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On the Generation of All Spectral Factors

***Abstract*—Two solutions are presented to the problem of finding from one spectral factorization all spectral factorizations of a non-negative para-Hermitian matrix. The first solution results from the theory of equivalent networks while the second is given through a derivation from first principles.**

A problem of considerable interest in view of its applications to filtering theory, stability theory, network synthesis, and other areas of system theory is that of spectral factorization. Since one would like to investigate all possibilities, Kalman[1] recently has posed, and solved for small order, the problem of finding all spectral factorizations of nonminimal size. Here we present two slightly different but complete and simple solutions, the first following immediately from the known theory of equivalent circuits and the second following a development independent of network considerations for which we give the details.

One is given an $n\times n$ matrix $\mathbf{A}(p)$ each element being a rational function, with real coefficients, of the complex variable $p = \sigma + j\omega$ (A is called real-rational) with the properties that A is para-Hermitian, that is $\tilde{\mathbf{A}}_* = \mathbf{A}$ [where the tilde denotes matrix transposition and the subscript asterisk denotes replacement of p by $-p$ (Hurwitz conjugation)], and that $\mathbf{A}(j\omega)$ is positive semidefinite for almost all ω, written $\mathbf{A} \geq 0$ ($\mathbf{A} \geq 0$ is called non-negative). One wishes to find a (or all) factorization(s) $\mathbf{A} = \tilde{\mathbf{K}}_*\mathbf{K}$ such that K is real-rational (the factorization is then also called real-rational). Various methods are available for finding one particular factorization[2]–[5] (Also see Newcomb[6] p. 89.) and methods for passing among ones of minimal size, $\mathbf{A} = \tilde{\mathbf{J}}_*\mathbf{J}$ with dimension $\mathbf{J} = r \times n$, $r = \text{rank } \mathbf{A}$, are well known (Belevitch,[7] p. 307 and Youla,[2] p. 176).

Oono and Yasuura through the theory of equivalent networks (Oono and Yasuura,[8] p. 137) and Youla's subsequent amplification of their work (Youla,[2] p. 178) have put forward the following result which gives a method of finding all spectral factorizations given any one minimal factor J. Here $\mathbf{1}_k$ is the $k \times k$ identity, $\mathbf{0}_{k-r,n}$ is the $(k - r) \times n$ zero matrix, and a $k \times k$ matrix $\mathbf{\Phi}(p)$ is called para-unitary if $\tilde{\mathbf{\Phi}}_*\mathbf{\Phi} = \mathbf{1}_k$.

Theorem 1

Let $\mathbf{A}(p)$ be an $n \times n$ real-rational matrix satisfying

$$\mathbf{A} = \tilde{\mathbf{A}}_* \geq 0 \tag{1}$$

and consider any two real-rational factorizations

$$\mathbf{A} = \tilde{\mathbf{J}}_*\mathbf{J} = \tilde{K}_*\mathbf{K} \tag{2}$$

having J and K, respectively, $r \times n$ and $k \times n$ with $k \geq r = \text{rank } \mathbf{A}$. Then there exists a $k \times k$ real-rational para-unitary $\mathbf{\Phi}$ such that

$$\mathbf{K} = \mathbf{\Phi}\begin{bmatrix} \mathbf{J} \\ \hline \mathbf{0}_{k-r,n} \end{bmatrix} \tag{3}$$

Proof: With slight changes in notation the proof is essentially given in Newcomb,[9] pp. 319, 186. The procedure can be outlined as a) forming $\mathbf{K}\mathbf{J}^{-1}$ by obtaining a right pseudo-inverse for J, and b) extending the rectangular $\mathbf{K}\mathbf{J}^{-1}$ to the square $\mathbf{\Phi}$. Q.E.D.

The important conclusion to be reached from Theorem 1 is that given any one factorization of minimal size, $\mathbf{A} = \tilde{\mathbf{J}}_*\mathbf{J}$, we can find all others by forming

$$\mathbf{\Phi}\begin{bmatrix} \mathbf{J} \\ \hline \mathbf{0} \end{bmatrix}$$

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where 0 ranges over all zero matrices of n columns and Φ ranges over all (real-rational) para-unitary matrices. All Φ can be readily generated through degree one or two factors (Newcomb,[9] p. 190) and a simple means of finding one J is through the Gauss factorization (such a J very simply yields J^{-1} mentioned in the theorem's proof). (See Newcomb,[9] p. 168.)

At this point we prove in detail an alternate result which is, however, a nontrivial but straightforward generalization of the result of Youla[2] (p. 176).

Lemma 1

Let $A(p)$ be an $n \times n$ real-rational matrix of rank n almost everywhere, and satisfying (1). Let $J(p)$ be an $n \times n$ real-rational matrix such that

$$A = \tilde{J}_* J. \quad (4)$$

Then $K(p)$ is a $k \times n$ matrix satisfying

$$A = \tilde{K}_* K \quad (5)$$

if and only if there exists a $k \times n$ matrix $V(p)$ with

$$\tilde{V}_* V = 1_n \quad (6)$$

and

$$K = VJ. \quad (7)$$

Proof: Suppose (4), (6), and (7) hold. That (5) is satisfied follows by direct calculation.

Conversely suppose (4) and (5) hold. Then it is easy to verify that a matrix $V(p)$ satisfying (6) and (7) is given by

$$V = KJ^{-1}. \quad (8)$$

[Note that $J^{-1}(p)$ exists by the hypothesis on the rank of $A(p)$]. Q.E.D.

Lemma 1 may now be generalized to the situation where $A(p)$ has rank $r < n$ almost everywhere.

Theorem 2

Let $A(p)$ be an $n \times n$ real-rational matrix of rank r almost everywhere, and satisfying (1). Let $J(p)$ be an $r \times n$ real-rational matrix such that

$$A = \tilde{J}_* J. \quad (9)$$

Then the $k \times n$ matrix $K(p)$ satisfies

$$A = \tilde{K}_* K \quad (10)$$

if and only if there exists a $k \times r$ matrix $V(p)$ such that

$$\tilde{V}_* V = 1_r \quad (11)$$

and

$$K = VJ. \quad (12)$$

Proof: Assuming that (9), (11), and (12) hold, (10) follows by direct calculation.

To prove the converse we assume that (9) and (10) hold. Let $X(p)$ be a nonsingular $n \times n$ matrix such that

$$JX = [\hat{J} \mid 0_{r,n-r}] \quad (13)$$

where $\hat{J}$ is a nonsingular $r \times r$ matrix. Then immediately

$$\tilde{X}_* \tilde{K}_* KX = \left[\begin{array}{c|c} \tilde{\hat{J}}_* \hat{J} & 0_{r,n-r} \\ \hline 0_{n-r,r} & 0_{n-r,n-r} \end{array}\right] \quad (14)$$

on using (9), (10), and (13). A simple argument using the nonnegativity of $A(j\omega)$ then implies

$$KX = [\hat{K} \mid 0_{k,n-r}] \quad (15)$$

for some $\hat{K}(p)$. Using Lemma 1 we conclude the existence of a $k \times r$ matrix $V(p)$ satisfying

$$\tilde{V}_* V = 1_r \quad (11)$$

and

$$\hat{K} = V\hat{J}, \quad (16)$$

or, using (13) and (15)

$$K = VJ \quad (12)$$

which completes the proof. Q.E.D.

Notice that since $J(p)$ has rank r (following from the rank of $A(p)$), it has a right inverse satisfying

$$JJ^{-1} = 1_r. \quad (17)$$

Thus the matrix V which relates any two matrices J and K satisfying (9) and (10) is simply, by (12)

$$V = KJ^{-1}. \quad (18)$$

It is not, however, immediate, but a nontrivial fact, that

$$VJ = K$$

if V is defined by (18). This is because J^{-1} is merely a right inverse of J.

We point out that the difference between the two theorems is in the fact that Φ is nonsingular while V generally is not. The two results can be reconciled by noting that any V can be extended to a (nonsingular) para-unitary Φ by known means (Newcomb,[9] p. 186). Theorem 1 has the advantage that the structure of para-unitary matrices is well investigated (Oono and Yasuura,[8] pp. 138–142) while Theorem 2 has the advantage of working with smaller matrices.

For filtering purposes it is often of interest to find all state variable realizations for (Kp) that is, all constant matrices A, B, C, D and integers m such that

$$K(p) = D + C(p1_m - A)^{-1}B \quad (19)$$

All realizations of K can be found using previous theories.[10]

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