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Extensions of quadratic minimization theory II. Infinite time results†

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Necessary and sufficient conditions are developed for the existence and calculation of well-defined Riccati differential equation solutions associated with infinite time quadratic loss minimization problems.

The significance of the results is that they not only extend optimal control theory but also have application in a number of areas other than optimal control, e.g. stability theory and time-varying spectral factorization theory.

1. Introduction

In a companion paper, Moore and Anderson (1968), a finite time quadratic loss minimization problem is considered, and the necessary and sufficient conditions are given for the optimal control law to exist. Also, the steps required in the calculation of the control law are given.

In this paper a simplified version of the same problem is considered and is extended to the case when the minimization is for an infinite time interval. The significance of the results is that they not only extend optimal control theory but also have application in a number of areas other than optimal control, e.g. stability theory, Moore and Anderson (1967 a), and time-varying spectral factorization theory, Moore and Anderson (1967 b). There is a correspondence between the various extensions and those given in Anderson (1967) for the time invariant problem.

2. Review

A specialization of the results given in the companion paper, Moore and Anderson (1968) is now reviewed. These are to be extended in the following section for the case when the time intervals of interest are infinite.

Consider the linear, n-dimensional system:

$$\dot{x} = Fx + Gu, \tag{1}$$

and performance index:

$$V(x_0, u, t_1, t_0) = \int_{t_0}^{t_1} (u'Ru + 2x'Hu)\, dt, \tag{2}$$

where

(A1) F, G, R, R^{-1} and H are finite valued and $R = R' > 0$.

For convenience we define:

$$\mathrm{R}(t,\tau) = R(t)\delta(t-\tau) + H'(t)\Phi(t,\tau)G(\tau)1(t-\tau) + G'(t)\Phi'(\tau,t)H(\tau)1(\tau-t); \tag{3}$$

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where $\Phi(t, \tau)$ is given from:

$$\frac{d}{dt}\Phi(t,\tau) = F\Phi(t,\tau); \quad \Phi(\tau,\tau) = I_n, \tag{4}$$

and label six conditions as follows:

(A 2) $R(t, \tau)$ is a covariance.

(A 3) $[F, G]$ is completely controllable at t_0, in the sense that for any state x_0 at t_0, a control u_c and time T_0 may be found taking the system from the zero state at time $t_0 - T_0$ to the state x_0 at time t_0.

(A 4) $R(t,\tau) - \eta I_n \delta(t-\tau)$ is a covariance for some positive constant η.

(A 5) $[F, G]$ is uniformly completely controllable, Silverman and Anderson (1968); F, G, H and R are bounded.

(A 6) F is asymptotically stable. For the case when the interval of interest is bounded on the right (but possibly not on the left), 'F is asymptotically stable and bounded' should be interpreted as meaning 'F is bounded and the transition matrix $\Phi(t, \tau)$ is bounded by a quantity of the form

$$a_1 \exp[-a_2(t-\tau)]$$

for some positive constants a_1, a_2'.

(A 7) F and H are bounded.

Control law 1. For the system (1) and performance index (2) with (A 1) satisfied on $[t_0, t_1]$, the necessary condition for a control law u_* to exist, and to minimize the index (2) is that (A 2) be satisfied on $[t_0, t_1]$. Sufficient conditions are *either* that (A 4) be satisfied on $[t_0, t_1]$ *or* that for some T_0, matrices F, G, R and H can be selected on $[t_0 - T_0, t_0]$ such that (A 2) is satisfied on $[t_0 - T_0, t_1]$ and (A 3) is satisfied.

The control law u_* is given by:

$$u_* = -R^{-1}(G'\Pi(t,t_1) + H')x \tag{5}$$

and the minimum index is:

$$V_*(x_0, u_*, t_1, t_0) = x_0'\Pi(t_0, t_1)x_0, \tag{6}$$

where $\Pi(t, t_1)$ is the solution of the Riccati differential equation:

$$-\dot{\Pi} = \Pi(F - GR^{-1}H') + (F' - HR^{-1}G')\Pi - \Pi GR^{-1}G'\Pi - HR^{-1}H', \tag{7a}$$

$$\Pi(t_1, t_1) = 0. \tag{7b}$$

Further, the minimum performance index $V_*(x_0, u_*, t_1, t_0)$ is bounded as follows:

$$-\infty < K(x_0, t_0) \le V_*(x_0, u_*, t_1, t_0) \le 0, \tag{8}$$

where K is $-V(0, u_c, t_0, t_0 - T_0)$, see (A 3), and depends on x_0, (because u_c depends on x_0), as well as t_0.

Some results for the case when the interval $[t_0, t_1]$ is infinite are also given in Moore and Anderson (1968). These are now reviewed.

For the limiting case as $t_1 \to \infty$, assuming that (A 1) is satisfied on $[t_0, \infty)$, the sufficient conditions for (8) to be satisfied are *either* that (A 4), (A 6) and (A 7) be satisfied on $[t_0, \infty)$ *or* for some positive T_0, matrices F, G, R and H can be selected on $[t_0 - T_0, t_0]$ such that (A 3) is satisfied and such that (A 2) be satisfied on $[t_0 - T_0, \infty]$.

For the case when t_0 is arbitrary we may require the $K(x_0, t_0)$ of (8) to be independent of t_0, i.e. we require $V_*(x_0, u_*, t_1, t_0)$ to be bounded as:

$$-\infty < K(x_0) \leq V_*(x_0, u_*, t_1, t_0) \leq 0. \tag{9}$$

For the limiting case as $t_0 \to -\infty$, assuming that (A 1) is satisfied on $(-\infty, t_1]$, sufficient conditions for (9) to be satisfied are *either* that (A 4), (A 6) and (A 7) are satisfied on $(-\infty, t_1]$ *or* that both (A 2) and (A 5) are satisfied on $(-\infty, t_1]$.

For the limiting case as both $t_0 \to -\infty$ and $t_1 \to \infty$, assuming (A 1) is satisfied on $(-\infty, \infty)$, sufficient conditions for (9) to be satisfied are *either* that (A 4), (A 6) and (A 7) are satisfied on $(-\infty, \infty)$ *or* that both (A 2) and (A 5) are satisfied on $(-\infty, \infty)$.

Another result that may be quoted from Moore and Anderson (1968) is that when the initial state is zero, then:

$$V(0, u, t_1, t_0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u'(t) \mathrm{R}(t, \tau) u(\tau) \, dt \, d\tau, \tag{10}$$

with $u(t)$ taken to be zero outside $[t_0, t_1]$.

3. Existence considerations

Two lemmas are now given regarding the limiting solution of (7) as $t_1 \to \infty$.

Lemma 1. Assume that (A 1) is satisfied on $[t_0, \infty)$ and *either* (A 4), (A 6) and (A 7) are satisfied on $[t_0, \infty)$ *or* a selection of F, G, R and H can be made in a finite interval $[t_0 - T_0, t_0]$ such that (A 3) is satisfied and (A 2) is satisfied on $[t_0 - T_0, \infty]$. Then a matrix $\overline{\Pi}(t)$ is defined by:

$$\lim_{t_2 \to \infty} \Pi(t, t_2) = \overline{\Pi}(t) \tag{11}$$

for all $t \geq t_0$.

Proof. Let u_1 be the control which minimizes $V(x_1, u, t_2, t_1)$ for some t_1 and t_2 with $t_2 > t_1 \geq t_0$ and some state x_1 at time t_1; let x_2 be the state at time t_2 resulting from x_1 and u_1, and let u_2 be the control which minimizes for some $t_3 > t_2$ the performance index $V(x_2, u, t_3, t_2)$. (Note that (A 2) implies $R + R'$ is non-negative definite, and thus by (A 1), the finite-valuedness of R^{-1} and the symmetry of R imply that R is positive definite. Then control law 1 guarantees the existence of both controls u_1 and u_2.) Let u_{12} denote the concantenation of u_1 and u_2. Then

$$\begin{aligned} x_1'\Pi(t_1, t_3)x_1 &\leq V(x_1, u_{12}, t_3, t_1) \\ &= V(x_1, u_1, t_2, t_1) + V(x_2, u_2, t_3, t_2) \\ &= x_1'\Pi(t_1, t_2)x_1 + x_2'\Pi(t_2, t_3)x_2. \end{aligned} \tag{12}$$

The application of control law 1 means that $x_2'\Pi(t_2, t_3)x_2$ is bounded above by zero; thus (12) may be simplified as follows:

$$x_1'\Pi(t_1, t_3)x_1 \leq x_1'\Pi(t_1, t_2)x_1. \tag{13}$$

That is, $x_1'\Pi(t_1,t)x_1$ is a decreasing function of t. This, together with the result that it is bounded below (see control law 1 with t_0 replaced by t_1, x_0 replaced by x_1, t_1 replaced by t) may be used to show that $\lim_{t_2\to\infty}\Pi(t_1,t_2)$ exits; since t_1 is arbitrary, the lemma is established.

Lemma 2. With conditions satisfied as in lemma 1 $\overline{\Pi}(t)$ given from (11) satisfies the equation:

$$-\dot{\overline{\Pi}}=\overline{\Pi}(F-GR^{-1}H')+(F'-HR^{-1}G')\overline{\Pi}-\overline{\Pi}GR^{-1}G'\overline{\Pi}-HR^{-1}H'. \tag{14}$$

Proof. Using the notation $\Pi(t,t_1;B)$ to denote the solution of (7 *a*) with B as the boundary condition at t_1, then for the case $t_0\le t<t_1<t_2$:

$$\Pi(t,t_2)=\Pi(t,t_1;\Pi(t_1,t_2)). \tag{15}$$

(Note that $\Pi(t,t_1)$ and $\Pi(t,t_1;0)$ are equivalent.) In the limit as $t_2\to\infty$, using the continuity of the solution of differential equations with respect to initial conditions:

$$\overline{\Pi}(t)=\Pi(t,t_1;\overline{\Pi}(t_1)). \tag{16}$$

Since then $\Pi(t,t_1;\overline{\Pi}(t_1))$ satisfies (7 *a*), $\overline{\Pi}(t)$ satisfies (14) and the lemma is established.

4. Infinite time control law

Further lemmas are now considered leading to a control law for the infinite time case.

Lemma 3. With (A 1), (A 6) and
(A 8) F and G bounded,
satisfied on $[t_0,\infty)$ and either (A 4) and (A 7) satisfied on $[t_0,\infty)$ or (A 3) and (A 4) satisfied on $[t_0-T_0,\infty)$ for suitable selection of F, G etc., on $[t_0-T_0,t_0]$ then the application of the control law:

$$\bar{u}_*=-R^{-1}(G'\overline{\Pi}+H')x \tag{17}$$

results in an asymptotically stable closed-loop system, i.e.

$$\dot{x}=[F-GR^{-1}(G'\overline{\Pi}+H')]x \tag{18}$$

is asymptotically stable.

Proof. If (A 4) and (A 7) hold on $[t_0,\infty)$, then because (A 6) also holds, a selection of T_0, F, G etc., may certainly be found on $[t_0-T_0,t_0]$ such that $\mathrm{R}(t,\tau)$ is a covariance on $[t_0-T_0,\infty)$ and F, G is completely controllable. See Appendix 1 of Moore and Anderson (1968). In fact, selections may be made so that $\mathrm{R}(t,\tau)-\hat{\eta}I\delta(t-\tau)$ is a covariance for any positive $\hat{\eta}<\eta$, as may readily be checked by examining the constructive procedure in detail. Thus we can always assume that (A 3) and (A 4) hold, with $\hat{\eta}$ replacing η, either by initial assumption, or because of (A 4), (A 6) and (A 7).

The first step in the proof of asymptotic stability is to show that $\bar{u}_*$ is square integrable. Define $\hat{u}$ to be the concatenation of u_c as defined in (A 3) and u_* as in (5), and having zero value ouside the time interval $[t_0-T_0,t_1]$; then using (10) we may write:

$$V(0,u_c,t_0,t_0-T_0)+V(x_0,u_*,t_1,t_0)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\hat{u}'(t)\mathrm{R}(t,\tau)\hat{u}(\tau)\,dt\,d\tau. \tag{19}$$

Now an application of (6) and (A 4) results in the inequality, holding for any $t_3 < t_1$,

$$V(0, u_c, t_0, t_0 - T_0) + x_0'\Pi(t_0, t_1)x_0 \geq \hat{\eta} \int_{t_0}^{t_3} u_*'u_* \, dt. \tag{20}$$

Taking the limit of both sides of (20) as $t_1 \to \infty$ gives, using (11) and (17):

$$V(0, u_c, t_0, t_0 - T_0) + x_0'\overline{\Pi}(t_0)x_0 \geq \hat{\eta} \int_{t_0}^{t_3} \bar{u}_*'\bar{u}_* \, dt. \tag{21}$$

The finiteness of the interval $[t_0, t_3]$ is important for taking the limit on the right-hand side of (20). Now since the left side of the inequality (21) is independent of t_3 and since both $V(0, u_c, t_0, t_0 - T_0)$ and $x_0'\overline{\Pi}(t_0)x_0$ are finite (see control law 1 and lemma 1), it follows that $\bar{u}_*$ is square integrable.

Because F is bounded and asymptotically stable, it is exponentially asymptotically stable. Consequently the response of:

$$\dot{x} = Fx + G\bar{u}_* \tag{22}$$

approaches zero as $t \to \infty$, using the boundedness of G and a slight modification of a result of Sandberg (1964), see theorem 9.

Lemma 3 does not establish that $\bar{u}_*$ is an optimal control law for the infinite time problem. It appears that to do this, stronger conditions are required.

Lemma 4. With (A 1), and either (A 2) and (A 5) or (A 4), (A 6), (A 7) holding on $(-\infty, t_2]$:

$$0 \geq \Pi(t_1, t_2) \geq \delta I \quad \text{for all} \quad t_1 \leq t_2 \tag{23}$$

for some negative δ. With (A 1), and either (A 2) and (A 5) or (A 4), (A 6) (A 7) holding on $(-\infty, \infty)$,

$$0 \geq \overline{\Pi}(t_1) \geq \delta I \quad \text{for all} \quad t_1, \tag{24}$$

for some negative δ.

Proof. Consider first the proof of (23). The result follows by noting that either set of assumed conditions imply the existence of $\Pi(t_1, t_2)$ and satisfaction of:

$$0 \geq x_1'\Pi(t_1, t_2)x_1 \geq K(x_1) > -\infty, \tag{25}$$

where K is independent of t_1. Arguments as in Silverman and Anderson 1968 yield, when (A 2) and (A 5) hold, that $K(x_1) \geq \delta\|x_1\|^2$ for some negative δ and all x_1, whence (23) follows.

Under (A 4), (A 6), (A 7) it is shown in Moore and Anderson (1968) that

$$K(x_1) = -\rho \int_{t_1 - T_0}^{t_1} u_c'u_c \, dt$$

where ρ and u_c are independent of t_1. It then follows again that $K(x_1) \geq \delta\|x_1\|^2$ for some negative δ and all x_1. Similarly (24) may be proved.

Equations (23) and (24) also hold for all $t_1 > T_1$, where T_1 is some fixed time, if (A 4), (A 6) and (A 7) hold on $[T_1, \infty)$ or (A 5) holds on $[T_1, \infty)$ and (A 2) on $[T_1 - T_1', \infty)$ where T_1' is greater than the minimum time required to transfer from the zero state to an arbitrary state. The proof is again a simple extension of the earlier material.

Lemma 5. If conditions (A 1), (A 4), (A 6), (A 7), (A 8) hold on $[t_0, \infty)$ the optimal control minimizing:

$$V(x_1, u, \infty, t_1) = \int_{t_1}^{\infty} (u'Ru + 2x'Hu)\, dt \tag{26}$$

for arbitrary $t_1 \geq t_0$ is $\bar{u}_*$ given from (17) and the optimal index is:

$$0 \geq V(x_1, \bar{u}_*, \infty, t_1) = x_1'\overline{\Pi}(t_1)x_1 \geq \delta \|x_1\|^2 \tag{27}$$

for some negative constant δ.

Proof. The substitution of (17) into (2) and the application of (7) gives for arbitrary t_2:

$$V(x_1, \bar{u}_*, t_2, t_1) = x_1'\overline{\Pi}(t_1)x_1 - x_2'\overline{\Pi}(t_2)x_2. \tag{28}$$

Taking the limit as $t_2 \to \infty$ and using the boundedness of $\overline{\Pi}$ which follows from lemma 4, together with the stability of the closed loop system, see lemma 3, yields (27). In order to show that $\bar{u}_*$ in fact minimizes (26) assume there is a different control u_0 with performance index less than $x_1'\overline{\Pi}(t_1)x_1$. Then for t_2 sufficiently large:

$$V(x_1, u_0, t_2, t_1) < x_1'\overline{\Pi}(t_1)x_1 \tag{29}$$

and since $x_1'\Pi(t_1, t_2)x_1$ is a decreasing function as t_2 increases, from (29) we have:

$$V(x_1, u_0, t_2, t_1) < x_1'\Pi(t_1, t_2)x_1, \tag{30}$$

i.e.

$$V(x_1, u_0, t_2, t_1) < V(x_1, u_*, t_2, t_1). \tag{31}$$

This contradicts control law 1 and the proof is complete.

A partial converse to the above result is that if $R(t, \tau)$ is not a covariance, lack of existence of an optimal control can be demonstrated along the lines adopted for the finite time case.

The preceding results may be summarized into a control law for the infinite time problem.

Control Law 2. For the system (1) and performance index (26) sufficient conditions for a control law $\bar{u}_*$ given by Riccati theory to exist and be well defined which minimizes (26) are that (A 1), (A 4), (A 6), (A 7) and (A 8) be satisfied on $[t_0, \infty)$. A necessary condition is that (A 2) be satisfied on $[t_0, \infty)$.

The control law $\bar{u}_*$ which minimizes (26) is given in (17); the minimum index is given by (27) where $\overline{\Pi}$ satisfies (14) and is given by (11) where, in turn, $\Pi(t, t_2)$ is the solution of (7 *a*) with the boundary condition $\Pi(t_2, t_2) = 0$. The closed-loop system (18) is also asymptotically stable.

5. Further properties of $\overline{\Pi}(t)$

Lemma 6. If the conditions of lemma 1 are fulfilled guaranteeing the existence of $\overline{\Pi}(t)$ and if the following condition is satisfied:

(A 9) $[F, H']$ is completely observable at time t, in the sense that if $H'(t_1)\Phi(t_1, t)x(t) = 0$ for all t_1 implies $x(t) = 0$, then $\overline{\Pi}(t)$ is negative definite.

If the stricter condition
(A 10) $[F, H']$ is uniformly completely observable on $[t_0, \infty)$ and F, G, R^{-1} and H are bounded on $[t_0, \infty)$ is satisfied, then:

$$\overline{\Pi}(t) \leq \gamma I < 0 \tag{32}$$

for some negative constant γ and all t.

Proof. Since $x'(t)\Pi(t,t_1)x(t)$ is a decreasing function as t_1 increases (see (13)), then if it can be shown that $\Pi(t,t_1)$ is negative definite, we may conclude using lemma 1 that $\overline{\Pi}(t)$ is negative definite. Certainly $\Pi(t,t_1)$ is non-positive definite (see (6) and (8)), and we now show that:

$$x^{*\prime}(t)\Pi(t,t_1)x^*(t) = 0 \tag{33}$$

for all t_1 and some non-zero $x^*(t)$ leads to a contradiction. Assumption (33) is equivalent to the assumption that the optimal performance index (6) is zero for the case $t_0 = t$, $x_0 = x^*(t)$, t_1 arbitrary. Since the optimal control u for a fixed t_1 is unique, Athans and Falb (1966), and a zero control results in a zero performance index (see (2)) then the optimal control u_* is zero whatever t_1 is, and (5) gives:

$$-R^{-1}(G'\Pi(\sigma,t_1) + H')x^*(\sigma) = 0, \tag{34}$$

where this equation holds for all t_1, and for all σ in the range $[t, t_1]$; also, $x^*(\sigma)$ is the response of the system with zero input and initial state $x^*(t)$. Setting $\sigma = t_1$ in (34) and using the boundary condition on Π leads to $H'(t_1)x^*(t_1) = 0$, which contradicts the complete observability of $[F, H']$, thus establishing the first part of the lemma.

The solution $\Pi_2(t,t_1)$ of

$$-\dot{\Pi}_2 = \Pi_2(F - GR^{-1}H') + (F' - HR^{-1}G')\Pi_2 - HR^{-1}H', \tag{35 a}$$

$$\Pi_2(t_1,t_1) = 0, \tag{35 b}$$

is easily verified to be:

$$\Pi_2(t,t_1) = -\int_t^{t_1} \Phi_1'(\lambda,t)HR^{-1}H'\Phi_1(\lambda,t)\,d\lambda, \tag{36}$$

where

$$\frac{d}{dt}\Phi_1(t,\lambda) = (F - GR^{-1}H')\Phi_1(t,\lambda); \quad \Phi_1(\lambda,\lambda) = I \tag{37}$$

and therefore if $[F - GH^{-1}H', H']$ is uniformly completely observable there exists a constant δ_0 such that for $t_1 - t > \delta_0$:

$$\Pi_2(t,t_1) \leq \gamma I < 0 \tag{38}$$

for some negative γ. The uniform complete observability condition may be stated in terms of its dual, viz. we required $[F' - HR^{-1}G', H]$ to be uniformly completely controllable.

This restatement may now be simplified using the result that for bounded linear systems, uniform complete controllability is preserved as bounded state feedback is applied, Silverman and Anderson (1967). That is, since F, H, R^{-1} and G are bounded we require simply that $[F', H]$ be uniformly completely controllable. This is in fact the dual of condition (A 9), and thus if (A 9) is satisfied (38) holds and the proof may proceed.

Substracting (7) from (35) gives a linear differential equation in the variable $\Pi - \Pi_2$:

$$-(\dot{\Pi} - \dot{\Pi}_2) = (\Pi - \Pi_2)(F - GR^{-1}H') + (F' - HR^{-1}G')(\Pi - \Pi_2) - HGR^{-1}G'\Pi, \tag{39 a}$$

$$\Pi(t_1, t_1) - \Pi_2(t_1, t_1) = 0, \tag{39 b}$$

which has a solution:

$$\Pi(t, t_1) - \Pi_2(t, t_1) = -\int_t^{t_1} \Phi_1'(\lambda, t)\Pi GR^{-1}G'\Pi\Phi_1(\lambda, t) \tag{40}$$

and this is non-positive definite. This result together with (38) imply that for $t_1 - t > \delta_0$:

$$\Pi(t, t_1) \leq \gamma I < 0. \tag{41}$$

The fact that $\Pi(t, t_1)$ is a decreasing function as t_1 increases (see (13)) and lemma 1 together imply that in the limit as t_1 approaches infinity (41) may be written as (32).

To conclude this section, we make some comments on the case $t_0 = -\infty$. Points of interest are the limiting behaviour as $t_0 \to -\infty$ of $\Pi(t_0, t_1)$ and $\bar{\Pi}(t_0)$, and properties of the closed-loop system for both the finite and infinite t_1 cases.

Lemma 7. If conditions (A 1), (A 4), (A 6), (A 7) and (A 8) are satisfied on $(-\infty, t_1]$, $\Pi(t, t_1)$ is bounded for all t and the transition matrix of $F - GR^{-1}(G'\Pi + H)$ is exponentially bounded on $(-\infty, t_1]$. With t_1 replaced by infinity, the same is true with $\Pi(t, t_1)$ replaced by $\bar{\Pi}(t)$.

Proof. Boundedness of $\Pi(t, t_1)$ and $\bar{\Pi}(t)$ follows from lemma 4; lemma 3 and the assumed boundedness of F, G, H and R^{-1} guarantees the exponential bound of the transition matrix of the closed-loop system for $t_1 = \infty$. For finite t_1, observe that eqn. (19) yields:

$$-K(x_0) + x_0'\Pi(t_0, t_1)x_0 \geq \hat{\eta}\int_{t_0}^{t_1} u_*'u_*\,dt, \tag{42}$$

where we have used the fact that K is independent of t_0 (see lemma 4). The bound on Π yields that

$$\int_{-\infty}^{t_1} u_*'u_*\,dt$$

is bounded. The boundedness of F and G and the asymptotic stability of F then yield as before that $\dot{x} = Fx + Gu_*$, i.e. the closed-loop system is asymptotically stable; exponential asymptotic stability follows because $F - GR^{-1}(G'\Pi + H)$ is bounded.

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