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A Simplified Viewpoint of Hyperstability

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***Abstract*—The definition of a hyperstable system according to Popov is given a network theoretic interpretation, and proofs are presented outlining the connection between passive and hyperstable systems.**

I. INTRODUCTION

In 1963, Popov introduced the notion of hyperstability for linear single-input single-output time-invariant systems[1] and subsequently extended his ideas.[2] The key feature of his results is the existence of an equivalence between hyperstable systems and systems which can be described by a transfer function that is positive real, i.e., the transfer function is like that which maps the input into the output of a passive system.

Using well-known ideas from the theory of passive systems and network theory, Popov's statements are interpreted in terms of energy flow and excitation of energy storage elements and are presented in what are believed to be more transparent proofs of his theorem. These results are also presented in multivariable format for greater generality.

As Popov himself points out, stability questions associated with a system containing memoryless nonlinear feedback can be interpreted as questions of hyperstability, thus emphasizing the tieup between positive real functions, or matrices, and regulating systems with nonlinear feedback.[1]

II. DEFINITION AND NOTATION

Consider systems described by state space equations of the form

$$\dot{x} = Fx + Gu \tag{1a}$$

$$y = H'x + Ju \tag{1b}$$

where throughout the paper it is assumed that the pair $[F, G]$ is completely controllable[3] and $[F, H]$ completely observable.[3] It is also assumed that the vectors u and y have the same dimension.

Hyperstability is a property of the system which requires the state vector to remain bounded if the inputs u are restricted to belonging to a subset of the set of all possible inputs. The subset of inputs u is defined by those $u(\cdot)$ which satisfy for all T

$$\int_0^T u'(t)y(t)dt \leq \delta[\|x(0)\|] \sup_{0\leq t\leq T} \|x(t)\|. \tag{2}$$

Here δ is a positive constant depending on the initial state $x(0)$ of the system but independent of the time T and the symbol $\|\ \|$ denotes the Euclidean norm.

The system (1) is termed "hyperstable" if for any $u(\cdot)$ in the subset defined by (2) the following inequality holds for some positive constant K and all t:

$$\|x(t)\| \leq K(\|x(0)\| + \delta). \tag{3}$$

The system is termed "asymptotically hyperstable" if for any $u(\cdot)$ in the subset defined by (2) *which is also bounded* the inequality (3) holds together with

$$\lim_{t\to\infty} x(t) = 0. \tag{4}$$

An important point is that the definitions of hyperstability and asymptotic hyperstability are "coordinate free" as may easily be checked, that is, if a new basis is chosen for the state space, causing replacement of x by $\hat{x} = Tx$, and of F by $\hat{F} = TFT^{-1}$, etc., then hyperstability in the first coordinate system implies hyperstability in the second and vice versa. The same holds, of course, for asymptotic hyperstability. Consequently, hyperstability and asymptotic hyperstability are properties of the transfer-function matrix

$$Z(s) = J + H'(sI - F)^{-1}G \tag{5}$$

as distinct from being properties of a particular minimal realization of $Z(s)$. A minimal realization of $Z(s)$ is a quadruple $\{F, G, H, J\}$ such that (5) holds with $[F, G]$ completely controllable and $[F, H]$ completely observable.

A rational transfer-function matrix as in (5) is termed "positive real" if the following three conditions are satisfied[4]:

1) $Z(s)$ has real elements for real s,
2) The elements of $Z(s)$ have no poles in $\mathrm{Re}[s]>0$ and poles on the $j\omega$ axis are simple, and such that the associated residue matrix is non-negative definite Hermitian,
3) For any real value of ω such that no element of $Z(j\omega)$ has a pole for this value, $Z(j\omega)+Z'^*(j\omega)$ *is non-negative definite* Hermitian.

For real rational $Z(s)$, 1) and 3) may be replaced by[4]:

2) $Z(s)+Z'^*(s)$ is non-negative definite Hermitian in $\mathrm{Re}[s]>0$.

III. THE RESULTS OF POPOV

Popov's two results, generalized to the multiple-input situation, are as follows.

Theorem 1

A necessary and sufficient condition for the transfer-function matrix $Z(s)$ of (5) to define a hyperstable system is that $Z(s)$ be positive real.

Theorem 2

A necessary and sufficient condition for the transfer-function matrix $Z(s)$ of (5) to define an asymptotically hyperstable system is that $Z(s)$ be strictly positive real, in the sense that $Z(s)$ is positive real, all poles of $Z(s)$ should lie in the half-plane $\mathrm{Re}[s]<0$ and $Z(j\omega)+Z'^*(j\omega)$ should be positive definite Hermitian for all real ω.

These two theorems have interesting interpretations in terms of energy constraints in passive systems. As is well known, appropriate choice of input variable u and output variable y of a passive system leads to a transfer-function matrix for the system which is positive real with $\int_0^T u'(t)y(t)dt$ having the interpretation of the energy delivered into the system from the external world over the time interval $[0, T]$.[4] With a correct choice of coordinate basis, it is also possible to interpret $x(t)$ as having entries which are measures of the excitation of the energy storage elements of the system.

Of the set of all possible inputs, the inputs in the subset defined by (2) are those for which the energy flow into the system up till time T is bounded by the maximum excitation over $[0, T]$ of the energy storage elements within the system. Hyperstability requires, [see (3)], that for such inputs the maximum excitations should be bounded in terms of the initial excitations. The preceding interpretation in fact will enable us to provide proofs of Theorems 1 and 2.

IV. PROOFS OF THE RESULTS

Positive Realness Implies Hyperstability

Since $Z(s)$ is positive real, one minimal realization $\{F, G, H, J\}$ of $Z(s)$ is provided through the medium of an interconnection of pas-

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sive inductors, passive resistors, ideal transformers, and gyrators which constitute a network synthesis of $Z(s)$.[4] For it is basic to network theory that associated with a positive real $Z(s)$ there is always a network made up of such passive components, the port impedance of which is precisely $Z(s)$. Capacitors may be eliminated by replacing them with a gyrator and inductor. As is also well known, a suitable state-variable vector x for $Z(s)$ is provided by the instantaneous inductor currents. Moreover, as can always be assumed, this state vector is of minimum dimension if the number of reactive elements in the synthesis of $Z(s)$ is minimal.

If all inductors have value unity, which can always be arranged using transformer normalizations, the stored energy in the network at time t is $\frac{1}{2}x'(t)x(t)$, and passivity requires that for all T

$$\int_0^T u'(t)y(t)dt + \tfrac{1}{2}x'(0)x(0) \geq \tfrac{1}{2}x'(T)x(T). \tag{6}$$

In other words, the sum of the initial stored energy of the network and the energy directed into the network terminals over $[0, T]$ must be no less than the stored energy at time T. The "greater than" sign in (6) will hold if the network dissipates energy during $[0, T]$.

If allowable inputs are those satisfying (2), then, for all T,

$$\tfrac{1}{2}x'(T)x(T) \leq \tfrac{1}{2}x'(0)x(0) + \delta[\|x(0)\|] \sup_{0\leq t\leq T}\|x(t)\|. \tag{7}$$

From (7) it follows that $\|x(t)\|$ is bounded, for if not, for a sequence of values of T, $T=T_1, T_2, \cdots$, the left-hand side of (7) would increase faster than the right-hand side, so that for sufficiently large $\|x(T_i)\|$ the inequality would fail. Let M denote $\sup_{0\leq t\leq\infty}\|x(t)\|$. Then (7) yields

$$\tfrac{1}{2}M^2 \leq \tfrac{1}{2}\|x(0)\|^2 + \delta[\|x(0)\|]M \tag{8}$$

or

$$\begin{aligned}\{M - \delta[\|x(0)\|]\}^2 &\leq \|x(0)\|^2 + \delta[\|x(0)\|]^2 \\ &\leq \{\|x(0)\| + \delta[\|x(0)\|]\}^2\end{aligned} \tag{9}$$

whence

$$M \leq \|x(0)\| + 2\delta[\|x(0)\|] \tag{10}$$

or for all t

$$\|x(t)\| \leq 2(\|x(0)\| + \delta) \tag{11}$$

which is a special case of (3).

Hyperstability Implies Positive Realness

It will actually be proved that the negative statement, lack of positive realness, implies lack of hyperstability. In particular, assume that the positive real definition of condition 2) does not hold, i.e., that $Z(s)+Z'^*(s)$ fails to be non-negative definite Hermitian for some value of s, for example, $\sigma_0+j\omega_0$ with $\sigma_0>0$. By continuity of the elements of $Z(\cdot)$ in $\mathrm{Re}[s]>0$, ω_0 is assumed to be nonzero. Then there exists a vector u_0 such that

$$u_0'^*[Z(\sigma_0 + j\omega_0) + Z'^*(\sigma_0 + j\omega_0)]u_0 < 0. \tag{12}$$

Moreover, there exists an initial state x_0 such that the response to the input

$$u(t) = \begin{cases} 0, & t < 0 \quad (13a)\\ [\mathrm{Re}\, u_0 e^{\sigma_0 t} e^{j\omega_0 t}], & t \geq 0 \quad (13b)\end{cases}$$

is given for $t\geq 0$ by

$$y(t) = \mathrm{Re}\, Z(\sigma_0 + j\omega_0)u_0 e^{\sigma_0 t}e^{j\omega_0 t}. \tag{14}$$

Explicit calculation set out for example in Newcomb ([4] ch. 4) yields

$$\begin{aligned}\int_0^T u'(t)y(t)dt = \tfrac{1}{2}\,\mathrm{Re}\Big[\frac{1}{\sigma_0} u_0'^* Z'^*(j\omega_0 + \sigma_0)u_0 e^{2\sigma_0 T} \\ + \frac{1}{\sigma_0 + \omega_0} u_0' Z'(j\omega_0 + \sigma_0)u_0 e^{2(\sigma_0 + j\omega_0)T}\Big] + a\end{aligned} \tag{15}$$

where a is a constant, independent of time (the details of the calculation of (15) are of no concern here). For suitably chosen T the second term is zero and the first term alone predominates; then the inequality (12) means that the integral on the left of (15) diverges to $-\infty$. Consequently the input $u(t)$ of (13) obeys the inequality (2). At the same time, it is clear that $\|x(t)\|$ is unbounded, increasing more or less as $e^{\sigma_0 t}$. Thus lack of hyperstability is demonstrated.

Strict Positive Realness Implies Asymptotic Hyperstability

Note first that strict positive realness implies that for a sufficiently small positive σ, $\hat{Z}(s)=Z(s-\sigma)$ is positive real: condition 1) of the positive real definition is satisfied for $\hat{Z}(s)$, for arbitrary real σ. Condition 2) is satisfied for sufficiently small σ because all poles of $Z(s)$ lie in $\mathrm{Re}\, s<0$. To see that condition 3) is satisfied, observe that the non-negativity of $\hat{Z}(j\omega)+\hat{Z}'^*(j\omega)$ is equivalent to the non-negativity for all ω of a series of polynomials which are the minors of the matrix $d^*d[\hat{Z}(j\omega)+\hat{Z}'^*(j\omega)]$, d being the lowest common denominator of the elements of $\hat{Z}(j\omega)$. As $\sigma\to 0$, these minors which approach the corresponding minors associated with $Z(j\omega)$ are strictly positive by assumption. So for sufficiently small σ, the minors associated with $\hat{Z}(j\omega)$ are non-negative. Observing that $\hat{Z}(s)$ is given by

$$\hat{Z}(s) = J + H'[sI - (F + \sigma I)]^{-1}G \tag{16}$$

the positive realness of $\hat{Z}(s)$ implies the existence of real constant matrices $P=P'>0$, L, and W_0 such that[5]

$$P(F + \sigma I) + (F' + \sigma I)P = -LL' \tag{17a}$$

$$PG = H - LW_0 \tag{17b}$$

$$W_0'W_0 = J + J'. \tag{17c}$$

It will now be shown that $V=x'Px$ is a Liapunov function for the system (1) with any bounded input u satisfying (2). Of course V is positive definite; also,

$$\begin{aligned}\dot{V} &= 2x'P\dot{x} = x'(PF + F'P)x + 2x'PGu \\ &= -2\sigma x'Px - (x'L - u'W_0')(L'x - W_0 u) + 2u'y.\end{aligned} \tag{18}$$

Thus

$$\dot{V} \leq -2\sigma V + 2u'y. \tag{19}$$

Setting $W=\int_{t_0}^t V(\lambda)d\lambda$ and integrating (19) yields

$$0 \leq \dot{W} < -2\sigma W + V(t_0) + 2\int_{t_0}^t u'y dt. \tag{20}$$

Now (1) is certainly hyperstable, thus (2) and (3) yield

$$\int_{t_0}^t u'y dt \leq K_1 \tag{21}$$

for some constant K_1 and all t. Hence

$$0 \leq \dot{W} \leq -2\sigma W + V(t_0) + K_1. \tag{22}$$

The definition of W and (22) imply W is bounded above, and that $\dot{W}$ is non-negative and bounded above. Hence $\dot{W}\in L^1$, the space of functions which are absolutely integrable on $[0, \infty)$. The boundedness of x, following from hyperstability as distinct from asymptotic hyperstability which has yet to be proven, and the assumed boundedness of u guarantee that $\dot{V}$ in (18) is bounded. Hence $\dot{W}$ is uniformly continuous. Since also $\dot{W}\in L^1$, it follows that $\dot{W}\to 0$ as $t\to\infty$, i.e., $\lim_{t\to\infty} x(t)=0$.

Asymptotic Hyperstability Implies Strict Positive Realness

It is a simple matter to modify the earlier argument that hyperstability implies positive realness. The strict positive realness can fail in two ways, with positive realness still being maintained.

Either $Z(s)$ can have a pole on the $j\omega$ axis, or there exists a vector u_0 and frequency ω_0 for which

$$u_0'^*[Z(j\omega_0) + Z'^*(j\omega_0)]u_0 = 0. \tag{23}$$

In the former case, there exists an initial state which is Liapunov stable but not asymptotically stable in the presence of zero input.

Since asymptotic stability is a requirement for asymptotic hyperstability, it is seen that failure of $Z(s)$ to be strictly positive real for the first reason results in a lack of asymptotic hyperstability.

In the latter case, arguments like those used immediately after (8) lead to the conclusion that there exists an initial state x_0 and sinusoidally varying ($\omega_0 \neq 0$) or constant ($\omega_0 = 0$) input $u(t)$ for which $\int_0^T u'(t)y(t)dt$ is bounded; moreover $x(t)$ does not decay to zero. Thus with inequality (2) holding, the hyperstability is not asymptotic.

V. Conclusions

The interpretation of the hyperstability concept in terms of properties of passive systems as given in Section III is straightforward but does not seem to have been given before. In particular it allows the writing down of (6), essentially by physical arguments. This equation, whose simple deduction is not to be found in Popov,[1] allows a quicker derivation of the result that positive realness implies hyperstability.

The stability of systems with nonlinear memoryless feedback can be studied with the aid of the hyperstability inequality (2). Further stability results could possibly be deduced by searching for other forms of feedback also allowing satisfaction of (2).

References

[1] V. M. Popov, "The solution of a new stability problem for controlled systems," *Automation and Remote Control*, vol. 24, pp. 1–23, January 1963.

[2] ——, "Hyperstability and optimality of automatic systems with several control functions," *Rev. Roumanian Sci. Tech.—Electrotech. and Engrg.*, vol. 9, no. 4, pp. 629–690, 1964.

[3] L. A. Zadeh and C. A. Desoer, *Linear System Theory*. New York: McGraw-Hill, 1963.

[4] R. W. Newcomb, *Linear Multiport Synthesis*. New York: McGraw-Hill, 1966.

[5] B. D. O. Anderson, "A system theory criterion for positive real matrices," *SIAM J. Control*, ser. A, vol. 5, pp. 171–182, May 1967.