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Extensions of quadratic minimization theory
I. Finite time results†

J. B. MOORE and B. D. O. ANDERSON
Department of Electrical Engineering, University of Newcastle, New South Wales, 2308, Australia

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Necessary and sufficient conditions are developed for the existence and calculation of well-defined Riccati differential equation solutions associated with quadratic loss minimization problems. Of particular interest is the fact that a covariance condition is involved. The disclosure of this condition not only extends the range of optimal control problems for which a solution, guaranteed to be well defined, may be calculated, but also introduces an approach for establishing the existence of well-defined solutions in other problems involving covariance conditions, as for example, in a time-varying spectral factorization procedure. This paper is concerned with finite time results, while a companion paper considers the infinite time case.

1. Introduction

This paper considers the determinations of a set of necessary and sufficient conditions for the existence of well-defined Riccati differential equation solutions to finite time quadratic loss minimization problems.

A companion paper, Anderson and Moore (1968), considers the infinite time case.

The systems considered are linear and finite dimensional, and may be represented by the state-space equation:

$$\dot{x} = Fx + Gu, \tag{1}$$

where x is an n vector (the state), u is an m vector (the input), and F and G are matrices which are of appropriate dimension and may be time varying. The quadratic performance index considered is that often used for state-regulator problems, i.e.

$$V(x_0, u, t_1, t_0) = x_1' A x_1 + \int_{t_0}^{t_1} (u'Ru + 2x'Su + x'Qx)\, dt, \tag{2}$$

where x_0 is the notation used for the initial state $x(t_0)$ and x_1 for the final state $x(t_1)$. The various matrices (possibly functions of time) satisfy the conditions:

(A 1) F, G, R, S and Q are finite valued on $[t_0, t_1]$, and without loss of generality $A = A'$, $R = R'$, and $Q = Q'$. Also $R > 0$ on $[t_0, t_1]$.

Here $X > Y$ [$X \geq Y$] means that $(X - Y)$ is positive [non-negative] definite.

It has been shown, Kalman (1960) and Athans and Falb (1966), that *sufficient* conditions for the existence of a well-defined solution to the minimization of the performance index (2) are that (A 1) be satisfied, $Q - SR^{-1}S' \geq 0$ on $[t_0, t_1]$, and $A \geq 0$. The sufficient conditions given in this paper are far more general. Moreover, for the case when (A 1) is satisfied it is shown that a covariance condition given below is a *necessary* condition for a well defined problem solution.

† Communicated by Professor B. D. O. Anderson.

The various developments in this paper to some extent parallel those of Anderson (1967), where the time invariant case (F, G, Q, R, S constant) is solved when $Q=A=0$, F is asymptotically stable, $(R/2-\eta I)+S'(sI-F)^{-1}G$ is positive real for some positive η, and $[F, G]$ is completely controllable.

A number of system theory results are now reviewed.

The response of system (1) to a control signal u may be written as:

$$x(t)=\Phi(t,t_0)x_0+\int_{t_0}^{t}\Phi(t,\lambda)G(\lambda)u(\lambda)\,d\lambda, \tag{3}$$

where $\Phi(t,t_0)$ is the state transition matrix given from the equation:

$$\frac{d}{dt}\Phi(t,t_0)=F\Phi(t,t_0), \tag{4}$$

$$\Phi(t_0,t_0)=I_n. \tag{5}$$

If the output of (1) is given by:

$$y=H'x+Ju, \tag{6}$$

where y is an m vector, and the initial state of the system is zero, then the impulse response $w(t,\tau)$ relating u to y is given by:

$$w(t,\tau)=J(t)\delta(t-\tau)+H'(t)\Phi(t,\tau)G(\tau)1(t-\tau) \tag{7}$$

where $\delta(t)$ is the Dirac delta function, and $l(t)$ is the unit step function at time t. Of interest in the next section is the matrix:

$$\mathrm{R}(t,\tau)=w(t,\tau)+w'(\tau,t). \tag{8}$$

In particular $\mathrm{R}(t,\tau)$ will be a covariance matrix if the following condition is satisfied for all u, T_0 and T_1:

$$\int_{T_0}^{T_1}\int_{T_0}^{T_1}u'(t)\mathrm{R}(t,\tau)u(\tau)\,dt\,d\tau\geq 0. \tag{9}$$

If (9) holds only for T_0 and T_1 contained within an interval $[t_0, t_1]$, we shall say that $\mathrm{R}(t,\tau)$ is a covariance on $[t_0,t_1]$, where t_0 and t_1 are not necessarily restricted to being finite.

2. Finite time quadratic minimization

In this section a number of preliminary results are first developed. These will enable the stating of necessary and sufficient conditions guaranteeing the existence of a solution to the finite time minimization problem.

An expansion of the $(x'Qx)$ term of the index (2) using a matrix P defined through the linear differential equation:

$$-\dot{P}=PF+F'P+Q, \tag{10}$$

$$P(t_1)=A, \tag{11}$$

is as follows (noting that (A 1) implies that $P=P'$):

$$\int_{t_0}^{t_1}(x'Qx)\,dt=-x'Px\Big|_{t_0}^{t_1}+2\int_{t_0}^{t_1}(x'P\dot{x}-x'PFx)\,dt. \tag{12}$$

An application of (1) and (11) gives the result:

$$x_1'Ax_1+\int_{t_0}^{t_1}(x'Qx)\,dt=x_0'P(t_0)x_0+2\int_{t_0}^{t_1}x'PGu\,dt. \tag{13}$$

This means that if we define H and J (see (6)) as:

$$H=S+PG;\quad J=R/2, \tag{14}$$

the index (2) may be written as:

$$V(x_0,u,t_1,t_0)=x_0'P(t_0)x_0+\int_{t_0}^{t_1}(u'Ru+2x'Hu)\,dt \tag{15}$$

and for the case $x_0=0$ (using (6)):

$$V(0,u,t_1,t_0)=\int_{t_0}^{t_1}(y'u+u'y)\,dt. \tag{16}$$

Since for the case $x_0=0$:

$$y(t)=\int_{t_0}^{t}w(t,\tau)u(\tau)\,d\tau, \tag{17}$$

and for $\tau>t$, $w(t,\tau)=0$, a rearrangement of (16) may be made as follows:

$$V(0,u,t_1,t_0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u'(t)[w(t,\tau)+w'(\tau,t)]u(\tau)\,dt\,d\tau, \tag{18}$$

where $u(t)$ is zero outside the time interval $[t_0,t_1]$. Application of (8) and (9) gives the important property that if:

(A 2) $\mathrm{R}(t,\tau)$ is a covariance on $[t_0,t_1]$, where

$$\mathrm{R}(t,\tau)=R(t)\delta(t-\tau)+H'(t)\Phi(t,\tau)G(\tau)1(t-\tau)+G'(t)\Phi'(\tau,t)H(\tau)1(\tau-t) \tag{19}$$

(see also (14), (10) and (11) for values of H) then:

$$V(0,u,t_1,t_0)\geq 0\quad\text{for and } u. \tag{20}$$

Under the assumption that there exists a control u_* minimizing (2), we can simply establish an upper bound on the minimum performance index

$$V_*(x_0,u_*,t_1,t_0).$$

Since $u\equiv 0$ is a legitimate control, (15) gives the result that:

$$V(x_0,0,t_1,t_0)=x_0'P(t_0)x_0 \tag{21}$$

and therefore we have immediately:

$$V_*(x_0,u_*,t_1,t_0)\leq x_0'P(t_0)x_0 \tag{22}$$

and an upper bound is established.

To establish a lower bound on $V_*(x_0,u_*,t_1,t_0)$ is not so straightforward:

Lemma. For the system (1) with performance index (2) and the condition (A 1) satisfied (except that R need not be positive definite on $[t_0,t_1]$, sufficient conditions for the existence of a lower bound on $V_*(x_0,u_*,t_1,t_0)$ (assuming existence of u_*) are *either* (i) that a selection of F, G, S, Q and R can be made in a finite interval $[t_0 - T_0, t_0]$ such that:

(A 3) $\mathrm{R}(t, \tau)$ is a covariance on $[t_0 - T_0, t_1]$

and

(A 4) $[F, G]$ is completely controllable at t_0 in the sense that for any state x_0 at t_0, a control u_c and time T_0 may be found taking the system from the zero state at time $t_0 - T_0$ to the state x_0 at time t_0;

or (ii) (and (ii) implies (i)) that:

(A 5) $\mathrm{R}(t, \tau) - \eta I_n \delta(t-\tau)$ is a covariance on $[t_0, t_1]$ for some positive constant η.

Proof. The assumption that a selection of F, G, Q, S and R can be made in a finite interval $[t_0 - T_0, t_0]$ such that (A 3) and (A 4) are satisfied yields that $V(0, u_c, t_0, t_0 - T_0)$ is finite; use of (20) appropriately modified yields that $V(0, u_c, t_0, t_0 - T_0)$ is non-negative. A further application of (20) gives the result that if the control u_* (assuming this exists) is applied in the time range $[t_0, t_1]$ then:

$$V(0, u_c, t_0, t_0 - T_0) + V_*(x_0, u_*, t_1, t_0) \geq 0. \tag{23}$$

The result establishing a lower bound for $V_*(x_0, u_*, t_1, t_0)$ is thus:

$$-\infty < K(x_0, t_0) \leq V_*(x_0, u_*, t_1, t_0), \tag{24}$$

where K is $-V(0, u_c, t_0, t_0 - T_0)$ and depends on x_0 (because u_c depends on x_0), and on t_0.

To complete the proof, it will be shown that if (A 5) is satisfied, a selection may be made of T_0 itself, and the matrices F, G, Q, S and R over $[t_0 - T_0, t_0]$ so that (A 3) and (A 4) are satisfied. First choose F, G, and T_0 such that (A 4) holds; if desired, F and G can be constant and T_0 arbitrarily small. Then select Q arbitrarily and S such that H in (14) is zero. Finally select R to be ρI where ρ is a positive constant, specified below. Then for arbitrarily u:

$$\begin{aligned}\int_{t_0 - T_0}^{t_1} \int_{t_0 - T_0}^{t_1} & u'(t)\, \mathrm{R}(t, \tau) u(\tau)\, d\tau\, dt \\ &= \int_{t_0 - T_0}^{t_1} u_1'(t) u_1(t)\, dt + \int_{t_0}^{t_1} \int_{t_0}^{t_1} u_2'(t) \mathrm{R}(t, \tau) u_2(\tau)\, d\tau\, dt \\ &\quad + \int_{t_0}^{t_1} u_2'(t) H'(t) \Phi(t, \tau_0)\, dt \int_{t_0 - T_0}^{t} \Phi(t_0, \tau) G(\tau) u_1(\tau)\, d\tau, \end{aligned} \tag{25}$$

where u_1 is u restricted to the interval $[t_0 - T_0, t_0]$, u_2 is u restricted to the interval $[t_0, t_1]$. Application of the Cauchy–Schwarz inequality and condition (A 5) now gives the inequality:

$$\begin{aligned}\int_{t_0 - T_0}^{t_1} \int_{t_0 - T_0}^{t_1} u'(t) \mathrm{R}(t, \tau) u(\tau)\, dt\, d\tau &\geq \rho \int_{t_0 - T_0}^{t_0} u_1' u_1\, dt + \eta \int_{t_0}^{t_1} u_2' u_2\, dt \\ &\quad - \left\{ \left[\int_{t_0}^{t_1} u_2' u_2\, dt \right]^{1/2} \left[\int_{t_0}^{t_1} \|H'(t) \Phi(t, t_0)\|^2\, dt \right]^{1/2} \right. \\ &\quad \left. \times \left[\int_{t_0 - T_0}^{t_0} \|\Phi(t_0, \tau) G(\tau)\|^2\, dt \right]^{1/2} \left[\int_{t_0 - T_0}^{t_0} u_1' u_1\, d\tau \right]^{1/2} \right\} \end{aligned} \tag{26}$$

and evidently there exists a choice of ρ such that the right-hand side of (26) is non-negative irrespective of $u(\cdot)$. To see this, observe that with z_1 identified with

$$\left[\int_{t_0-T_0}^{t_0} u_1'u_1\,dt\right]^{1/2}$$

and z_2 identified with

$$\left[\int_{t_0}^{t_1} u_2'u_2\,dt\right]^{1/2},$$

the right side of (26) is of the form:

$$\rho z_1^2+\eta z_2^2-\gamma z_1 z_2, \tag{27}$$

where η and γ are fixed constants. Variation of u corresponds to variation of z_1 and z_2, and the choice of ρ is made so that the quadratic form (27) is non-negative for all z_1 and z_2. Note that if η were zero, this could not be done.

We conclude that the covariance condition (A 5) implies that matrices S and R on $[t_0-T_0, t_0]$ can be found for any choice of Q, F and G on $[t_0-T_0, t_0]$ such that the covariance condition (A 3) is satisfied. Moreover, since an F and G selection on $[t_0-T_0, t_0]$ can readily be made such that (A 4) is satisfied, then with (A 5) satisfied the application of the first part of the proof of lemma 1 gives the lower bound for $V_*(x_0, n_0, t_1, t_0)$ as in (24).

The Appendix considers extensions of the lemma to be used in Anderson and Moore (1967) for the limiting cases when t_0 approaches $-\infty$ and/or t_1 approaches $+\infty$.

As remarked earlier, commonly the minimization of (2) is considered with $Q-SR^{-1}S'$ non-negative definite. Under this condition, it is not difficult to show that (A 5) holds; then in this case no explicit assumption of complete controllability is required in order to obtain optimal performance index bounds.

We now return to the mainstream of the development.

Under condition (A 1), application of Hamiltonian–Jacobi theory, Kalman (1960) and Athans and Falb (1966), gives a Riccati differential equation that may be used to give the minimum value of the index (2), or equivalently (15) *if the Riccati equation solution exists and is well defined.* Since the term $x_0'P(t_0)x_0$ in (15) is constant for fixed t_0, it is permissible to seek to minimize just the integral on the right side of the equation. Denoting by $\tilde{V}_*$ the minimum of this integral, the associated Hamilton–Jacobi equation is:

$$\frac{\partial \tilde{V}_*(x,t)}{\partial t}+\mathrm{H}_*\left(x, \frac{\partial \tilde{V}_*}{\partial x}(x,t), t\right)=0, \tag{28a}$$

$$\tilde{V}_*(x,t_1)=0 \quad \text{for all} \quad x, \tag{28b}$$

where H_* is the unique minimum value (with respect to u) of the function:

$$\mathrm{H}(x,p,t,u)=u'Ru+2x'Hu+p'Fx+p'Gu \tag{29}$$

(p being the adjoint variable), found by selecting an appropriate control $u_*(x,p,t)$. The control u_* which minimizes (29) is given by:

$$u^*=-R^{-1}\left(G'\frac{p}{2}+H'x\right) \tag{30}$$

and the minimized Hamiltonian is thus:

$$\mathrm{H}_*(x,p,t) = -\tfrac{1}{4}p'GR^{-1}G'p - x'HR^{-1}H'x + p'(F-GR^{-1}H')x. \tag{31}$$

It is readily checked that (28) has a solution $\tilde{V}_*(x,t) = x'\Pi(t,t_1)x$, which gives the minimum performance index (2) as:

$$V_*(x_0,u_*,t_1,t_0) = x_0'[\Pi(t_0,t_1)+P(t_0)]x_0, \tag{32}$$

if $\Pi(t,t_1)$ is the solution of the Riccati differential equation:

$$-\dot{\Pi} = \Pi(F-GR^{-1}H') + (F'-HR^{-1}G')\Pi - \Pi GR^{-1}G'\Pi - HR^{-1}H', \tag{33 a}$$

$$\Pi(t_1,t_1) = 0. \tag{33 b}$$

The theory of Kalman (1960) and Athans and Falb (1966), also yields the result that the optimal control (30) may be found by replacing p by the gradient of $\tilde{V}^*(x,t)$. Thus one has, always assuming the existence of $\Pi(t,t_1)$,

$$u^* = -R^{-1}[G'\Pi(t,t_1)+H']x. \tag{34}$$

The existence result is as follows:

Theorem. Consider the system (1) and assume that (A 1) holds. Sufficient conditions for the solution of the matrix Riccati equation (33) to exist on $[t_0,t_1]$ are that *either* (A 5) holds *or* that selections of T_0, F, G, R, Q and S may be made on $[t_0-T_0,t_0]$ such that (A 3) and (A 4) hold. A necessary condition is that (A 2) holds.

Proof. Note first that the sufficiency conditions in the above hypothesis are identical to those required for the lemma to hold, except that for the theorem it is required that $R>0$ on $[t_0,t_1]$. Proof of the theorem proceeds by assuming that (33) does not have a well-defined solution and then reasoning as in Kalman (1960) follows. In brief, differential equation theory way be used to show that the solution is well defined in the neighbourhood of t_1. But if the solution is not extendable an arbitrary distance to the left, then for some τ_1 in the range $[t_0,t_1]$ but not in the neighbourhood of t_1, some x_0 and some positive ϵ, as $\epsilon\to 0$, $V_*(x_0,u_*,t_1,\tau_1+\epsilon)\to+\infty$ or $-\infty$ (see (33)). This contradicts either (22) or the lemma; thus the sufficient conditions for the lemma to hold, together with the requirements that $R>0$ on $[t_0,t_1]$, are shown to be sufficient conditions for (33) to exist on $[t_0,t_1]$.

Condition (A 2) is a necessary condition, since if it is not satisfied (8), (9) and (18) may be used to show that there is some $\tilde{u}$ for which $V(0,u,t_1,t_0)$ is negative, and this is in contradiction to the result that if the Riccati eqn. (33) has a well-defined solution, the minimum index $V_*(0,u_*,t_1,t_0)$ from (32) is zero.

The results given may be summarized in the following control law.

Control law. (Solution to the 'state regulator' problem.) Given the linear, finite dimensional system (1) and performance index (2) where (A 1) is satisfied, then a necessary condition for a minimizing control law u_* given from the Riccati theory to exist and be well defined is that (A 2) be satisfied (see also (19), (4), (5), (14), (10) and (11)).

The sufficient conditions are that either (A 5) be satisfied *or* for some T_0, that F, G, Q, R and S matrices can be selected on $[t_0 - T_0, t_0]$ such that (A 3) and (A 4) are satisfied.

The control law u_* is given by (34) and the minimum index is given by (32) where $\Pi(t, t_1)$ is the solution of the Riccati differential eqn. (33).

Existence results for (33) with a different boundary condition may be found in Appendix 2.

3. Concluding remarks

The introduction of necessary and sufficient conditions for the solution of finite time quadratic minimization problems has considerably extended the range of such problems for which it is known that well-defined solutions exist. A control law has been given for the simplest problem usually referred to as the state regulator problem, but extensions to the tracking problem and the case when the systems are stochastic are immediate.

In the companion paper, infinite time problems are considered for the case $A = Q = 0$ based on the results of this paper. The infinite time results appear to have application in time-varying spectral factorization problems and generalized Popov stability problems.

Appendix 1

An extension of the lemma of § 2 is now stated with proof. The extended lemma has application in the companion paper which extends the control law to the cases when the time interval $[t_0, t_1]$ is infinite.

Lemma (*Infinite Time*). Consider the system (1) with the performance index written as:

$$V(x_0, u, t_1, t_0) = \int_{t_0}^{t_1} (u'Ru + 2x'Hu)\,dt, \tag{35}$$

where F, G, R and H are finite values on $[t_0, t_1]$ and without loss of generality $R = R'$. For three cases when $[t_0, t_1]$ is an infinite time interval, sufficient conditions are given for $V_*(x_0, u_*, t_1, t_0)$ (assuming existence of u_*) to be bounded independently of t_1 as follows:

$$-\infty \le K(x_0, t_0) \le V_*(x_0, u_*, t_1, t_0) \le 0 \tag{36}$$

and independently of t_0 and t_1 as follows:

$$-\infty \le K(x_0) \le V_*(x_0, u_*, t_1, t_0) \le 0. \tag{37}$$

For convenience we label four conditions:

(B 1) $R(t, \tau)$ is a covariance

(B 2) $R(t, \tau) - \eta I_n \delta(t - \tau)$ is a covariance for some positive constant η.

(B 3) F is asymptotically stable; F and H are bounded†.

(B 4) $[F, G]$ is uniformly completely controllable, Silverman and Anderson (1968); F, G, R and H are bounded.

Case 1: Interval $[t_0, \infty]$. Equation (36) is satisfied in the limit for arbitrarily large t_1 if *either* (i) a selection of F, G, R and H can be made in a finite interval

† For the case when the interval of interest is bounded on the right (but possibly not on the left), 'F is asymptotically stable and bounded' should be interpreted as meaning 'F is bounded, and the transition matrix $\Phi(t, \tau)$ is bounded by a quantity of the form $a_1 \exp[-a_2(t - \tau)]$ for some positive constants a_1, a_2'.

$[t_0-T_0, t_0]$ such that (A 4) is satisfied and (B 1) is satisfied on $[t_0-T_0, \infty)$ *or* (ii) (and (ii) implies (i)) that (B 2) and (B 3) are both satisfied on $[t_0, \infty)$.

Case 2: Interval $(-\infty, t_1]$. Equation (37) is satisfied for arbitrarily negative t_0 if *either* (i) (B 1) and (B 4) *or* (ii) (B 2) and (B 3) are satisfied on $(-\infty, t_1]$.

Case 3: Interval $(-\infty, \infty)$. Equation (37) is satisfied for arbitrarily negative t_0 and arbitrarily positive t_1 if *either* (i) (B 1) and (B 4) *or* (ii) (B 2) and (B 3) are satisfied on $(-\infty, \infty)$.

Proof. With u_c a control taking the zero state at t_0-T_0 to state x_0 at time t_0, the bound $K(x_0, t_0)$ or $K(x_0)$ is precisely $-V(0, u_c, t_0, t_0-T_0)$. The existence of u_c can follow by (A 4), in which case K is not necessarily independent of t_0, or by (B 4) when K is independent of t_0. (The uniform complete controllability of $[F, G]$ ensures, see Silvermann and Anderson (1968), that T_0 and the bound of u_c of the § 2 lemma are independent of t_0, and this, together with the boundedness of F, G, R and H ensures that $V(0, u_c, t_0, t_0-T_0)$ or $-K(x_0)$ is independent of t_0.) Part (i) of each of the three cases is then established.

To check part (ii), we shall show that selections of T_0, ρ and a completely controllable pair $[F, G]$ on $[t_0-T_0, t_0]$ can be made (independently of t_0 in cases 2 and 3) so that $R(t,\tau)$ is a covariance on $[t_0-T_0, t_1]$ or $[t_0-T_0, \infty]$. The procedure of the main lemma is followed, while we note from eqn. (26) that in eqn. (27), the constant γ is:

$$\gamma=\left[\int_{t_0-T_0}^{t_0}\|\Phi(t_0,\tau)G(\tau)\|^2 d\tau\right]^{1/2}\left[\int_{t_0}^{t_1}\|H'(t)\Phi(t,t_0)\|^2 dt\right]^{1/2} \qquad (38\,a)$$

for case 2 or

$$\gamma=\left[\int_{t_0-T_0}^{t_0}\|\Phi(t_0,\tau)G(\tau)\|^2 d\tau\right]^{1/2}\left[\int_{t_0}^{\infty}\|H'(t)\Phi(t,t_0)\|^2 dt\right]^{1/2} \qquad (38\,b)$$

for cases 1 and 3. The first integral can be made independent of t_0 by choosing fixed T_0, F and G. In all cases, the second integral exists and is bounded *independently* of t_0 in cases 2 and 3 because of the boundedness of $H(\cdot)$ and the exponential asymptotic stability of F, following from the boundedness and asymptotic stability of F. Call $\hat{\gamma}$ the resulting bound on γ; it follows that the right side of (26) is bounded below by:

$$\rho z_1{}^2+\eta z_2{}^2-\hat{\gamma} z_1 z_2$$

and thus ρ can be chosen to ensure the non-negativity of this expression for all z_1, z_2. Because η and $\hat{\gamma}$ are independent of t_0, ρ is also.

The matrix H is of course taken as zero on the interval $[t_0-T_0, t_0]$. It follows that $K(x_0)$, or $-V(0, u_c, t_0, t_0-T_0)$ is

$$-\rho\int_{t_0-T_0}^{t_0} u_c{}'u_c\,dt.$$

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