What though of the resulting dynamics of the system? Different initial choices by the designer of Q and R lead to different control laws (K_{opt}) and in turn to different closed loop system dynamics. The way in which Q and R affect the end dynamics is complicated but some general remarks may be found in reference 12. One of the principal conclusions is that the smaller the eigenvalues of R in relation to those of Q, the further in the left half plane (from the imaginary axis) lie the poles of the closed loop transfer function and thus the faster the response of the system.

6. Kalman, R. E., "Contributions to the Theory of Optimal Control", Boletin de la Sociedad Matematica Mexicana, Mexico, 1960, p. 102.
7. Athans, M. and Falb, P. L., "Optimal Control" McGraw-Hill, New York (1966).
8. Kalman, R. E., "When is a Linear Control System Optimal?", *J. Basic Engineering, Trans. A.S.M.E.*, Series D, Vol. 86, March 1964, p. 1.
9. Anderson, B. D. O., "The Inverse Problem of Optimal Control", Report No. SEL-66-038, (TR No. 6560-3), Stanford Electronics Laboratories, Stanford, California, May 1966.
10. Anderson, B. D. O., "Sensitivity Improvement Using Optimal Design", *Proc. I.E.E.*, Vol. 113, No. 6, June 1966, p. 1084.
11. Moore, J. B. and Anderson, B. D. O., "Applications of the Multi-dimensional Popov Criterion", *Intl. J. Control*, Vol. 5, No. 4, 1967, pp. 345-353.
12. Kalman, R. E., Englar, T. S. and Bucy, R. S., "Fundamental Study of Adaptive Systems", Technical Report ASD-TR-61-27, Vol. I, Aeronautical Systems Division, Air Force Systems Command, Wright-Patterson AFB, Ohio.

*In fact, for a single-input system the phase margin is always greater than 60° and the gain margin virtually infinite.

7. Methods of Achieving Arbitrary Dynamics

The pole positions of the system transfer function matrix determine its transient response, insofar as initial conditions remain unspecified. It is not hard to show that when a feedback law, K, is implemented, the resulting transfer function of the closed loop system becomes

$$W_c(s) = H(s1 - F + GK)^{-1}G \tag{14}$$

in contrast to

$$W(s) = H(s1 - F)^{-1}G \tag{15}$$

for the open loop system.

The poles of $W_c(s)$ are the zeros of the determinant of $[s1 - F + GK]$, which are the same thing as the eigenvalues of the matrix $[F - GK]$. Thus the pole positioning problem can be expressed as follows: given F and G, find a K such that $[F - GK]$ has arbitrary eigenvalues.

In the event that [F, G] is "completely controllable" (see reference 2), which is a condition that is normally satisfied in physical system, it is possible to show that K exists and to give a constructive procedure for obtaining K. See reference 2 for the single loop case, where a unique K exists, and reference 13 for the harder, multiloop case where there are many suitable solutions for K. The multiplicity of the possible solutions for K in the multi-loop case, gives rise to a secondary problem of determining which is the most suitable from some other point of view than simply the pole positions of the closed loop system.

As an example, consider again the system:

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{5a}$$

$$y = [b_1 \quad b_2]x \tag{5b}$$

with the general feedback law:

$$K = [k_1 \quad k_2] \tag{7}$$

We recall that the closed loop transfer function is then

$$W_c(s) = \frac{b_2 s + b_1}{s^2 + (a_2 + k_2)s + (a_1 + k_1)} \tag{8c}$$

How to choose K to achieve arbitrary poles in $W_c(s)$ when a_1 and a_2 are known is then evidently quite clear; if the denominator of $W_c(s)$ is to be $(s^2 + d_2 s + d_1)$, we set $k_2 = (d_2 - a_2)$ and $k_1 = (d_1 - a_1)$.

In general the procedure for selecting K is nowhere near as easy as above and digital computation will often be required for high order systems.

It is helpful to note that the appropriate choice of K leads to arbitrarily fast dynamics (the corresponding pole positions have arbitrarily large negative real parts). Such a K however also leads to a system which requires arbitrarily large amounts of control energy to drive the system from a non-zero initial state to a final zero state (the control energy associated with an input, u, over the time interval from t_0 to t_1 is $\int_{t_0}^{t_1} u'(t)u(t)dt$, by analogy with many physical systems), see reference 14.

A second difficulty associated with choosing arbitrary dynamics derives from the fact that there is no control over the sensitivity improvement (or lack of it) of the closed loop system. This constitutes another argument for design via the optimal control technique, though, to be sure, the latter does not allow initial arbitrary prescription of the closed loop poles.

A difficulty with both the optimal control technique and the technique discussed above for choosing a feedback control law is that little control is exercised over the zeros of the closed loop system. When the closed loop system is used as a signal processor (as distinct from, for example, a regulator), this difficulty may be very serious.

Finally, we note that when the states are not available for measurement, we may construct a state-variable estimator and follow it by a control law, K, chosen to give $[F - GK]$ arbitrary eigenvalues and conclude that: there exists a feedback controller operating on a system output and a controller design procedure, so that the resulting closed-loop system has arbitrary poles.

8. Reduction of the Effects of Noise

The preceding remarks have all dealt with noiseless situations or deterministic problems. As both designers and users of control systems know, noise can constitute a major problem in systems.

The discussion of this section will be based around a particular characterisation of noise, the characterisation being particular for two reasons:

(a) the noise model reflects accurately many physical situations;

(b) the problem of specifying how to reduce the effect of the noise is tractable.

It is assumed that stationary white noise enters additively at the input (v in fig. 4) and at the output (n in fig. 4). It is further assumed that v and n are uncorrelated. Finally, it is assumed that v and n are Gaussian, with zero mean and with covariances V and N respectively.

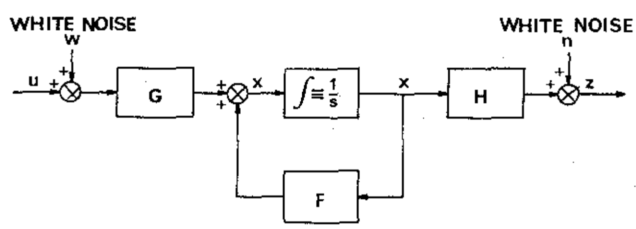

Figure 4.—Prototype noisy system.

For a multiple-input system, v is a vector and V a matrix. For a multiple output system, n is a vector and N a matrix.

The only variables available for measurement are the

13. Anderson, B. D. O., and Luenberger, D. G., "Design of Multivariable Control Systems"; *Proc. I.E.E.*, Vol. 114, No. 3, March 1967, p. 395.

14. Kalman, R. E., Ho, Y. C. and Narendra, K. S., "Controllability of Linear Dynamical Systems", from "Contributions to Differential Equations", Vol. 1, John Wiley, New York (1963).

input, u, and the noisy output, z. Kalman and Bucy[15] have shown that under a very broad range of conditions, it is possible to construct a state estimator comprising a linear time-invariant system whose input consists of u and z and whose output is $\hat{x}$, the best possible estimate (in a certain sense) of the state variable, x. Other workers[16, 17] have shown that when the linear regulator problem is considered with the complication of input and output additive noise, an optimal arrangement derives from feeding back $K_{opt}\hat{x}$ as in the deterministic case. The feedback law K_{opt} is thus independent of the noise covariances. The same is true for the servomechanism problem. Thus, a feedback system designed from Kalman and Bucy's filtering theory and deterministic optimal control theory is overall optimal with respect to both noise performance and minimisation of a trivial variant of the performance index considered earlier (the trivial variant referred to, is the replacement of the earlier performance index by a mean performance index, the mean being defined over all possible performance indices resulting from the random noise).

It is useful to note that Kalman and Bucy's filtering theory incorporates Wiener's filtering theory and sets the latter on a basis which is exceedingly appropriate for control systems.

Finally we remark that lack of knowledge of the noise covariances V and N may lead to a somewhat incorrect state estimator; also the feedback law, K_{opt}, of the deterministic theory may be for some other reason incorrectly implemented. Reference 18 discusses both these non-optimal situations.

9. The State-Variable Estimator

When state variables are not available for direct measurement, it is necessary to obtain them by some other means. A device that recovers the state variables from inputs, that are the only directly measurable variables associated with the plant, is termed a state-variable estimator.

Two classes of estimators exist: those designed to operate in a noisy environment and those designed to operate in an environment free from noise. In actual fact some noise can usually be tolerated in the latter class. On occasions it would also seem possible to recover state variables by differentiation of the plant output; such a procedure is not considered here however, owing to the extreme sensitivity to noise of such an estimator.

The estimator of Kalman and Bucy has been discussed in the previous section. It actually consists of a linear, multi-loop, time-invariant system, the input being the input together with the noisy output of the main plant and the output of the estimator being, naturally, the state estimate. It is usable in a non-stochastic environment if desired.

The second class of estimators have been discussed in references 12, 19 and 20. These estimators fall into two categories, those which are of the simplest possible design and those which have arbitrary dynamics (but are not necessarily of the simplest possible design). All are linear and time-invariant.

An estimator of the simplest possible design type requires knowledge of the feedback law operating on the state estimates for its design. It is simple in the sense that it contains a minimum number of poles, this number being determined in a complex way by the properties of the matrices in the state space description of the system. Some arbitrary assignment of pole positions is possible.

An estimator in the remaining category is in a certain sense a model of the plant, with a feedback law implemented round it. Just as for an arbitrary plant a feedback law can be chosen to give arbitrary dynamics for the closed loop system, so the feedback law inside the estimator can be chosen to give arbitrary dynamics for the estimator. The techniques for designing the estimator are accordingly closely related to the techniques for selecting feedback laws in the pole positioning problem of section 7. Though the two problems of plant feedback law design and estimator design are physically distinct, mathematically they are essentially the same.

In the multi-loop case the same difficulty arises as occurs in the pole positioning problem, namely, the inability to distinguish between the merits of the non-unique solutions to the problem of achieving arbitrary dynamics.

Whereas it is understood how the pole positions of a closed-loop transfer function affect the behaviour of the system, it may not be clear what effect the pole positions of an estimator have. To be strictly accurate in a description of estimator operation, it is necessary to note that the estimator produces an estimate, $\hat{x}$, of the state, x, where $(\hat{x} - x)$ is exponentially decaying. The rate of this exponential decay is precisely determined by the pole positions of the estimator. Provided the decay of $(\hat{x} - x)$ is governed by a time constant far shorter than any time constant associated with the closed loop system, initial inaccuracies in the state estimate are immaterial (even if the time constant relationship is not satisfied, reasonably satisfactory operation of the overall system can be expected). As a general rule therefore, one expects estimator poles obtained from the filtering theory of Kalman and Bucy to lie further to the left of the imaginary axis than the poles of the closed loop system.

15. Kalman, R. E. and Bucy, R. S., "New Results in Linear Filtering and Prediction Theory", *J. Basic Engineering, Trans. A.S.M.E.*, Series D, Vol. 83, March 1961, p. 95.
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10. Directions for Future Research

Throughout the preceding survey, a number of inadequacies of the present theory have been mentioned. As specific questions which might be investigated, we would suggest:

(a) To what extent can an optimally designed system achieve arbitrary pole positions?

(b) To what extent can the especially simple state estimators of references 19 and 20 be applied in stochastic situations?

(c) Can a design procedure giving sensitivity reduction be presented independently of optimal design methods?

(d) In the multi-loop situation, how may different solutions to the pole positioning problem be compared?

(e) How should one cope with a system where noise is generated at all points in the system?

(f) How can linear system design be extended to adaptive system design so that a system could be built which takes into account plant parameter variations other than incremental ones?

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Signalling Considerations for P.C.M. Systems

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Summary

This article compares a number of alternative signalling arrangements for p.c.m. junction signalling and discusses the considerations underlining the choice of a particular design under various conditions. The possible signalling arrangements are grouped for comparison according to the type of line signalling facilities provided by the p.c.m. terminal and according to the time allocation plan adopted for the time division multiplex interleaving of signalling and speech digits.

In contrast with previous practice it is attempted to adapt the signalling facilities provided by the transmission equipment to the requirements of the existing telephone network. The problems of developing a basic design, which, in connection with interchangeable plug-in signalling equipment, is capable of interworking with exchanges of different types, are discussed.

1. Introduction

1.1 *General*

Pulse code modulation systems transmit information in the form of groups of pulses. The main advantage of this mode of operation is that the receiving equipment needs to decide only whether pulses are present or absent at specific instances. This decision can be made with a great deal of reliability, even when the noise level is so high that the channel would be practically useless for the transmission of analogue signals. Once the presence or absence of pulses is correctly detected, a new pulse train can be generated in the receiver which is an exact replica of the one transmitted, that is, completely free from degradation. This regenerated pulse train can be transmitted and regenerated again. Thus, at least in theory, p.c.m. offers the possibility of transmitting signals over long links containing many repeaters, without suffering from cumulative degradation caused by noise and distortion.

It was first envisaged to employ p.c.m. over microwave links but it was soon realised that p.c.m. systems offer similar advantages when employed over voice frequency (v.f.) cables. Indeed, the first practical application of p.c.m. was in the telephone network for the provision of multiplexed speech channels over physical pairs in short-haul junction cables.

This application has raised some special and sometimes rather difficult requirements. A transmission system employed in the telephone network must transmit not only speech but also a variety of telephone signals. These telephone signals are required because the switching centres linked by the transmission system must exchange information to facilitate the setting-up, supervision and clearing of calls.

While the requirements for the transmission of speech are essentially the same all over the world, the signalling requirements can vary a great deal even within one country, due to the large variety of exchange types employed. It is highly desirable to design the transmission equipment so that it offers signalling facilities directly suitable for or easily adaptable to most types of exchanges. With analogue type transmission equipment, this was often not possible and special adaptation circuits (signalling relay sets) have been required in most cases. The flexibility of the digital techniques employed with p.c.m. could often avoid this need.

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